Zeitschrift für Angewandte Mathematik und Mechanik, 12 April 2013

Scaling and hierarchies of wave motion in solids

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Received XXXX, revised XXXX, accepted XXXX
Published online XXXX

Key words wave propagation, hierarchy, scaling.
MSC (2010) 74J05, 74J30, 74F05, 35L25

The concept of wave hierarchies in the Whitham's sense is generalized to hierarchies of second order wave operators. Based on Mindlin's model of microstructured solids, the scaling procedure is described and the corresponding hierarchical equation derived which includes two wave operators. It is shown that waves in the Cosserat' medium are described by a similar hierarchical equation. These results are generalized to a multiscale case (a scale within a scale) and to nonlinear media. It is shown also how to construct hierarchies for waves in elastic ferroelectrics. The results obtained by Scott for hierarchies in thermoelasticity are presented in the similar framework. Finally, the cases with first order wave operators are described.

1 Introduction

Solids are interpreted in classical theories as homogeneous media although we know that there are many scales in solids starting from scales of their crystal structure to scales characteristic in structural mechanics. In many practical applications (statics, slow dynamical loading) the assumption of the material homogeneity works well. However, the wave motion in solids should always take scales against wavelengths (frequencies) into account. As attempting as it sounds, it is impossible to construct an overwhelming theory of wave motion over the many scales. Nevertheless, all the theories which are more or less satisfactory at certain scales must form a whole with some areas of usage overlapping and some areas not overlapping.

Intuitively it is clear that scales form a certain hierarchy from smaller to larger. This idea is used by Whitham [1] who formulated the principles of wave hierarchies. If it is possible to determine wave operators which govern the wave motion at a certain scale then by using a proper scaling it is possible to construct mathematical models which involve many wave operators together with scaling parameters. Beside general ideas on hierarchies, Whitham [1] has analysed respective models of traffic flows and bores in channels and rivers. In mathematical terms, the one-wave operators $L_i = \partial/\partial t + a_i \partial/\partial x$ have been used. Note that every operator has its own velocity a_i. Recently the interest to hierarchies of waves has been increased because the problems of wave motion in fluidized bed [2], in bubbly liquids [3], in granular media [4], in oceans [5], in microstructured solids [6], in thermoelasticity [7], etc.

In this paper an attempt is presented to generalize these results into a systematic description of wave hierarchies where the proper scaling plays an important role. Our general description is based on the Mindlin-type theory of microstructured solids which after the seminal Mindlin's paper [8] has recently got more attention because of wide interest to contemporary materials [9]– [13]. As a result, an efficient conceptual tool is described to construct the backbone of governing equations for wave motion in materials with internal structures.

The paper is organized as follows. In Section 2 the basic model of the micromorphic continua is briefly described in a general form and then governing equations are presented. After explaining the scaling procedure, the hierarchical equation is derived and analysed. This result is then generalized in Section 3 to a multiscale case. The hierarchies in the presence of nonlinearities are demonstrated in Section 4. Further the influence of other fields is studied. First in Section 5 for elastic ferroelectrics and second, in Section 6 for thermoelasticity. Then in Section 7 a brief summary is presented for hierarchies of evolution (one-wave) equations. Finally, in Section 8 the discussion and final remarks are given.

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2 Basic model – microstructured material

2.1 Theoretical considerations

The continuum approach for microstructured solids is elaborated by incorporating the intrinsic microstructural effects into governing equations (Mindlin, [8]; Eringen and Suhubi, [14]). Leaving aside the regular structures, we focus here on irregular microstructures like polycrystalline solids or functionally graded materials. A leading concept is to separate the macro- and microstructure (or microstructures) and to formulate the balance laws for both structures separately [8]. Another possibility is to introduce the microstructural quantities into one set of balance laws [15] which gives an explicit description of interaction forces between macro- and microstructure. Following Maugin [15], the balance of the canonical (material) momentum on the material manifold $\mathcal{M}^3$ reads

$$\left.\frac{\partial \mathbf{P}}{\partial t}\right|_{\mathbf{X}} - \mathrm{Div}_R \mathbf{b} = \mathbf{f}^{\mathrm{int}} + \mathbf{f}^{\mathrm{ext}} + \mathbf{f}^{\mathrm{inh}}. \tag{1}$$

