19th INTERNATIONAL CONGRESS ON ACOUSTICS
MADRID, 2-7 SEPTEMBER 2007

PIANO STRING VIBRATION AND THE ROLE OF THE BRIDGE

PACS: 43.75.Mn

Stulov, Anatoli[1]; Erofeyev, Vladimir[2]; Lissenkova, Elena[2]
[1]Institute of Cybernetics at TUT, Akadeemia 21, Tallinn, 12618, Estonia; stulov@ioc.ee
[2]Mechanical Engineering Research Institute of Russian Academy of Sciences, Nizhny Novgorod Branch, 603024, IMASh RAN, 85 Belinskogo str., Nizhny Novgorod, Russia; erf04@sinn.ru

ABSTRACT
We investigate the bridge influence on the fundamental natural vibrating frequency of piano string in two ways. First, we consider a perfectly flexible string, which one end is rigidly clamped and another one is terminated on the bridge. The bridge is considered as a simple linear oscillating system, and the influence of the system parameters on a string vibrations spectrum is investigated. Also the problem of interaction of the wave traveling along the string with viscoelastic inertial support is considered. The both reflected and transmitted waves are described. The influence of a stiffness of a string on amplitude of its vibration is also estimated. It is shown, that the growth of the string stiffness moves the spectrum of the string vibrations in direction of high frequencies.

INTRODUCTION
The problem of determination of natural frequencies of the distributed systems arises in connection with an occasion of occurrence of a resonance in these systems under the forced oscillations. The resonance appears, when the frequency of external force is the exact integer multiplies of one of the natural frequencies of the distributed system [1]. The amplitude of the system oscillations at the absence of the dissipative forces theoretically can rise to infinity. However, in practice, the dissipative forces (external and internal), which prevent the unlimited increase of the amplitude are always exist. Nevertheless, the phenomenon of a resonance can affect the rising of inadmissible large amplitudes of oscillations and, as result the origin of the undesirable nonlinear phenomena, especially in case of a resonance at one of the lowest natural frequency. Thus, the determination of natural frequencies is interesting from the practical point of view, for example, for a rating and control of a technical condition of elements of the piano, and also for theoretical studies of process of the sound formation.

STRING WITH VISCOELASTIC SUPPORT
The dynamic behavior of the systems, which is described by the partial differential equation of a second-order, can be represented in the form of d'Alembert's waves travelling in both directions along the string. The interaction of a wave with a support, which is differing from the free end, creates a phase shift between the incident and reflected wave. As the termination of the string possesses elastic, inertial or dissipative properties, and, hence, it is the power-consuming too, the energy of an incident wave transforms in potential energy of the support, and then it is transferred to the system. Thus, the termination of the string acts like transmitter, and a little time is needed to provide this process. As a result of this effect the phase of the reflected wave is changed. On a basis of the mentioned above, the natural frequencies of system can be determined from equation:

$$2\frac{\omega \ell}{c} - \varphi_1 - \varphi_2 = 2\pi n \qquad \text{(Eq. 1)}$$

where ℓ is the system length, c is the speed of traveling wave, φ_i is the phase delay at reflection $(i = 1,2)$, $2\frac{\omega\ell}{c}$ is the phase shift for a double traveling time $\tau = \frac{\ell}{c}$.
Let's consider the dynamic behavior of the string with supports, which one is rigid and another one is the oscillating system (bridge) consisting of the mass, spring and damper (Figure 1).

