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PIANO STRING SCALE OPTIMIZATION ON THE BASIS OF DYNAMIC MODELING OF PROCESS OF THE SOUND FORMATION

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Abstract

Several different approaches are described to the problem of the design of the piano string scale, starting from physical models of hammer and string. Optimization of piano scale can be carried out by comparison of the simulated spectra of the string vibrations excited by the hammer. The string is assumed to be perfectly flexible. The nonlinear hysteretic model describes the piano hammer. This model is based on the assumption that the hammer felt (made of wool) is a microstructural material possessing history-dependent properties, i.e. a material with memory. The parameters of the model were determined by numerical simulation of experimental data. The computer modeling is used for the systematical approach of the structure of piano scale to its optimum value. The main attention is devoted to choosing of suitable linear mass density and tension of piano strings, and determination of position of the striking point. The problem of a choice of a tension for the neighbor strings terminated on the separate bass and treble bridges is considered in detail.

INTRODUCTION

The grand piano is a musical string instrument with a more than two hundred year history. The design of a modern grand piano is a masterpiece of art. Its musical features have been empirically reached, as a result of many hundreds of experiments and long-term practical experience. But till now by manufacture of grand pianos the exact science was not practically used due to absence of good mathematical models

describing the processes of the sound formation.

The sound of the grand piano depends mostly on the detailed motion of strings excited by the impact of the hammers. So, the creation of good theoretical models of the hammer and the hammer-string interaction are important problems for determining the sound produced by a piano. The mathematical modeling of this problem allows to predict the spectrum of the piano string motion, which is very important for piano string scale design.

This paper is focused on the numerical simulation of the hammer-string interaction based on the new physical models of piano hammer [1, 2]. These models are based on assumption that the hammer felt made of wool is a microstructural material possessing history-dependent properties. Such a physical substance is called the hereditary material or material with memory.

The elastic and hereditary parameters of piano hammers were obtained using a special piano hammer testing device that was developed and built in the Institute of Cybernetics at Tallinn University of Technology [3].

The optimization of the piano string scale is considered here as altering of old scales or designing of new piano scales to improve the efficiency and quality of the sound production of the instrument. The main performance criterion, which is used here, is the simple and reasonable physical assumption about the spectra similarity of the neighbor notes. Another type of optimization presented below also is used for choosing suitable masses of bass strings to avoid such negative phenomenon as multiple contacts.

PIANO STRING SCALE AND BASIC FORMULAE

Piano string scale is a summary table of the full collection of the string lengths, string diameters, diameters of wrapping wires for the bass strings, and the distance along the string from the hammer striking point to the nearer string termination. The piano string scale computation is based on the more or less sufficient practical requirements and pure empirical original data.

Usually the total number of notes of grand piano is equal to eighty-eight. The number of strings is much larger, because the number of strings per note changes from one and two string for the bass notes to three strings in treble.

The main frequency of each note of the grand piano is equal to

$$f_{n+1} = f_n \sqrt[12]{2} = 1.05946 f_n, \quad n = 1 \dots 88, \quad (1)$$

thus the region of notes frequency is exactly determined from $f_1 = 27.5$ Hz for the first note A_0 to $f_{88} = 4186$ Hz for the last note c_5 .

The relationships connecting the transverse wave velocity of string c , string vibration frequency f , string length L , string tension T , and the linear mass density of string μ are the following

$$c = 2fL, \quad T = \mu c^2. \quad (2)$$

The string tension distribution calculated in according to (2) must be more or less smooth function of a key number N to provide a uniform loading to the cast-iron frame. The mass of the string M is given by

$$M = \mu L = 0.25\pi\rho Ld^2, \quad (3)$$

where d is the string diameter and ρ is the mass density of the string material. The case of the wrapped strings is not considered here.

STRING AND HAMMER MODELS

The displacement $y(x,t)$ of the ideal (flexible) string obeys the simple wave equation

$$\frac{1}{c^2} \frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}. \quad (4)$$

Similar to [4], we have the system of equations describing the hammer-string interaction

$$\begin{aligned} \frac{dz}{dt} &= -\frac{2T}{cm} g(t) + V, \\ \frac{dg}{dt} &= \frac{c}{2T} F(t), \end{aligned} \quad (5)$$

where $g(t)$ is the outgoing wave created by the hammer strike, $F(t)$ is the acting force; m , $z(t)$, and V are the hammer mass, the hammer displacement, and the hammer velocity. The hammer felt compression is determined by $u(t) = z(t) - y(0,t)$. Function $y(0,t)$ describes the string deflection at the contact point and is given by

$$y(0,t) = g(t) + 2 \sum_{i=1}^{\infty} g\left(t - \frac{2iL}{c}\right) - \sum_{i=0}^{\infty} g\left[t - \frac{2(i+a)L}{c}\right] - \sum_{i=0}^{\infty} g\left[t - \frac{2(i+b)L}{c}\right]. \quad (6)$$

It is assumed that the string of length L extends from $-bL$ to aL with $b = 1-a$. The initial conditions at the moment when the hammer first contacts the string, are taken to be $g(0) = z(0) = 0$, and $dz(0)/dt = V$.

