WAVES IN MICROSTRUCTURED SOLIDS: DISPERSION AND THERMAL EFFECTS

Arkadi Berezovski, Jüri Engelbrecht[a)]
Centre for Nonlinear Studies, Institute of Cybernetics, Tallinn University of Technology, Tallinn, 12618, Estonia

Summary A consistent framework is outlined to describe wave propagation in solids with internal structure. The dual internal variables concept generalizes the single internal variable theory. The evolution equations for internal variables yield from the dissipation inequality. Specific results are presented for microdeformations and microtemperatures.

INTRODUCTION

Dispersion plays a prominent role in the nonlinear water waves theory. Similarly, the dispersion accompanies wave propagation in heterogeneous solids. The modeling of such a dispersion goes back to Maxwell, Rayleigh, and Boussinesq. Two research lines in the modeling of dispersion effects can be clearly separated: (i) "bottom-up" approach and (ii) "top-down" framework. In the bottom-up approach, one starts with the description at a micro-scale (usually atomic scale) to derive the macroscopic equations of motion from the "first principles". Alternatively, the top-down framework is based on the continuum description at the macro-scale introducing additional variables for the accounting for the micro-scale influence. Any combination of the two descriptions gives a multi-scale method. As shown recently [1], the bottom-up approach and the top-down framework can be "matched" on the level of the micromorphic continuum description. However, thermal effects are not a part of the micromorphic theory of microstructured solids. The main difficulty in the description of microstructure follows from the fact that the balance laws for microfields are considered as granted [2, 3] and may be thermodynamically inconsistent. The adopted phenomenological approach is based on the material formulation of continuum mechanics [4] and provides the full thermodynamic consistency due to the dual internal variables concept [5], which describes the influence of microstructure by internal fields. The material formulation takes internal variables into account naturally, consecutively, and consistently. The dual internal variables approach provides several possibilities for evolution equations for internal fields that describe effects of microdeformations or microtemperatures (and their gradients). The corresponding evolution equations can be hyperbolic even for microtemperatures, which in its turn can induce wave-like propagation for macrotemperature due to the coupling of equations.

THEORETICAL BACKGROUND

In the framework of the phenomenological continuum theory, it is assumed that the influence of a microstructure on the overall macroscopic motion of a body can be taken into account by the introduction of internal variables, which we associate with the integral distributed effect of the microstructure. In the considered for simplicity one-dimensional case (for 3D case see [6]), the free energy W depends (in addition to the deformation gradient u_x and temperature θ) on two internal variables, φ, ψ, and their space derivatives, $W = \overline{W}(u_x, \theta, \varphi, \varphi_x, \psi, \psi_x)$. The intrinsic dissipation inequality reduces to [6]

$$\Phi = (\tau - \eta_x)\varphi_t + (\xi - \zeta_x)\psi_t \geq 0, \quad \text{where} \quad \tau := -\frac{\partial \overline{W}}{\partial \varphi}, \quad \eta := -\frac{\partial \overline{W}}{\partial \varphi_x}, \quad \xi := -\frac{\partial \overline{W}}{\partial \psi}, \quad \zeta := -\frac{\partial \overline{W}}{\partial \psi_x}. \tag{1}$$

The dissipation inequality (1) can be satisfied by several means. We consider three main cases for choosing the evolution equations for internal variables: (i) fully dissipative case; (ii) non-dissipative case; and (iii) intermediate (composite) case. Using a quadratic function as the free energy dependence

$$\overline{W} = \frac{1}{2}\left(\lambda + 2\mu\right) u_x^2 - \frac{\rho_0 c_p}{2\theta_0}\left(\theta - \theta_0\right)^2 + m\left(\theta - \theta_0\right) u_x + A\varphi u_x + A'\varphi_x u_x + \frac{1}{2}B\varphi^2 + \frac{1}{2}C\varphi_x^2 + \frac{1}{2}D\psi^2, \tag{2}$$

we neglect for simplicity the contribution of the gradient of second internal variable. Here ρ_0 is the density of the material, c_p is heat capacity, θ_0 is the reference temperature, the thermoelastic coefficient m is related to the dilatation coefficient α, and the Lamé coefficients λ and μ by $m = \alpha(3\lambda + 2\mu)$, A, A', B, C and D are additional material parameters.

MATHEMATICAL MODELS

Example 1: Fully dissipative case

In the fully dissipative case, equations of motion and energy are reduced to [6]

$$\rho_0 u_{tt} = \left(\lambda + 2\mu\right) u_{xx} + m\theta_x + A'\varphi_{xx}, \quad \rho_0 c_p \theta_t - \left(k\theta_x\right)_x = m\theta_0 u_{xt} + \frac{1}{R_1}\varphi_t^2. \tag{3}$$

a) Corresponding author. E-mail: je@ioc.ee.