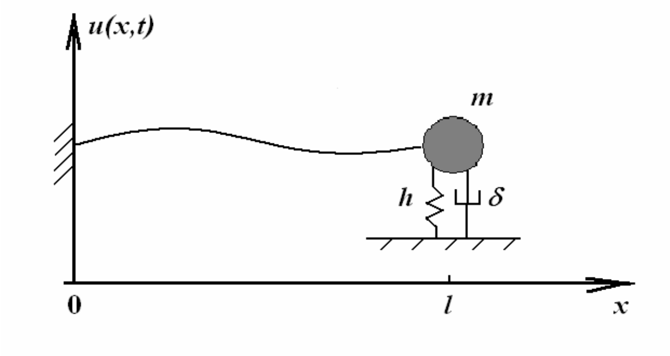

Figure 1.- Scheme of model I

In this case due to reflection from the absolutely rigid support the phase changes on π ($\varphi_1 = \pi$), and due to the reflection from the bridge
$\varphi_2 = 2arctg\left\{\left(Z_- Z_+ - z^2 + |z_0|^2\right)/2z|z_0 - \delta|\right\} \varphi_2 = 0$,
where $Z_- = \sqrt{(z-\delta)^2 + |z_0 - \delta|^2}$, $Z_+ = \sqrt{(z+\delta)^2 + |z_0 - \delta|^2}$, $z_0 = \delta - im\left(\Omega_0^2 - \omega_n^2\right)/\omega_n$.
Here $z = \sqrt{N\rho}$ is the wave resistance of the infinite string (string impedance); ρ, N are the linear mass density and the string tension; $\Omega_0 = \sqrt{k/m}$ is the main natural frequency of the bridge; m, k, δ are the inertial, the elastic, and the dissipative coefficients of support. Substituting these expressions into Eq. (1), we find

$$tg\frac{\omega_n \ell}{c} = -2z|z_0 - \delta| / \left(Z_- Z_+ - z^2 + |z_0|^2\right) \qquad \text{(Eq. 2)}$$

This is the equation for the resonant frequencies. From Eq. (2) it is possible to derive all limiting cases. Because the first natural frequency of piano string is the most important frequency, here we consider its behavior depending on a kind of the right termination. It is known [2], and Eq. (2) it confirms that the first natural frequency changes in an interval from $\pi/2\tau$ (for the free end) up to π/τ (in a case of the absolutely rigid clamp). The stiffness of the support moves the frequency value in the direction of the right side of this interval, the presence of the mass - to the left side. If the damper factor is less than the string impedance, the resonant frequency moves to the range of low frequencies, otherwise - to the range of high frequencies.
Let's note, that, as well as for the longitudinal oscillations of a rod [3], rigidly fixed at one end and with the damper at another end, at the initial stage of the string excitation (when the time does not exceed the double traveling time $t < 2\tau$), there are a such initial conditions, at which on this interval of time the amplitude of the string vibration is increased. After the moment $t > 2\tau$ in each cross-section of the string the transverse displacement of the string decays in according to exponential law. The rate of this decay is characterized by the logarithmic decrement, which is determined by expression $D = \ln\left|(1 + z/\delta)/(1 - z/\delta)\right|$.
This decrement depends on the ratio of the string impedance z to the dissipative coefficient δ, which we shall name here as the support impedance. The functional dependence of this decrement on the impedances ratio is shown in Figure 2. Thus, if the string impedance is equal to the support impedance, the value of logarithmic decrement tends to infinity. It means that at

any initial disturbances after the moment $t = 2\tau$ the amplitude of the string vibrations practically descends instantly.

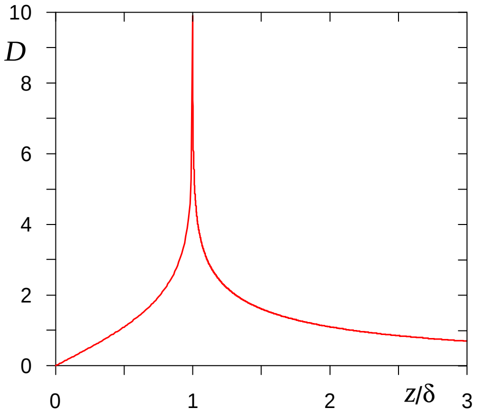

Figure 2.- The logarithmic decrement as a function of the impedances ratio

SINGLE INTERACTION OF THE TRANSVERSE WAVE TRAVELING ALONG THE STIFF STRING WITH A VISCOELASTIC INERTIAL SUPPORT

Before to study the most interesting problem of the string stiffness influence on the natural frequencies of the system, we shall consider in detail the modeling task shown in Figure 3, and describing the single interaction of a traveling wave with a bridge.