The corresponding dissipation inequality is

$$S\dot{\theta} + \mathbf{S} \cdot \nabla_R \theta \leq h^{\mathrm{int}} + \nabla_R(\theta \mathbf{K}). \tag{2}$$

Here the following notation is used: $\mathbf{P}$ is the material momentum (pseudomomentum), $\mathbf{b}$ is the material Eshelby stress, $\mathbf{f}^{\mathrm{inh}}$, $\mathbf{f}^{\mathrm{ext}}$, $\mathbf{f}^{\mathrm{int}}$, are the material inhomogeneity force, the material external (body) force and the material internal force, respectively; S is the entropy density per unit reference volume, $\mathbf{S}$ is the entropy flux, θ is the absolute temperature, h^{int} – the source term (if any) and $\mathbf{K}$ – the extra entropy flux (if any).

The next step is to determine the function of free energy W which in general terms may be formulated as

$$W = W\left(\mathbf{F}, \theta, \boldsymbol{\varphi}, \nabla_R \boldsymbol{\varphi}, \ldots\right), \tag{3}$$

where $\mathbf{F}$ is the deformation gradient and $\boldsymbol{\varphi}$ denotes microdeformation according to Mindlin [8] or the internal variable according to Berezovski et al. [16]. Then it is possible to determine stress tensor $\mathbf{b}$ and forces in balance law (1), while the governing equation for the internal variable $\boldsymbol{\varphi}$ is determined by satisfying dissipation inequality (2). Here we omit the details (see, for example, [16]) and focus further on the 1D setting.

2.2 Governing equations

Based on principles briefly described in Section 2.1: one balance law for the canonical momentum plus the dissipation inequality, the governing equations for wave motion are easily derived. As far as our main aim is to focus on scaling, we restrict here ourselves to the linear case and take $\theta =$ const, i.e. we deal with a pure elastic case. The free energy function W is then the following:

$$W = \frac{1}{2}\left(\lambda + 2\mu\right) u_x^2 + A\varphi u_x + \frac{1}{2} B\varphi^2 + \frac{1}{2} C\varphi_x^2, \tag{4}$$

where u is the macrodisplacement, φ is the microdeformation (or the internal variable – see Section 2.1), λ and μ are the Lamé parameters and A, B, C are additional material parameters. Here and further, the indices denote partial differentiation. Bearing in mind that the kinetic energy K in this simple case is

$$K = \frac{1}{2}\rho_0 u_t^2 + \frac{1}{2} I \varphi_t^2, \tag{5}$$

where ρ_0 is the macrodensity and I - the microinertia. it is possible to derive the governing equations (for details, see [10], [16]):

$$\rho_0 u_{tt} - (\lambda + 2\mu) u_{xx} - A\varphi_x = 0, \tag{6}$$

$$I\, \varphi_{tt} - C\varphi_{xx} + Au_x + B\varphi = 0. \tag{7}$$

There are two wave operators in the model:

$$L_{\mathrm{ma}}(u) = \rho_0 u_{tt} - (\lambda + 2\mu) u_{xx}, \tag{8}$$

$$L_{\mathrm{mi}}(\varphi) = I\, \varphi_{tt} - C\varphi_{xx}. \tag{9}$$

Suppose the initial and boundary conditions are given

$$u(x,t=0) = u_t(x,t=0) = 0, \tag{10}$$

$$u(x=0,t) = f(t), \tag{11}$$

$$\varphi(x,t=0) = 0, \tag{12}$$

$$\lim_{x\to\infty} u(x,t) = \lim_{x\to\infty} \varphi(x,t) = 0. \tag{13}$$

The wave operators are coupled in case $A \neq 0$ which is of our primary interest. The question is which of the wave operators prevails or are they competing with each other in wave motion generated by (10) – (13). In order to answer this question we have to find a suitable scaling procedure.