Here R_1 is an appropriate constant, k is the thermal conductivity [6]. The evolution equation for the primary internal variable becomes the parabolic one

$$\varphi_t = R_1 C \varphi_{xx} + A' u_{xx}, \tag{4}$$

and we can identify the internal variable with a microtemperature. Its influence on the macrotemperature manifests itself in the source term in the right hand side of Eq. $(3)_2$.

Example 2: Purely non-dissipative case

In the non-dissipative case, we can represent the equations of motion in the form, which includes only the primary internal variable

$$\rho_0 u_{tt} = (\lambda + 2\mu)\, u_{xx} + m\theta_x + A\varphi_x + A'\varphi_{xx}, \quad I\varphi_{tt} = C\varphi_{xx} + A' u_{xx} - A u_x - B\varphi, \tag{5}$$

where $I = 1/R^2 D$ is an internal inertia measure. The single dispersive wave equation that combines the equations of motion

$$u_{tt} = c^2 u_{xx} + \frac{C}{B}\left(u_{tt} - c^2 u_{xx}\right)_{xx} - \frac{I}{B}\left(u_{tt} - c^2 u_{xx}\right)_{tt} + \frac{A'^2}{\rho_0 B} u_{xxxx} - \frac{A^2}{\rho_0 B} u_{xx}, \tag{6}$$

unifies and generalizes all known approaches [7] (here $c^2 = (\lambda + 2\mu)/\rho_0$).

Example 3: Intermediate case (microtemperature effects)

In the intermediate case, we have for the equation of motion and for the evolution equation of the primary internal variable φ, respectively, [6]

$$\rho_0 u_{tt} = (\lambda + 2\mu)\, u_{xx} + m\theta_x + A'\varphi_{xx}, \quad I\varphi_{tt} + \frac{R_2}{R^2}\varphi_t = (C\varphi_{xx} + A' u_{xx}). \tag{7}$$

Here R, R_2 are appropriate constants. Clearly, Eq. $(7)_2$ is a Cattaneo-Vernotte-type hyperbolic equation [8]. Note that the energy conservation equation in this case is parabolic [6]

$$\rho_0 c_p \theta_t - \left(k\theta_x\right)_x = m\theta_0 u_{xt} + R_2 D^2 \varphi_t^2. \tag{8}$$

As in the first case, the equation for the macrotemperature (8) is influenced by a source term which depends on the internal variable. As it was shown [6], the primary internal variable can be identified in this case with the microtemperature. The latter may induce a wave-like propagation of the macrotemperature even in the case of a parabolic equation for the macrotemperature [6].

RESULTS

On the basis models described above, wave motion in microstructured solids has been studied, The focus of studies has been on physical effects: interaction forces between macro- and micro-fields, nonlinear effects, and dispersion. It is shown that the free energy function should be modified in some cases in order to catch properly interaction forces and nonlinearities. The role of microtemperature accompanying wave motion is explained with full thermodynamic consistency. The numerical simulations of wave fields demonstrate the applicability of such an approach [9]. It is proposed to use such models also for solving inverse problems. The talk will summarize theoretical and numerical results obtained over last years.

ACKNOWLEDGMENTS

The research was supported by the EU through the European Regional Development Fund and by the Estonian Science Foundation (grant No. 8702).

References

[1] Wang, X., Lee, J. D.: Micromorphic theory: a gateway to nano world. *Int. J. Smart Nano Mater.* **1**:115–135, 2010.
[2] Capriz, G.: Continua with Microstructure. Springer, Heidelberg 1989.
[3] Mariano, P. M.: Multifield theories in mechanics of solids. In: Advances in Applied Mechanics, **38**:1-93, 2002.
[4] Maugin, G. A.: Material Inhomogeneities in Elasticity, Chapman & Hall, London 1993.
[5] Ván, P., Berezovski, A., Engelbrecht, J.: Internal variables and dynamic degrees of freedom. *J. Non-Equilib. Thermodyn.* **33**:235–254, 2008.
[6] Berezovski, A., Engelbrecht, J., Maugin, G. A.: Thermoelasticity with dual internal variables. *J. Thermal Stresses*, **34**:413–430, 2011.
[7] Berezovski, A., Engelbrecht, J., Berezovski, M.: Waves in microstructured solids: a unified viewpoint of modeling. *Acta Mech.* **220**:349–363, 2011.
[8] Joseph, D. D., Preziosi, L.: Heat waves. *Rev. Mod. Phys.*, **61**:41–73, 1989.
[9] Engelbrecht, J., Berezovski, A., Berezovski, M.: Deformation waves in microstructured materials: theory and numerics, in: IUTAM Symposium on Recent Advances of Acoustic Waves in Solids. Wu, T.-T.; Ma, C.-C. (Eds.), Springer, IUTAM Bookseries, Vol. 26, Part 1, pp. 19–29, 2010.