The experimental testing of piano hammers that consist of a wood core covered with several layers of compressed wool felt demonstrates that all hammers have the hysteretic type of the force-compression characteristics. It was shown, that nonlinear hysteretic models can describe the dynamic behavior of the hammer felt [1, 2]. In according to a four-parameter hereditary model of the hammer presented in [1], the nonlinear force $F(t)$ exerted by the hammer is related to the felt compression $u(t)$ by the following expression

$$F(u(t)) = F_0 \left[u^p(t) - \frac{\varepsilon}{\tau} \int_0^t u^p(\xi) \exp\left(\frac{\xi-t}{\tau}\right) d\xi \right]. \quad (7)$$

Here the instantaneous hammer stiffness F_0 and compliance nonlinearity exponent p are the elastic parameters of the felt, and ε and τ are the hereditary parameters.

Another three-parameter hereditary model of the hammer is presented in [2] in the form

$$F(u(t)) = Q_0 \left[u^p + \alpha \frac{d(u^p)}{dt} \right]. \quad (8)$$

In this case the parameter Q_0 is the static hammer stiffness; compliance nonlinearity exponent p is also the elastic parameter, and α is the retarded time parameter.

According to these hysteretic models, a real piano hammer possesses history-dependent properties, or in other words, its felt made of wool is a material with memory. The parameters of the hammers in according to these models were obtained experimentally in [5], and for numerical simulation these values (and the hammer masses also) may be approximated as the functions of the hammer number N

$$\begin{aligned} F_0 &= 15.5 \exp(0.059N), & Q_0 &= 183 \exp(0.045N), & p &= 3.7 + 0.015N, \\ \varepsilon &= 0.9894 + 8.8 \cdot 10^{-5} N, & \tau &= 2.72 - 0.02N + 9 \cdot 10^{-5} N^2, \\ \alpha &= 248 + 1.83N - 5.5 \cdot 10^{-2} N^2 + 8.5 \cdot 10^{-4} N^3, & m &= 11.074 - 0.074N + 10^{-4} N^2. \end{aligned} \quad (9)$$

The spectrum of the string motion excited by the hammer is calculated directly from the force history $F(t)$ [4]. The general expression for the string mode energy level is

$$\begin{aligned} EL_n &= 10 \log \frac{2M\omega_n^2}{mV^2} (A_n^2 + B_n^2), \\ A_n &= \frac{\sin(\pi\alpha n)}{\pi n c \mu} \int_0^{t_0} F(s) \cos(\omega_n s) ds, & B_n &= -\frac{\sin(\pi\alpha n)}{\pi n c \mu} \int_0^{t_0} F(s) \sin(\omega_n s) ds, \end{aligned} \quad (10)$$

where $\omega_n = \pi n c L^{-1} = n\omega_0$ is the string mode angular frequency; t_0 is the contact time.

The presented models of piano string and hammer are used here as a tool for simulation of the hammer-string interaction.

BASS STRINGS TENSION OPTIMIZATION

The process of piano string scale optimization is presented here on an example of a scale of *Baby-Grand* piano designed by Tallinn Piano Factory together with the Department of Mechanics and Applied Mathematics at the Institute of Cybernetics of the Estonian Academy of Sciences.

The *Baby-Grand* piano is a small instrument. Its length is only 163 cm (5'4). The maximum lengths of the bass strings and location of the bass bridge are strongly determined by technological conditions. The form and location of the treble bridge is firmly determined by its position near the main diagonal of the soundboard. As the value of frequency for each note is known, and the length of the strings is also prescribed, to complete the piano scaling we must calculate the values of the appropriate linear mass density and tension of the strings.

As was mentioned above, the total number of notes in the piano is eighty-eight. According to the construction of this instrument, the first ten notes ($A_0 - F_1^\sharp$) have only one string per note. The notes from eleven to twenty nine ($G_1 - C_3^\sharp$) have two strings per note, and the other notes consist of three strings. Besides, those strings that structure the first twenty six notes ($A_0 - A_2^\sharp$) terminate on the bass bridge, and the other strings terminate on the treble bridge.

It is well known that due to the technical conditions and the recommendations of experienced piano makers the tension of the strings in treble does not depend on the key (or note) number. Thus, in our piano the string tension distribution for notes with $N \geq 27$ is a constant. The value of string tension for these strings is chosen approximately 680 N per string. Therefore, due to the fact that with increasing of the key (note) number N the diameters of these strings decrease smoothly and continuously as well as the hammer parameters and the note frequency changing, the oscillation spectra of the neighbor strings are alike and very similar. Such near resemblance of spectra characterizes also the high sound quality of the piano.

