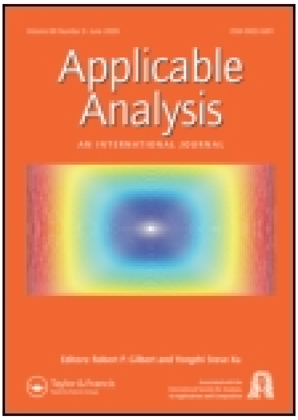

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Soliton ensembles and solitonic structures

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Soliton ensembles and solitonic structures†

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The article deals with the analysis of three complicated one-dimensional evolution equations, the solutions of which can be classified as soliton ensembles or solitonic structures. The case studies from the physical viewpoint are: (i) martensitic–austenitic alloys; (ii) hyperelastic rod and (iii) granular materials. The corresponding evolution equations governing the propagation of longitudinal waves include higher order nonlinear and dispersive terms and are nonintegrable. Numerical simulation is carried out by the pseudospectral method. The first class of solutions involves solitons with nonvanishing oscillatory tails and wave packets called solitonic structures. The second class includes special soliton ensembles or more exactly – plaited solitons. The emergence of such entities and their interaction demonstrate the solitonic character of waves.

Keywords: dispersion; nonlinearities; evolution equations; solitonic structures; soliton ensembles

AMS Subject Classifications: 35Q74; 74J30; 74J35; 65M70

1. Introduction

This article explains the role of complicated evolution equations in the analysis of wave motion and, as a result, reveals some complicated solitonic solutions. The classical wave equations are hyperbolic and within the linear theory well-understood from a long time (see, e.g. [1]). In the simplest one-dimensional case the wave equation describes two waves, propagating to the right and to the left and possesses the d'Alembert solution. This is a cornerstone of mathematical physics and many practical cases are solved using such a model. However, during the last half a century, the attention has been turned also towards evolution equations which describe just one wave propagation along a properly chosen characteristic. Such an approach permits explicitly with a certain accuracy to account also for nonlinear, dispersive, dissipative a.o. effects which accompany the wave motion and can

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sometimes bring in novel phenomena. The highlight of nonlinear evolution equations is without any doubt the concept of soliton – a solution to an evolution equation with a specific nonlinear and dispersive (higher order) terms. Soliton is a wave in which the nonlinear and dispersive effects are balanced and therefore it propagates as a stable entity and interacts with another similar entity without losing its shape and velocity.

There is no need to describe here the history of solitons. First seen and described by John Scott Russel in 1834 as a wave in a narrow canal next to Edinburgh, the mathematical model of this phenomenon was derived by Korteweg and de Vries [2] and bears now their name – the Korteweg–de Vries (KdV) equation. The real meaning of this equation and its richness was understood only after the seminal paper by Zabusky and Kruskal [3] who demonstrated the emergence of solitons from a harmonic input and coined the concept of soliton. This article is an excellent example of a fruitful computational experiment. Since then, the solitons are widely studied theoretically (see, history by Weissert [4]) and many practical applications have followed.

One of the first overviews on the essence and the analysis of the KdV equation was the overview by Jeffrey and Kakutani [5]. In 1980s, the general theory was developed in order to derive evolution equations from rather general models of wave motion, see [6–8] etc. and the references therein. Clearly, the attention was focused to nonlinear models which have added a new breath to understanding of physical phenomena.

The description of a soliton is based on the iconic KdV equation which involves the quadratic nonlinearity and cubic dispersion. As mentioned by Ablowitz and Clarkson [9], this is the simplest nonclassical partial differential equation (PDE) possessing the minimum number of independent variables (one), the lowest order of the derivative not considered classically (three), the simplest such term (an unmixed derivative), the simplest additional term to make the equation nonlinear (quadratic), etc. Also the KdV equation is not only a rich model, it possesses specific mathematical beauty [10].

However, the underlying physics of wave processes can be much more complicated than that described by nonlinearity and dispersion embedded into the KdV equation and therefore evolution equations with much more complicated nonlinearities and dispersive terms can arise. We limit here ourselves to conservative models and leave dissipative effects aside. The interesting question is whether such complicated evolution equations can also support solitons or soliton-like solutions (solitary waves).