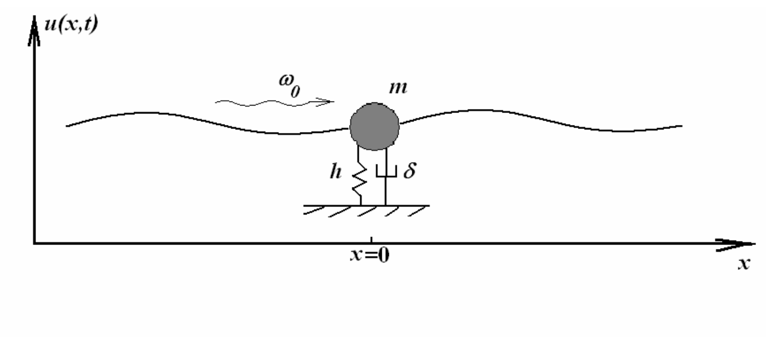

Figure 3.- Scheme of model 2

The distinctive features of such interaction in many respects can define the probable phenomena at the multiple reflections too, which takes place in the bounded systems.
The transverse displacement of a stiff string is described by the governing equation

$$\rho u_{tt} - N u_{xx} + IE u_{xxxx} = 0 \qquad \text{(Eq. 3)}$$

The boundary conditions at $x = 0$ can be written as

$$u(-0,t) = u(+0,t) = {}^0u(t) \qquad \text{(Eq. 4)}$$

$$u_{xx}(-0,t) = u_{xx}(+0,t) = 0 \qquad \text{(Eq. 5)}$$

$$m\,{}^0\ddot{u} + h\,{}^0u + \delta\,{}^0\dot{u} = \left[-Nu_x + IEu_{xxx}\right]_{x=0} \qquad \text{(Eq. 6)}$$

Here, as well as above, ρ, N, IE are the linear mass density, the string tension, and string stiffness; h, m, δ are the elastic, the inertial, and the dissipative coefficients of support.
Usually, the analysis of linear tasks can be divided into two separate stages: 1) to solve a kinematical problem (determination of frequencies ω and wave numbers k of the waves

excited in the distributed system); 2) to search the amplitudes of these waves and the forces of interaction.

Let the source located on the infinite distance on the left from the object radiates a simple-harmonic wave $u_f(x,t) = A_0 \exp[i(\omega_0 t - k_0 x)]$, which due to the interaction with the object creates the reflected waves $u_r(x,t) = A_1 \exp[i(\omega_1 t - k_1 x)] + A_2 \exp[i(\omega_2 t - k_2 x)]$, and the transmitted waves $u_p(x,t) = A_3 \exp[i(\omega_3 t - k_3 x)] + A_4 \exp[i(\omega_4 t - k_4 x)]$, where A_j are the amplitudes, ω_j are the frequencies, and k_j are the complex constants (*j*=1- 4).

Substituting expression

$$u(x,t) = \begin{cases} u_f(x,t) + u_r(x,t), & x < 0 \\ u_p(x,t), & x > 0 \end{cases} \qquad \text{(Eq. 7)}$$

into Eq. (3) and into boundary conditions Eq. (4) - Eq. (6), the following dispersion equation results

$$\omega^2 - c^2 k^2 - \alpha^2 k^4 = 0 \quad (c^2 = N/\rho,\ \alpha^2 = IE/\rho\ ) \qquad \text{(Eq. 8)}$$