2.3 Scaling procedure

On the one side, a characteristic scale of the microstructure (the size of an element) must be known, let us denote it by l. On the other side, let the excitation be characterized by its amplitude U_0 and wavelength L. The dimensionless variables are then introduced by

$$U = u/U_0 \ , \ \ X = x/L \ , \ \ T = c_o t/L \ , \tag{14}$$

where $c_o^2 = (\lambda + 2\mu)/\rho_0$. Two nondimensional parameters are introduced by

$$\delta = l^2/L^2 \ , \ \ \epsilon = U_0/L \ . \tag{15}$$

Concerning the coefficients of Eqs (6), (7), we suppose that $I = \rho_0 l^2 I^*$, $C = l^2 C^*$ where I^* is dimensionless and C^* has the dimension of stress. It must be noted that I is scaled against ρ_0 so that the difference between the densities of the macro- and microstructure is embedded in I^*.

We rewrite the system (6), (7) in its dimensionless form and apply the slaving principle: the variable φ related to the microstructure will be determined in terms of U using a series representation. The ideas of such an approach are envisaged by Whitham [1] and elaborated by Porubov [17], see also [18].

Two steps are needed for such a procedure. Firstly we consider that

$$\varphi = \varphi_0 + \delta\varphi_1 + \ldots \ , \tag{16}$$

and secondly, we determine φ from Eq. (7) in its dimensionless form

$$\varphi = -\frac{\epsilon A}{B} U_X - \frac{\delta}{B}\left((\lambda+2\mu) I^* \varphi_{TT} - C^* \varphi_{XX}\right) . \tag{17}$$

Then it is possible to determine

$$\varphi_0 = -\frac{\epsilon A}{B} U_X \ , \tag{18}$$

$$\varphi_1 = \epsilon \frac{A}{B^2}\left((\lambda+2\mu) I^* U_{XTT} - C^* U_{XXX}\right) . \tag{19}$$

Inserting (18), (19) into Eq. (6) in its dimensionless form, we get finally in terms of the macrodisplacement U the equation

$$U_{TT} - \left(1 - \frac{c_A^2}{c_0^2}\right) U_{XX} = \frac{c_A^2}{c_B^2}\left(U_{TT} - \frac{c_1^2}{c_0^2} U_{XX}\right)_{XX} . \tag{20}$$

Here $c_1^2 = C/I, \ c_A^2 = A^2/\rho_0 B, \ c_B^2 = BL^2/I$. A more close look to the velocities reveals that c_B^2 includes the interaction effects between macro- and microstructure. It is possible to establish that

$$\frac{c_A^2}{c_B^2} = \delta I^* \frac{A^2}{B^2} \, . \tag{21}$$

2.4 Hierarchies

A more compact presentation of Eq. (20) together with expression (21) reads

$$U_{TT} - k_1^2 U_{XX} = \delta m_1 \left(U_{TT} - p_1^2\, U_{XX} \right)_{XX} \,, \tag{22}$$

where

$$k_1^2 = 1 - \frac{c_A^2}{c_0^2} \;\;,\;\; m_1 = I^* \frac{A^2}{B^2} \;\;,\;\; p_1^2 = \frac{c_1^2}{c_0^2}. \tag{23}$$

Equation (22) involves two wave operators

$$L_{\mathrm{ma}}(U) = U_{TT} - k_1^2 U_{XX}, \tag{24}$$

$$L_{\mathrm{mi}}(U) = U_{TT} - p_1^2 U_{XX}. \tag{25}$$

As far as they are weighted by the scale parameter δ with other parameters being of order of $O(1)$, the hierarchical nature of wave propagation is clearly revealed. If δ is small, then waves are governed by the properties of macrostructure. If, however, δ is large, then waves “feel” more the properties of microstructure. It is in full accordance with the wave hierarchy principle described by Whitham [1]. As far as Eq. (22) involves the second derivative of the wave operator $L_{mi}(U)$, the microstructural effects are of the dispersive character governed by terms U_{TTXX} and U_{XXXX}. Both wave operators (24) and (25) contrary to the initial idea of Whitham [1] are of the second order and involve the velocities k_1 and p_1, respectively like in the standard wave equation (k_1^2 and p_1^2).