In this article we describe some of such complicated models which all are similar to the celebrated KdV equation and study the properties of solutions. In Section 2 the models are described and in Section 3 the main ideas of the pseudospectral method used for numerical simulation are briefly presented. Section 4 is devoted to the results and analysis. A short summary is given in Section 5.

2. Mathematical models

As it is well-known, the KdV equation

$$u_t + uu_x + du_{xxx} = 0 \tag{1}$$

is written in terms of dependent variable u and independent variables x and t and involves the dispersion parameter d. Note that x and t are usually scaled variables and x is actually a moving coordinate [7,8]. The balance of nonlinear and dispersive effects gives rise to solitons which interact with each other elastically, i.e., after interaction they retain their amplitude and velocity. There are many physical examples for which the one-dimensional wave process can be modelled by the KdV equation. But quadratic nonlinearity and cubic dispersion is just one case and there are many other physical examples with different nonlinearities and dispersion terms involved. In principle, the corresponding evolution equations can be of the KdV-type retaining the time derivative u_t but involving more complicated other terms. The interesting question is whether such 'modified' equations also possess soliton-type solutions and whether the properties of solitons are also retained. Here we present three interesting cases in order to demonstrate the richness of the soliton-type family of solutions.

Case i: martensitic–austenitic alloys. This case is characterized by higher order nonlinearity and higher order dispersion. Several such models are known to govern also other interesting problems. The cubic and quintic dispersive terms appear in the case of magneto-acoustic waves in a cold collision-free plasma [11], in the case of capillary-gravity water waves [12], etc. The existence of a quintic term changes radically the character of phase and group velocities for short waves [13]. Higher order nonlinearities can appear in models of waves in electronic transmission line [14], see also [15]. We focus here on a case of martensitic–austenitic shape memory alloys [16–18]. In this case the evolution equation is as follows [13,19]:

$$u_t + [P(u)]_x + du_{xxx} + bu_{xxxxx} = 0, \tag{2}$$

where $P(u)$ is the elastic potential and d and b are the third-order and fifth-order dispersion parameters, respectively. The elastic potential $P(u)$ is determined by a polynomial

$$P(u) = \frac{u^4}{4} - \frac{u^2}{2}. \tag{3}$$

This potential depicts quartic nonlinearity in its simplest (symmetrical) form that possesses two minima. The higher order dispersion can be caused by the dislocations in the crystal structure of an alloy. Because of the orders of nonlinearity and dispersion equation (2) is referred as the KdV435 below.

Case ii: hyperelastic rod. Small but finite amplitude travelling waves in a compressible hyperelastic rod are described by the evolution equation derived by Dai [20,21]

$$u_\tau + \sigma_1 uu_\xi + \sigma_2 u_{\xi\xi\tau} + \sigma_3(2u_\xi u_{\xi\xi} + uu_{\xi\xi\xi}) = 0, \tag{4}$$

where σ_1, σ_2 and σ_3 are functions of material parameters. Note that here the dispersion is described by a mixed third-order derivative and besides the usual quadratic nonlinearity more complicated nonlinearities are involved reflecting the coupling effect of the material nonlinearity and the geometry of the rod.

Case iii: granular materials. Here the waves are governed by a hierarchical evolution equation. The evolution equation governing the waves propagating near the region of an equilibrium of a dilatant granular material is derived by Giovine and Oliveri [22]

$$u_t + uu_x + \alpha_1 u_{xxx} + \beta(u_t + uu_x + \alpha_2 u_{xxx})_{xx} = 0, \tag{5}$$

where α_1 denotes the macrostructure (matrix) dispersion parameter and α_2 – the microstructure (grains) dispersion parameter. Parameter β involves besides the other material characteristics involves the ratio of the grain size and the wavelength. What should be stressed here, is that parameter β can be either positive or negative depending on the ratio of kinetic and potential energy of grains. For lower values of kinetic energy, parameter β is positive and for higher values it is negative.

Equation (5) consists of two KdV operators: the first describes the influence of the macrostructure and the second (in the parentheses) – the influence of motion in the microstructure. This equation is clearly hierarchical in Whitham's [23] sense and is referred to as the hierarchical KdV (HKdV) equation below. If parameter β is small, then the influence of the microstructure can be neglected and the wave 'feels' only the macrostructure. If, however, parameter β is large, then the influence of the microstructure is dominant.