Special case of strings length discontinuity

However, there is the specific point of piano scale, where the smooth continuity of all functions is broken. This is a situation of the strings transition from bass bridge on another treble bridge. In this case two strings of the note A_2^\sharp ($N = 26$), which terminate on the bass bridge are much shorter than the strings of note $N = 27$ terminated on the treble bridge. The length of the strings $N = 26$ is equal to 831.2 mm, and the length of the strings $N = 27$ is equal to 1031.0 mm. The length of the next string $N = 28$ is equal to 1007.4 mm, so with increasing of the note number N the continuity of lengths of strings goes on. Therefore, to avoid a rough discontinuity of the sound spectra for neighbor notes which strings are terminated on the different bridges, we must choose the mass and the tension of the string $N = 26$ very accurately.

The procedure of this string tension optimization is based on the computer simulation of the hammer-string interaction using the basic formulae presented above. The three-parameter hammer model (8) was explored. The values of hammer

parameters were computed by formulae (9). The results for notes $N = 26, 27,$ and 28 are presented in Fig. 1. The force histories were obtained by solving the system of equation (5) for initial hammer velocity $V = 2$ m/s. The string oscillation spectra were calculated in according to (10).

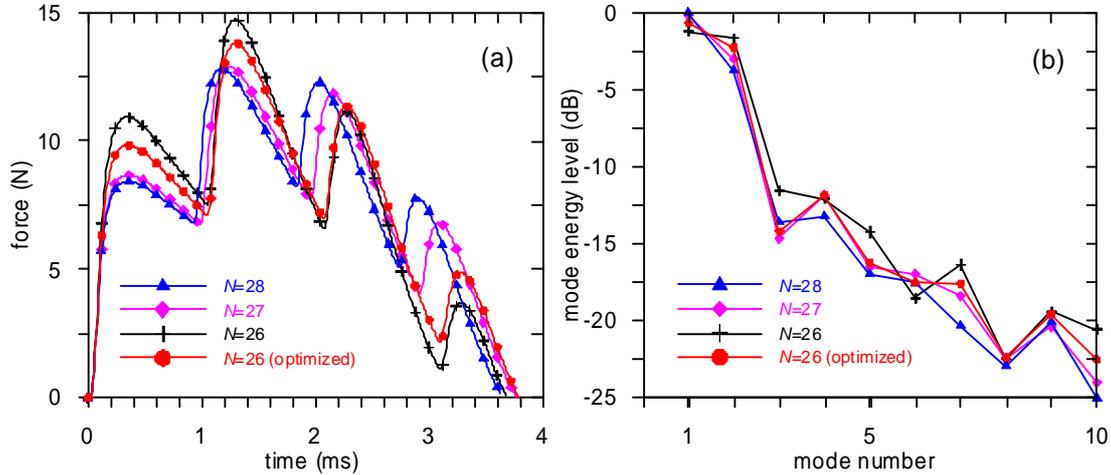


Figure 1 – Force histories and spectra envelopes for strings terminated on different bridges

The tension of strings for notes $N = 27$ and $N = 28$ is equal to 680 N. The corresponding curves are marked by diamond and triangle signs. The presented result demonstrates that the oscillation spectra of these neighbor strings are very similar indeed. On the contrary, the spectrum for note $N = 26$ calculated for the string tension 680 N (curve marked by crosses) differs from previous spectra in particular for mode numbers 3, 5, and 7. Repeating calculation for this string ($N = 26$) for the different string tension, the near resemblance of all three spectra was found for the string tension 600 N. It seems this value is the optimum string tension for the note $N = 26$. The corresponding curves are also displayed in Fig. 1, and marked by circle signs.

First string case

The first bass string of the grand piano is the longest string of the instrument. In our case its length is 1239.2 mm, and the note frequency is equal to 27.5 Hz. All bass strings terminate on a bass bridge, and as was mentioned above, the string tension distribution must be almost linear function of a key number N to provide a uniform loading to the cast-iron frame. The tension of the last bass string $N = 26$ is now determined, and it is equal to 600 N. Now the question is how to choose or determine the tension of the first bass string?

This problem can be solved also by numerical simulation of the string excitation by the hammer. In Fig. 2(a) are presented the force – time dependencies calculated for various string tensions. The initial hammer velocity was chosen rather high – 5 m/s. It is well known that for the hard blow the hammer has left the string just before the first reflection returns from the agraffe (the nearest string termination). This reflection pulse may catch up with the hammer and make renewed contact. This

undesirable phenomenon is referred to as multiple contacts. Therefore, to avoid this unwanted event we must choose such value of a string tension to minimize the influence of the reflected pulse.