2.5 Hierarchies in Cosserat media

The mathematical models in this case include rotation of the internal elements. Actually the Mindlin theory [8] includes also rotation when the cells are assumed to be rigid. Maugin [19] has derived a simple Cosserat model from the lattice model which has the form of a chain of dumbbells that exhibits both transverse displacements V and rotations ψ. His model (see [19], Eq. (4.55)) has the form:

$$\rho_0 V_{tt} - (\mu + \kappa) V_{xx} - \kappa \psi_x = 0, \tag{26}$$

$$j \psi_{tt} - \alpha \psi_{xx} + \kappa V_x + \kappa \psi = 0, \tag{27}$$

where k is the stiffness of springs, α and κ are micropolar densities and j is the microinertia density. The lattice spacing is a and particles have mass M. Then

$$\rho_0 = M/a^3, \;\; j = I/a^3, \;\; \mu = k/a. \tag{28}$$

The similarity of systems (6), (7) and (26), (27) is obvious. Introducing the dimensionless variables ($U = V/V_0, \psi$ is dimensionless) and parameters like in Section 2.3 and following the same idea of the slaving procedure, Eqs. (26) and (27) yield

$$U_{TT} - \hat{k}_1^2 U_{XX} = \delta \hat{m}_1 \left(U_{TT} - \hat{p}_1^2\, U_{XX} \right)_{XX} \,, \tag{29}$$

where

$$\hat{k}_1^2 = 1 - \frac{c_1^2}{c_0^2} \;\;,\;\; \hat{p}_1^2 = \frac{c_2^2}{c_0^2} \;\;,\;\; \hat{m}_1 = \frac{\epsilon L^2}{j c_0^2}. \tag{30}$$

and

$$c_0^2 = (\mu + \kappa)/\rho_0, \;\; c_1^2 = \kappa/\rho_0, \;\; c_2^2 = \alpha/j. \tag{31}$$

The resulting hierarchical equation (29) coincides with accuracy of coefficients with eq (22).

3 Multiscale hierarchies

The results described in Sections 2.2–2.4 correspond to the micromorphic solid with one microstructure. It is possible to follow the similar approach when dealing with multiscale models. For example, Engelbrecht et al. [6] and Berezovski et al. [20] have derived a model for a micromorphic solid if there are two scales of microstructure with scales l_1 and l_2 and second microstructure with a scale $l_2 \ll l_1$ is embedded into the first microstructure with a scale l_1. Such a situation is sometimes called “a scale within another scale”. By introducing microdeformation φ (as before) and ψ (the microstructure within φ, the governing equations following [6], [20] are:

$$\rho_0 u_{tt} - (\lambda + 2\mu)\, u_{xx} - A_1\varphi_x = 0, \tag{32}$$

$$I_1\, \varphi_{tt} - C_1\varphi_{xx} + A_1 u_x + B_1\varphi - A_2\psi_x = 0, \tag{33}$$

$$I_2\, \psi_{tt} - C_2\psi_{xx} + A_2\varphi_x + B_2\psi, \tag{34}$$

where I_1, I_2 are the corresponding microinertia, $A_i, C_i, B_i, i = 1, 2$ are coefficients (see Engelbrecht et al. [6]). Note that A_1 expresses the coupling of u and φ while A_2 expresses the coupling of φ and ψ. In order to carry on scaling, the dimensionless variables U, X, T are introduced as before (see Eqs (14)) together with parameters