3. Pseudospectral method

Let a function $u(x, t)$ (periodic in space) be given on the interval $0 \leq x \leq 2\pi$ and the space grid be formed by N points with space step $\Delta x = 2\pi/N$. Now the discrete Fourier transform (DFT) can be defined by

$$U(k, t) = \mathrm{F}u = \sum_{j=0}^{N-1} u(j\Delta x, t)\exp\left(-\frac{2\pi ijk}{N}\right) \tag{6}$$

and the inverse DFT (IDFT) by

$$u(j\Delta x, t) = \mathrm{F}^{-1}U = \frac{1}{N}\sum_k U(k, t)\exp\left(-\frac{2\pi ijk}{N}\right). \tag{7}$$

Here F denotes the DFT and F^{-1} the IDFT, i is the imaginary unit and wave numbers $k = 0, \pm 1, \pm 2, \ldots, \pm(N/2 - 1), -N/2$. Making use of properties of the DFT space derivatives of function $u(x, t)$ can be calculated as

$$\frac{\partial^n u(x, t)}{\partial x^n} = \mathrm{F}^{-1}[(ik)^n \mathrm{F}u]. \tag{8}$$

If the length of the space period for $u(x, t)$ is not 2π, but $2m\pi$, then one must use quantity k/m instead of k in formulae (8).

In a nutshell the idea of the pseudospectral method (PsM) is very simple. Let a PDE be given in a general form

$$u_t = \Phi(u, u_x, u_{xx}, \ldots). \tag{9}$$

Making use of formula (8) (which can be considered as numerical differential operator) one can formally reduce the original PDE (9) to ordinary differential equation (ODE)

$$u_t = \Psi(u). \tag{10}$$

Now Equation (10) can be solved making use of standard ODE solvers. However, because of the usage of DFT boundary conditions must be periodic and the length of the space period must be $2m\pi$, where m is a positive integer [24].

It is clear, that the DFT deals with derivatives and nonlinear terms are dealt by the ODE solvers. Calculations are carried out with the package SciPy [25] using: the FFTW library [26] for the DFT and the F2PY [27] generated Python interface to the ODEPACK Fortran code [28] for the ODE solver. In particular, for solving of resulting ODE in explicit form (10) solver LSODA is applied. LSODA automatically selects between nonstiff (Adams) and stiff (BDF) methods. It uses the nonstiff method initially, and dynamically monitors data in order to decide which method to use [28]. Thereafter, in the case of problems considered in this article, the instability of numerical scheme was not a problem. The integrals of conserved quantities (momentum and energy) [29] permit to check the accuracy of numerical results over calculation period [24]. For example, for Equation (5) the first and the second conservation laws [30] are as follows:

$$(u+\beta u_{xx})_t+\left[\frac{u^2}{2}+\alpha_1 u_{xx}+\beta\left(\frac{u^2}{2}+\alpha_2 u_{xx}\right)_{xx}\right]_x=0 \tag{11}$$

and

$$\left\{\frac{1}{2}\alpha_1 u^2+\beta\left[(u_x)^2+uu_{xx}\right]\right\}_t + \left\{\frac{1}{3}\alpha_1 u^3+uu_{xx}-\frac{1}{2}(u_x)^2+\beta\left[\frac{1}{3}\alpha_2 u^3+uu_{xx}-\frac{1}{2}(u_x)^2\right]_{xx}\right\}_x=0. \tag{12}$$

4. Results

4.1. *Wave propagation in martensitic–austenitic alloys*

In order to simulate the one-dimensional wave propagation in martensitic–austenitic alloys (characterized by quartic nonlinearity and positive values of third- and fifth-order dispersion) the KdV435 equation is integrated numerically under harmonic as well as localized initial conditions:

$$u(x,0)=\sin(x), \tag{13}$$

and

$$u(x,0)=A\,\mathrm{sech}^2\frac{x}{\Delta},\quad \Delta=\sqrt{\frac{12d}{A}}. \tag{14}$$

Here A is the amplitude and Δ the width of the initial solitary pulse. The proposed localized initial excitation (14) is the analytical solution of the KdV equation (1) and is known as the KdV soliton [3]. In our case the initial wave (14) is related to the model Equation (2) through the third-order dispersion parameter d (neglecting the quartic nonlinearity and the fifth-order dispersion). Here we demonstrate two typical solutions of the KdV435 equation (see [31–34] for additional examples).