and also the kinematical invariant

$$\omega = \omega_0 \qquad \text{(Eq. 9)}$$

that express the condition of equality of frequencies of waves in the system of coordinates connected with the concentrated object. The kinematics problem (Eqs. (8), (9)) defines eight pairs of values of ω and k at the left and the right sides of the object, one pair of which corresponds to the incident wave. However, we have only four equations i.e. the boundary conditions for determination of the wave amplitudes. Thus, it is necessary to find the additional conditions. Among all the waves that are determined by Eqs. (8), (9), are realized only what satisfy the condition of bounding of the amplitude of the string vibration at infinity

$$|u(x,t)| < \infty, \text{ as } x \to \pm\infty \qquad \text{(Eq. 10)}$$

and what satisfy the Sommerfeld radiation condition (the secondary waves should take away the energy from the object)

$$\begin{cases} V_e < 0 & if \quad x < 0 \\ V_e > 0 & if \quad x > 0 \end{cases} \qquad \text{(Eq. 11)}$$

where $V_e = S/H$ is the speed of the energy transmission, H is the energy density, and S is the energy flow. Let's note, that inside of many models of elastic systems the speed of the energy transmission V_e coincides with the group velocity of waves $V_e = V_{gr} = d\omega/dk$.

The frequencies ω and the wave numbers k determined by Eqs. (8), (9), can be both real and complex quantities. In case of complex ω and k, it is necessary to keep only that solutions determined by Eq. (7) what satisfy the condition (10). In case of real ω and k, the choice of the actual physical solutions can be executed with the help of the condition (11).

Thus, due to the interaction of the incident wave with an object the following waves are realized: one reflected wave with $\omega_4 = \omega_0$, $k_4 = -\gamma$, one transmitted wave with $\omega_3 = \omega_0$, $k_3 = \gamma$, and the waves with the non-uniform amplitude. These waves are the exponentially damped oscillations with $\omega_1 = \omega_0$, $k_1 = i\beta$, $\omega_2 = \omega_0$, $k_2 = -i\beta$,

where $\gamma = \sqrt{\left(c^4/4\alpha^4 + \omega_0^2/\alpha^2\right)^{1/2} - \dfrac{c^2}{2\alpha^2}}$, $\beta = \sqrt{\left(c^4/4\alpha^4 + \omega_0^2/\alpha^2\right)^{1/2} + \dfrac{c^2}{2\alpha^2}}$

The amplitudes of the secondary waves can be found from the system of the algebraic equations

$$
\begin{aligned}
&A_1 + A_2 - A_3 - A_4 = -A_0 \\
&k_1^2 A_1 + k_2^2 A_2 = -k_0^2 A_0 \\
&k_3^2 A_3 + k_4^2 A_4 = 0 \\
&ik_1 (N + IEk_1^2) A_1 + ik_2 (N + IEk_2^2) A_2 - (ik_3 (N + IEk_3^2) - \\
&- \omega_0^2 m + h + i\omega_0 \delta) A_3 - (ik_4 (N + IEk_4^2) - \omega_0^2 m + h + i\omega_0 \delta) A_4 = -ik_0 (N + IEk_0^2) A_0
\end{aligned}
$$

These equations are obtained after substitution of the solution into the Eq. (3), and boundary conditions (4-6). The solutions of this system are

$$
\begin{aligned}
&A_1 = A_0 i\left(\left(\beta^2 + \gamma^2\right)\left(h - \omega_0^2 m + i\omega_0 \delta\right) - 2IE\gamma^2 \beta^3 + 2N\gamma^2 \beta\right)/\Delta\,, \\
&A_2 = A_4 = 2A_0 \gamma^3 \left(IE\gamma^2 + N\right)/\Delta\,,\ A_3 = 2A_0 \gamma\beta^2 \left(IE\gamma^2 + N\right)/\Delta\,, \\
&\Delta = 2\beta\gamma(IE\beta\gamma + iN) - (i\gamma + \beta)\left(h - \omega_0^2 m + i\omega_0 \delta\right)
\end{aligned}
$$

The dependences of the dimensionless amplitude of the string vibrations as a function of the dimensionless frequency calculated for the ideal string (without stiffness, curve 1), and for the stiff string (curve 2) are presented in Figure 4.