$$\delta_1 = l_1^2/L^2 \quad , \quad \delta_2 = l_2^2/L^2 \quad . \tag{35}$$

With scaling of $I_i, C_i, i = 1, 2$ like in Section 2.3 and using the series representation, we obtain the following governing equation of motion in terms of macrodisplacement U (cf. Eq. (22)):

$$U_{TT} - k_{11}^2(A_1)U_{XX} = \delta_1 m_{11}\left(U_{TT} - p_{11}^2(A_2)\, U_{XX}\right)_{XX} + \delta_2^2 m_{12}\left(U_{TT} - p_{22}^2 U_{XX}\right)_{XXXX} . \tag{36}$$

where k_{11}, p_{11}, p_{22} are the corresponding velocities and m_{11}, m_{12} reflect the coupling effects.

Now we have three are wave operators

$$L_{ma}(U) = U_{TT} - k_{11}^2(A_1)U_{XX}, \tag{37}$$

$$L_{mi(1)}(U) = U_{TT} - p_{11}^2(A_2)U_{XX}, \tag{38}$$

$$L_{mi(2)}(U) = U_{TT} - p_{22}^2 U_{XX}, \tag{39}$$

which are scaled by δ_1 and δ_2. In this way the operators describe the wave motion over many scales and the influence of each of them is regulated by δ_1 and δ_2.

4 Hierarchies in the presence of nonlinearities

The mathematical models of micromorphic media (Engelbrecht et al. [6] permit also to account for nonlinear effects both at macro- and microlevel [21]. In this case, dealing with a single scale, free energy function (4) must be replaced by

$$W = \frac{1}{2}(\lambda + 2\mu)u_x^2 + A\varphi u_x + \frac{1}{2}B\varphi^2 + \frac{1}{2}C\varphi_x^2 + \frac{1}{6}Nu_x^3 + \frac{1}{6}M\varphi_x^3\,, \tag{40}$$

where N and M are the additional nonlinear parameters (cf. also Pastrone and Engelbrecht, [22]). Then the corresponding system of governing equation is

$$\rho_0 u_{tt} - (\lambda + 2\mu)u_{xx} - Nu_{xx} - A\varphi_x = 0, \tag{41}$$

$$I\, \varphi_{tt} - C\varphi_{xx} - M\varphi_x\varphi_{xx} + Au_x + B\varphi = 0. \tag{42}$$

Following the procedure described above, this system yields

$$U_{TT} - \left(1 - \frac{c_A^2}{c_0^2}\right) U_{XX} - \frac{1}{2}q_1\left(U_X^2\right)_X = \frac{c_A^2}{c_B^2}\left(U_{TT} - \frac{c_1^2}{c_0^2}\, U_{XX}\right)_{XX} - \frac{1}{2}q_2\left(U_{XX}^2\right)_{XX}\,, \tag{43}$$

where $q_1 = N\epsilon/(\lambda + 2\mu)$, $\quad q_2 = \delta^{3/2}\left(A^3 M^*\epsilon\right)/(\lambda + 2\mu)B^3$, $\quad M = M^* l^3$. Clearly, Eq. (43) permits to distinguish two wave operators

$$L_{\mathrm{ma}}(U) = U_{TT} - k_1^2 U_{XX} - \frac{q_1}{2}(U_X^2)_X, \tag{44}$$

$$L_{\mathrm{mi}}(U) = U_{TT} - p_1^2 U_{XX} - \frac{q_3}{2}\delta^{1/2}(U_{XX}^2). \tag{45}$$

Here $q_3 = q_2 c_B^2/c_A^2$. In this case Eq. (43) may lead to solitary waves due to the balance of dispersion and nonlinearity/ies [23].