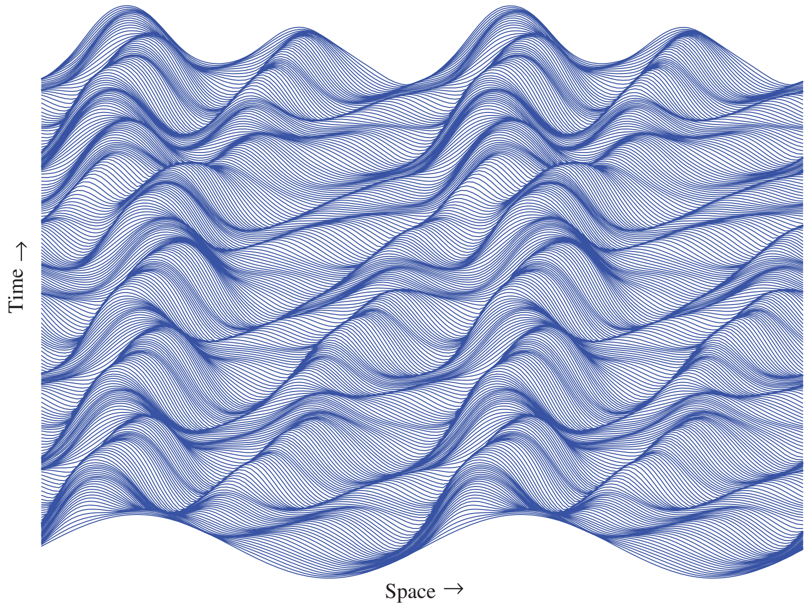

Figure 1. Solution of the KdV435 equation: time-slice plot over two space periods for $d = 10^{-2.4}$, $b = 10^{-2.8}$ and $t = 0, 0.2, 0.4, \ldots, 60$.

The first is the train of KdV-type solitons (Figure 1) that is emerged from the initial harmonic wave (13). Here, like in the case of the KdV solitons, solitons are phase-shifted and their amplitudes decrease during interactions. However in the present case – contrary to the KdV solitons – the lower the soliton the faster it propagates to the left.

The second solution type is a plaited soliton (Figure 2) that is emerged from sech^2-type localized initial pulse (14) (the amplitude of the initial localized pulse is $A = 0.37$ and the length of the space period is 16π). In the present case, two solitary waves and an oscillating tail emerge from the initial excitation. These two solitary waves interact with (i) each other, and (ii) the oscillating tail. Furthermore, these solitary waves form a plaited solitary entity that propagates at constant speed. On can call these solitary waves solitons, because they propagate with constant speed, and restore their amplitude after interactions.

4.2. *Wave propagation in hyperelastic rods*

The model equation (4) is integrated under localized initial conditions

$$u(x, 0) = A\,\mathrm{sech}^2 Bx. \tag{15}$$

Several examples of different solution types are presented in [35], here we consider two of them.

At first the formation of soliton train from the initial pulse is demonstrated. In the case of $A = 0.1$ and $B = 0.5$ a train of six solitons is formed (Figure 3).