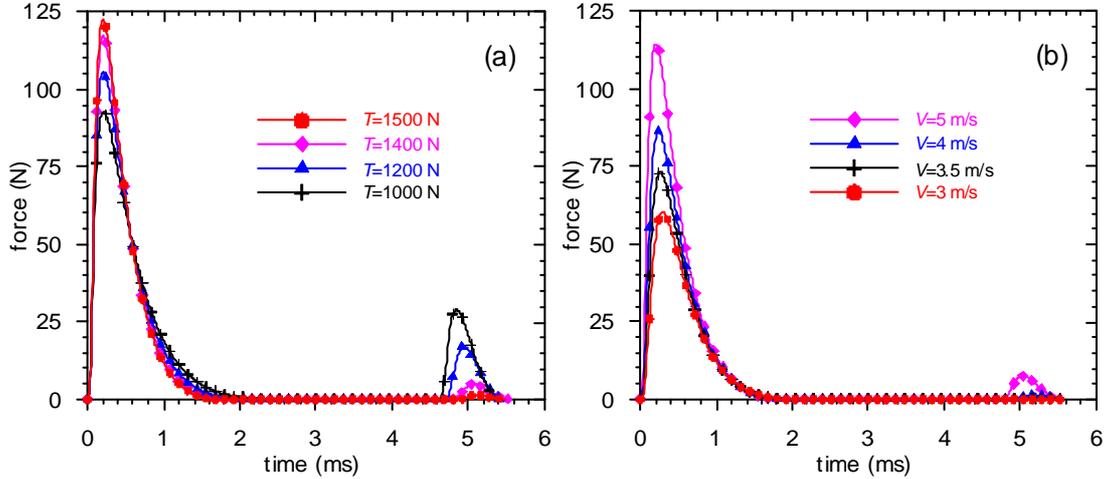


Figure 2 – Force histories computed for first bass string $N = 1$;
(a) - constant hammer velocity $V = 5$ m/s; (b) - constant string tension $T = 1350$ N

The results presented in Fig. 2 demonstrate that increasing of the string tension reduce the second pulse appearance. As a result, for the value of tension $T = 1350$ N selected for this string, the second pulse magnitude is a really minute for all values of hammer velocity. So, having defined a tension of the first and the last strings, we can obtain the linear law of tension distribution for all bass strings, and hence, in according to (2) and (3) the masses of strings are also defined.

STRIKING POINT POSITION OPTIMIZATION

Due to the fact that the string vibration spectrum is very sensitive to the position of the fractional striking point a , especially in treble, the numerical simulation of the hammer-string interaction gives the possibility to choose the piano string scale with a more uniform spectrum.

For approximately the sixty upper notes of the grand piano the position of the striking point little by little become displaced from $1/8$ to $1/24$ of the whole string length in the direction to the high notes. For example, for note $G_5^\#$ ($N = 60$) the recommended value of striking point position is close to $1/8.8$ (0.1136).

In Fig. 3 are displayed the spectra for this note calculated for initial hammer velocity $V = 1.5$ m/s, and for the different values of a . The recommended (used in practice) value of the fractional striking point for this note is $a = 0.114$. The visual inspection of these spectra shows that the even harmonics in the spectrum corresponding to recommended striking point are emphasized significantly. Probably, such spectrum really was the purpose for the designers of a grand piano. However, if we want to have a more uniform spectrum, we can choose another point of impact.

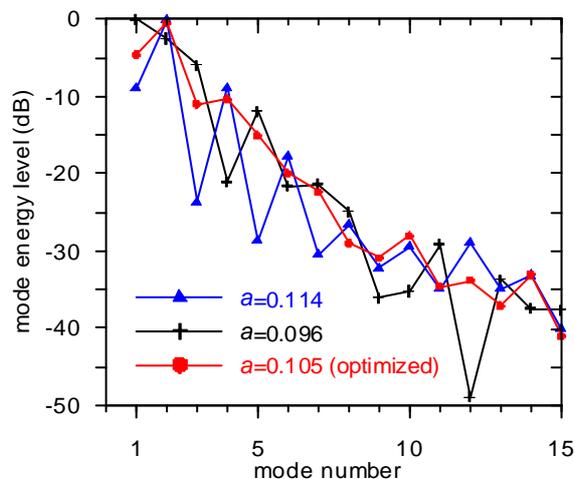


Figure 3 – Spectra for note $G_5^\#$ ($N = 60$) calculated for different striking point

If the criterion of optimization is the spectrum homogeneity, then the better choice is the fractional striking point $a = 0.105$.

CONCLUSIONS

Simple and efficient procedures, based on the numerical simulation of the hammer-string interaction, are available for piano string scale optimization. We conclude that application of physical models can be useful at designing new models of grand pianos.

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