5 Hierarchies in elastic ferroelectrics

In ferroelectrics the elastic displacements are coupled with the rotation of dipoles. The corresponding mathematical model is derived by Maugin [19] (see also the analysis in [24]):

$$u_{tt} - c_L^2 u_{xx} = \alpha_L (\cos\phi)_x \ , \tag{46}$$

$$v_{tt} - c_T^2 v_{xx} = -\alpha_T (\sin\phi)_x \ , \tag{47}$$

$$\phi_{tt} - \phi_{xx} - \chi \sin\phi = \alpha_L u_x \sin\phi + \alpha_T v_x \cos\phi \ , \tag{48}$$

expressed in dimensionless space – time coordinates. Here u and v are longitudinal and transverse displacements and $\phi = 2\widehat{\theta}$ is twice of the true angle $\widehat{\theta}$ of rotation of dipoles. The constants α_L and α_T are piesoelectric coefficients, χ is the electric susceptibility and c_L, c_T are acoustic speeds. If the system is linearized about $\phi = \phi_0 = 0$ then the longitudinal and transverse displacements are decoupled and Eqs. (47) , (48) yield

$$u_{tt} - c_T^2 v_{xx} = -\alpha_T \phi_x \ , \tag{49}$$

$$\phi_{tt} - \phi_{xx} - \chi\phi = \alpha_T v_x \ . \tag{50}$$

The similarity with system (6), (7) is obvious. We suppose $\alpha_T \sim \vartheta(\epsilon)$ and $\chi \sim \vartheta(1)$. Then ϕ_x in the first approximation is

$$\phi_x^0 \sim \frac{\alpha_T}{\chi} v_{xx} \tag{51}$$

and in the next

$$\phi_x^1 \sim \frac{1}{\chi}\left(\phi_{tt} - \phi_{xx}\right)_x = \frac{\alpha_T}{\chi^2}\left(v_{tt} - v_{xx}\right)_{xx} \ . \tag{52}$$

Then it is easy to derive

$$v_{tt} - \left(c_T^2 - \frac{\alpha_T^2}{\chi}\right) v_{xx} - \frac{\alpha_T}{\chi^2}\left(v_{tt} - v_{xx}\right)_{xx} = 0, \tag{53}$$

which is again a hierarchical equation with the operators

$$L_{\mathrm{ma}}(v) = v_{tt} - \left(c_T^2 - \frac{\alpha_T^2}{\chi}\right) v_{xx} \ , \tag{54}$$

$$L_{\mathrm{mi}}(v) = v_{tt} - v_{xx} \tag{55}$$

and the strength of the second operator is governed by α_T. Note that like in the case of the micromorphic model (Section 2.4) the influence of the microstructure (dipoles) is already in the main wave operator (54) (cf $k_1^2 = 1 - c_A^2/c_0^2$ and $c_T^2 - \alpha_T^2/\chi$).

6 Hierarchies in thermoelasticity

Classical models of thermoelasticity are of the parabolic type [25] and govern the diffusive process. The modified theories include also the thermal relaxation time τ_0 and then the resulting models are hyperbolic. The question whether it is possible to derive also hierarchical models of thermoelastic wave propagation. Engelbrecht [26] has shown that accounting of τ_0 is important for high-frequency processes. Thermoelasticity viewed as wave hierarchies is studied by Scott [7] and here we represent his results in the format used in previous sections.

In order to compare the results with those described above, we use 1D setting. Then the governing equations in case of $\tau_0 = 0$ [7] in terms of displacement U and temperature θ are:

$$\rho_0 u_{tt} - (\lambda + 2\mu) u_{xx} + \kappa\theta_x = 0 \ , \tag{56}$$

$$\rho_0 c_E \theta_t - k_0 \theta_{xx} + T_0 \kappa u_{xt} = 0 \ , \tag{57}$$

where $\kappa = (3\lambda + 2\mu)\alpha_T,\ T_0$ is the reference temperature, c_E - the specific heat, k_0 – the conductivity coefficient and α_T – the thermal expansion coefficient. In terms of temperature θ, Scott [7] has derived