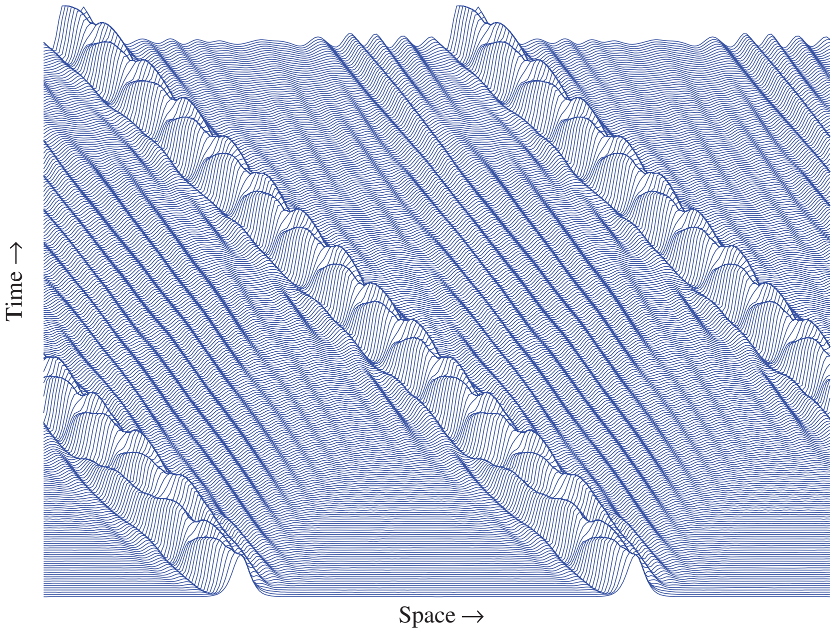

Figure 2. Solution of the KdV435 equation: time-slice plot over two space periods for $d = 10^{-1.2}$, $b = 10^{-2.0}$ and $t = 0,2,4,\ldots,500$.

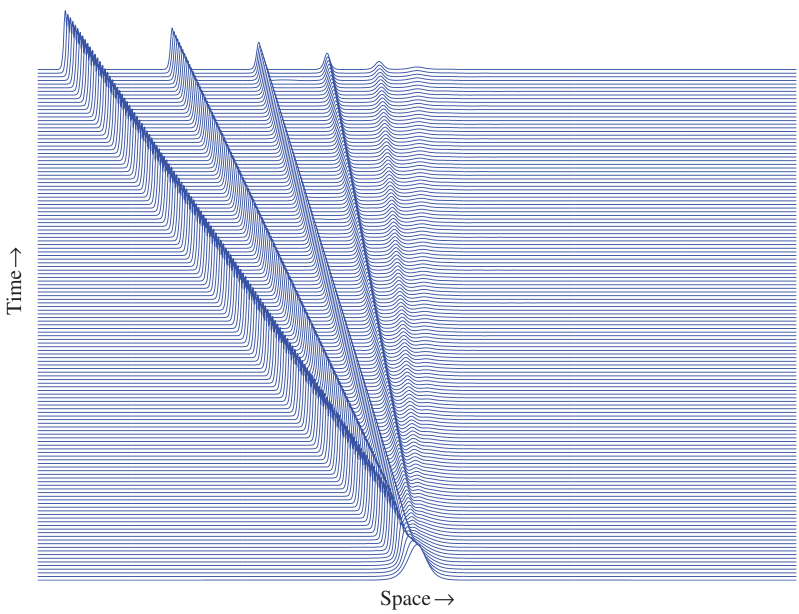

Figure 3. Solution of Equation (4): time-slice plot over one space period for $\sigma_1 = -0.7205$, $\sigma_2 = -0.0026$, $\sigma_3 = -0.0010$; $A = 0.1$; $B = 0.5$; $t = 0, 10, 20, \ldots, 1400$; the length of the space period is 40π.