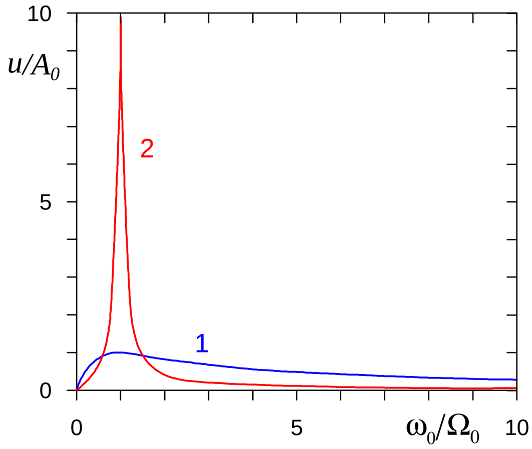

Figure 4.- Amplitude of the string vibrations vs. frequency

Let's notice, that the resonant action consisting in the incident wave as though does not “feel” the termination is possible in case when the dissipative losses are absent, and the natural frequency of the support $\Omega_0 = \sqrt{h/m}$ is equal to $\Omega_0 = \sqrt{\omega_0^2 + 2\beta\gamma^2\left(IE\beta^2 - N\right)/m}$. In this case the transverse displacement of the bridge will be maximal.

NATURAL FREQUENCIES OF THE STIFF STRING

Let l be the length of the stiff string between the supports, which one is rigid and another one is the oscillating system having the elastic and inertial properties. The natural frequencies of this string can be found from the system of equations

$$
\begin{aligned}
&u(0,t) = u_x(0,t) = 0 \\
&u_{xx}(l,t) = 0 \\
&m^0 \ddot{u} + h^0 u = \left[- Nu_x + IEu_{xxx}\right]_{x=\ell}
\end{aligned}
$$

In Figure 5 is shown the dependence of the dimensionless natural frequency of the string vibrations as a function of the string stiffness.

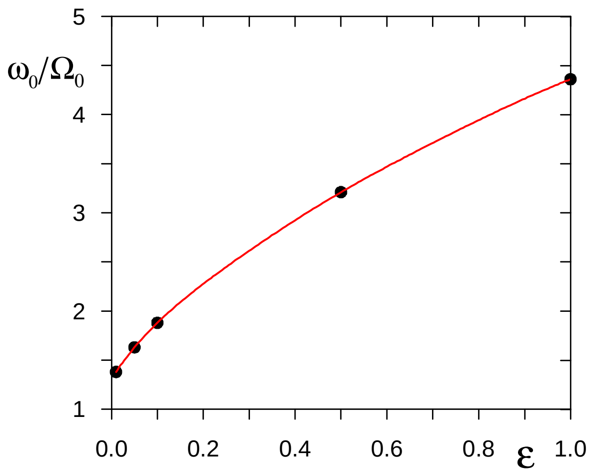

Figure 5.- Frequency of the string vs. string stiffness

Here the values of frequency have been normalized as above in Figure 4, by the natural frequency of the support $\Omega_0 = \sqrt{h/m}$, and $\varepsilon = \pi d^4 E / 64 l^2 N$ is the dimensionless parameter characterizing the string stiffness in terms of its diameter d and Young's modulus E.
The calculations show that the growth of the string stiffness moves the spectrum of the string vibrations to the high frequencies.

References: [1] E. Skudrzyk: Simple and Complex Vibratory Systems. University Park London (1981) 514 p.
[2] Nondestructive Testing Handbook V.7: Ultrasonic Testing. American Society for Nondestructive Testing (1991) 893 p.
[3] A. I. Vesnitskij, I. V. Miloserdova: Optimal damper longitudinal oscillations of a rod. J. Math. Appl. Mech. **61,** No.3 (1997) 523-525.