$$\epsilon^2\left(\theta_{tt} - c_k^2 \theta_{xx}\right) + \left(\theta_t + \delta\theta_x\right) = 0 \ , \tag{58}$$

where $c_k^2 = k_0(\rho_0 c_E \epsilon^2)$. From Eq (58) for $\epsilon \to 0,\ \delta \to 0$, the uncoupled equation

$$k_0\theta_{xx} - \rho_0 c_E \theta_t = 0 \ , \tag{59}$$

follows. Here the wave operators are

$$L_1 = \theta_t + \delta\theta_x \ , \tag{60}$$

$$L_2 = \theta_{tt} - c_k^2\theta_{xx} \ . \tag{61}$$

Operator L_1 is of the first order and operator L_2 is of the second order. In case of $\tau_0 \neq 0$, the governing system for an isotropic material is

$$\rho_0 u_{tt} - (\lambda + 2\mu)u_{xx} + \kappa\theta_x = 0 \ , \tag{62}$$

$$\rho_0 c_E \theta_t + \tau_0\rho_0 c_E \theta_{tt} - k_0\theta_{xx} + T_0\kappa u_{xt} + \tau_0 T_0 \kappa u_{xtt} \ . \tag{63}$$

In this case the hierarchical system is rather complicated [7]:

$$\tau_0(\partial_t^2 - c_1^2\partial_x^2)(\theta_{tt} - c_2^2\theta_{xx}) + (\theta_{tt} - c_3^2\theta_{xx})_t = 0 \ , \tag{64}$$

where c_1, c_2, c_3 are velocities. Note that here all the operators are of the second order.

7 One-wave hierarchies

The basic idea on hierarchies of waves introduced by Whitham [1] is related to operators of the first order. Here the models described above involve the full second-order wave operators and enlarge the concept of such hierarchical operators. As a matter of fact, any second-order operator describes two waves – one moving to the right, another to the left. However, the different scaling procedures may be used in order to derive first the evolution equations for unidirectional wave motion.The celebrated Korteweg-de Vries (KdV) equation derived by Korteweg and de Vries [27] is the best example from this class of equations. Nowadays the methods for deriving such evolution equations are widely known, see for example, Taniuti and Nishihara [28] and Engelbrecht [26]. If the evolution equations are first derived then as a result of proper scaling, the hierarchies can be constructed where every operator is of the first order.

Oliveri [3] has derived such a hierarchy for nonlinear waves in bubbly liquids. In this case the evolution equation reads

$$u_\tau + uu_\xi + \gamma\alpha u_{\xi\xi} + \delta\alpha u_{\xi\xi\xi} + \gamma\beta\left(u_\tau + uu_\xi\right)_\xi + \\ +\delta\beta\left(u_\tau + uu_\xi\right)_{\xi\xi} = 0\,, \tag{65}$$

where τ, ξ is a moving frame as usually taken for evolution equations and $\alpha, \beta, \gamma, \delta$ are constants [3]. If $\beta \to 0$ then the classical Korteweg-de Vries-Burgers equation is recovered. In a general case the governing model is constituted by the hierarchy of nonlinear one-wave operators $(u_\tau + uu_\xi)$.

Another interesting case is described by Giovine and Oliveri [4] for waves in dilatant granular materials. Here the evolution equation takes the form

$$u_\tau + uu_\xi + \alpha_1 u_{\xi\xi\xi} + \beta\left(u_\tau + uu_\xi + \alpha_2 u_{\xi\xi\xi}\right)_{\xi\xi} = 0, \tag{66}$$

where α_1 and α_2 are the dispersion parameters and β involves the ratio of the grain size and the wavelength. Equation (66) like Eq. (65) above is written in a moving frame τ, ξ. Here the model involves two Korteweg-de Vries (KdV) operators – one for motion in the macrostructure, another – in the microstructure while β regulates the weight of two operators. The parameter β depends on the ratio of kinetic and potential energies and can be either positive or negative. The consistent analysis of solutions to Eq. (66) is presented by Ilison and Salupere [29].