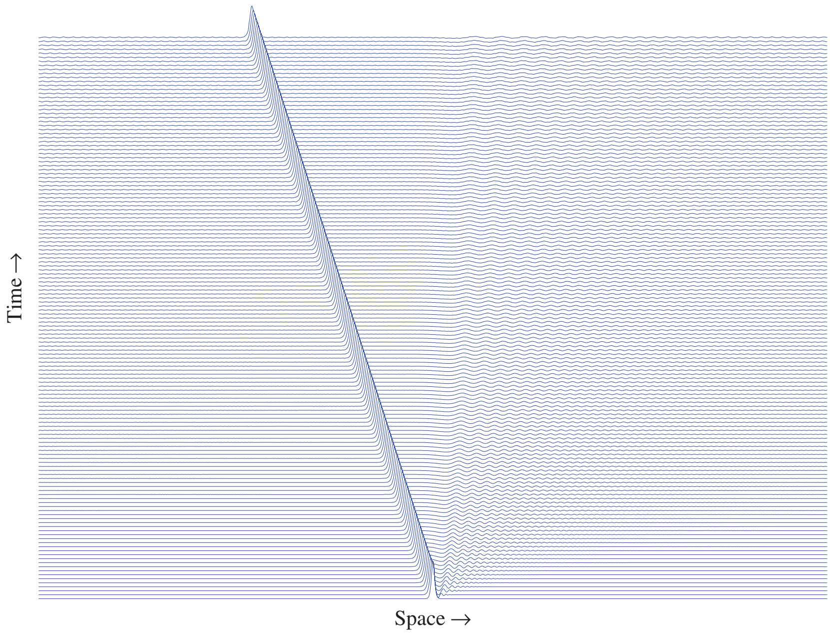

Figure 4. Solution of Equation (4): time-slice plot over one space period for $\sigma_1 = -0.7205$, $\sigma_2 = -0.0026$, $\sigma_3 = -0.0010$; $A = 0.1$; $B = 3.0$; $t = 0, 10, 20, \ldots, 1400$; the length of the space period is 40π.

However, in the case of narrower pulse ($B = 3$) only one soliton and an oscillating tail is evolved (Figure 4).

The second example demonstrates again the plaited soliton (Figure 5). Like in the case of the KdV435 equation a pair of interacting solitary waves and a tail form a solitonic structure.

4.3. *Wave propagation in granular materials*

In order to simulate numerically the propagation of solitary waves in dilatant granular materials HKdV equation (5) is integrated numerically under sech^2-type localized initial conditions

$$u(x, 0) = A\,\mathrm{sech}^2\frac{x}{\delta}, \quad \delta = \sqrt{\frac{12\alpha_1}{A}}, \tag{16}$$

where A is the amplitude and δ the width of the initial pulse. It is clear that the latter is the analytical solution of KdV equation that corresponds to the first KdV operator in Equation (5). In this article $A = 4$ and the length of the space period is 16π.

The following four solution types are found for $\alpha_1 \neq \alpha_2$: (i) train of KdV solitons (Figure 6), (ii) train of KdV solitons with a weak tail (Figure 7), (iii) solitary wave with a strong tail (Figure 8) and (iv) solitary wave with tail and wave packets (Figure 9). The type of an emerging solution depends on the parameters of Equation (5) [30]. More examples of different solution types can be found in [30,36,37].

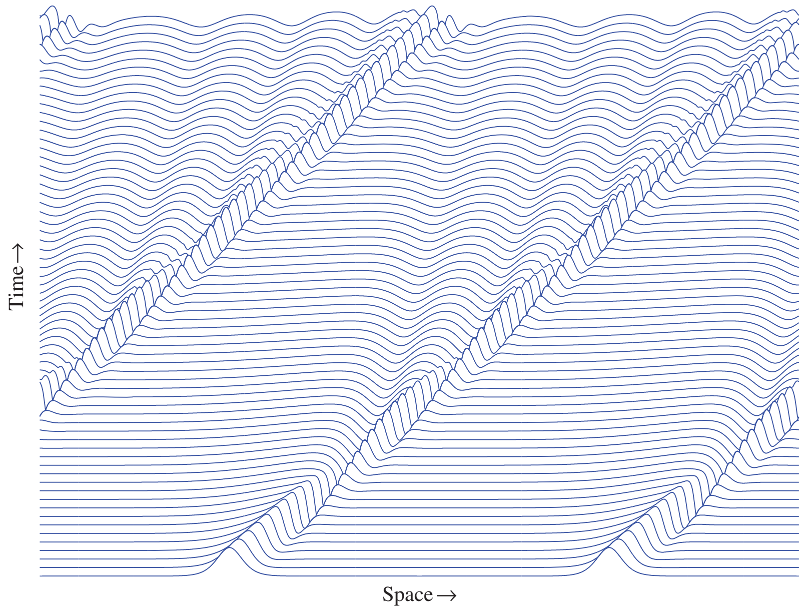

Figure 5. Solution of Equation (4): time-slice plot over two space periods for $\sigma_1 = 1.857$, $\sigma_2 = -0.036$, $\sigma_3 = -0.002$; $A = 0.1$; $B = 3.58$; $t = 0, 2, 4, \ldots, 130$; the length of the space period is 2π.

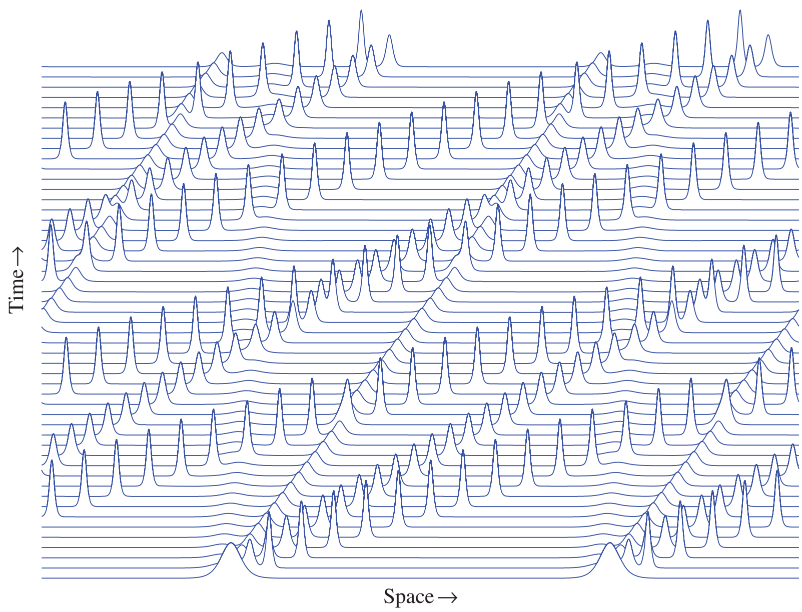

Figure 6. Solution of the HKdV equation: time-slice plot over two space periods for $\alpha_1 = 1.0$, $\alpha_2 = 0.1$, $\beta = 111.11$, $A = 4$, $t = 0, 2, 4, \ldots, 100$.

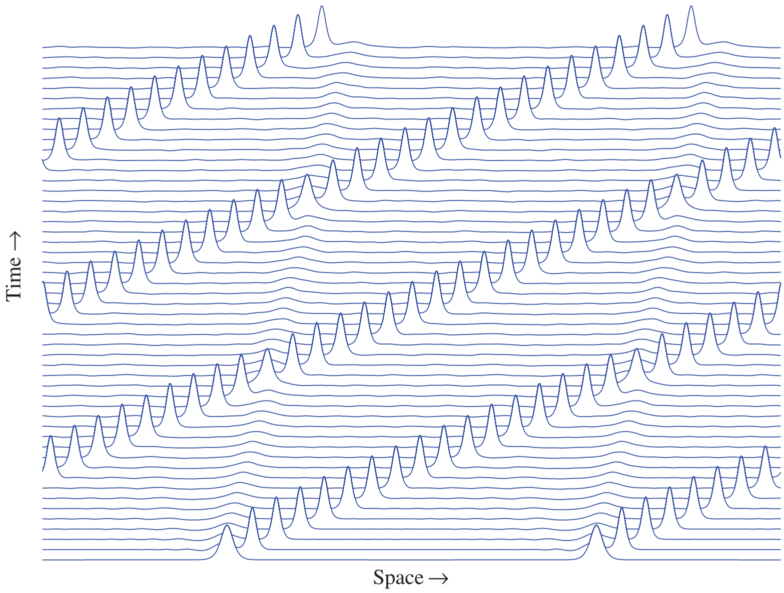

Figure 7. Solution of the HKdV equation: time-slice plot over two space periods for $\alpha_1 = 0.4$, $\alpha_2 = 0.2$, $\beta = 111.11$, $A = 4$, $t = 0, 2, 4, \ldots, 100$.