8 Discussion

It has been shown above that in modelling of wave motion in complex media with internal scales, the hierarchical governing equations involve several wave operators. While the initial idea of Whitham [1] is based on the first order (one-wave) operators, here the 1D problems involve the second order operators $L(U) = \left(\partial^2/\partial t^2 + c^2\partial^2/\partial X^2\right)U$ and their derivatives with respect to space coordinate X. One of the crucial problems in this context is the stability analysis. For Eq. (22) the analysis of the corresponding dispersion relation demonstrates the stability of the solution in case of given initial and boundary conditions (10) – (13) for the initial system (6), (7) – see Berezovski et al. [20]. In a general case, the stability analysis needs a special attention.

Hierarchies derived in Sections 2 – 8 are based on weakly nonlocal theories of mechanically structured (micromorphic) or physically structured (ferroelectric) materials. These hierarchies model the behaviour of 1D longitudinal waves either in microstructured media (internal field is either microdeformation or microrotation) or in media with other fields (electrical or thermal fields) in action. Characteristically to all these cases, the velocities of hierarchical operators are different from those of initial systems. This shows that the coupling of fields is not only important in modelling of dispersion but influences also the leading velocity of propagation in the macrostructure. This phenomenon is demonstrated also by direct numerical analysis [10]. However, it must be noted that the hierarchical equations are derived by using an asymptotic procedure (Section 2.3) and are correct with a certain accuracy which depends on values of physical parameters. This problem is in detail analysed by Peets et al. [30] for the case of the Mindlin model, i.e. of system (6), (7) and Eq. (22) by making use of the dispersion analysis and by Tamm and Salupere [31] by making use of numerical experiments.

The structure of the hierarchical equations includes the wave operator for the macrostructure followed by another operator (or operators) describing the microstructure. As a rule, this operator is included in the form of its second order derivative with respect to X. In this way, the most interesting cases when both operators are in force, the derivatives like U_{TTXX} and U_{XXXX} appear. If nonlinearity in the macroscale is included like operator (44) then the result is of the Boussinesq type [32], [33]. However, for the Maxwell-Rayleigh model in case of anomalous dispersion, the second wave operator is included in the form of its second derivative with respect to T (see Maugin [34]). Then the higher order terms are U_{TTTT} and U_{XXTT}.

Some special cases should be mentioned within the general framework of hierarchies. The effect of nonlinearities is reflected by corresponding operators like in Section 4 for both macro- and microstructures. It is possible, however, that nonlinearity is considered only at one level like demonstrated by Engelbrecht and Pastrone [22] who studied the microstructured solids with nonlinearities in microscale. It is also possible that hierarchies are described by wave operators of different order like in Section 6 for thermoelasticity which actually combines the diffusive effects with wave propagation.

Most cases described above involve asymptotic analysis for distinguishing the separation of scales. It is also possible to separate waves that in 1D case means instead of two waves propagating to the left and right to follow just one wave. This needs a different asymptotic analysis like presented in detail by Taniuti and Nishihara [28] and Engelbrecht [26]. Such an approach is used by Oliveri [3] and Giovine and Oliveri [4] resulting in hierarchies of operators where the leading term is of the first order. Here the operators describing microstructural effects enter also into the governing equation in the form of their derivatives but characteristic to evolution equations, with respect to moving coordinate ξ.

Like Whitham's original idea, the hierarchical equations derived by a scaling procedure permit to understand the wave motion in the microstructured materials and dispersion effects in a transparent way. We are tempted to paraphrase Salençon [35] who actually analysed the virtues of the principle of virtual power: the clarity of hierarchical equations helps a better understanding of the constructed models.

Acknowledgements This research was supported by the EU through the European Regional Development Fund (Centre for Nonlinear Studies (CENS) – Estonian Centre of Excellence in Research), by the Estonian Ministry of Education and Research (SF 0140077s08), and by the Estonian Science Foundation (grant No 8658).

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