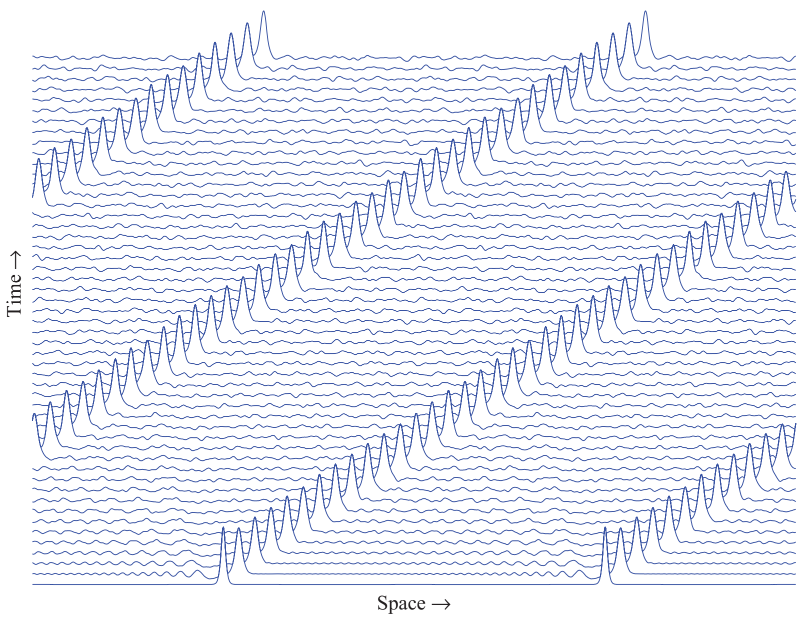

Figure 8. Solution of the HKdV equation: time-slice plot over two space periods for $\alpha_1 = 0.05$, $\alpha_2 = 0.09$, $\beta = 11.111$, $A = 4$, $t = 0, 2, 4, \ldots, 100$.

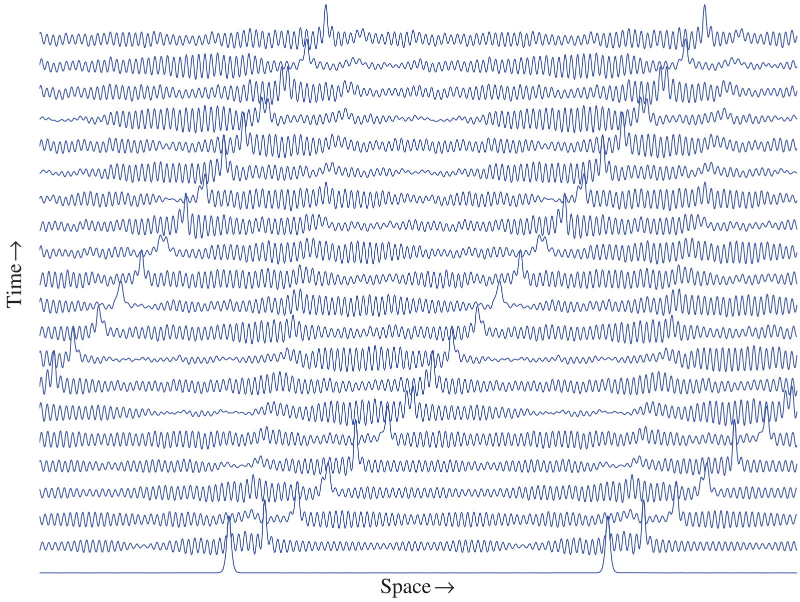

Figure 9. Solution of the HKdV equation: time-slice plot over two space periods for $\alpha_1 = 0.05$, $\alpha_2 = 0.09$, $\beta = 0.0111$, $A = 4$, $t = 0, 5, 10, \ldots, 100$.

5. Summary

The model equations like the KdV equation, the nonlinear Schrödinger equation, the sine-Gordon equation a.o. have revealed many fundamental properties of solitons. The real world is much more complicated that is why there is a growing interest to other soliton-bearing systems including the evolution equations with complicated (often higher order) nonlinear, dispersive and/or dissipative terms. In this context the question of integrability is important in order to obtain closed analytical solutions. For solving complicated nonintegrable systems powerful numerical methods are used which permit to obtain and analyse solutions and show their solitonic behaviour if it exists. This way or another, the soliton 'zoo' is large and special types of solitary waves like compactons [38], breathers [39], peakons, cuspons and pulsons [40,41], kovatons [42] etc., reflect rich physics of waves.

Here our results are based on numerical integration of governing equations (see Section 2) by the pseudospectral method. The accuracy and stability of the method [24] are prerequisites for the analysis. The conservation laws are checked at every time step of calculations. The number of space-grid points is chosen so high that the relative error of conserved densities does not exceed the limit which is fixed earlier (as a rule, the relative error is less than 10^{-6}). We demonstrated the existence of certain solitonic solutions without assuming the integrability of underlying models. These solutions have a complicated structure compared to single solitary waves or classical soliton ensembles or soliton trains (Figures 1, 3 and 6). The first typical class of such solutions involve solitons or solitary waves with nonvanishing oscillatory tails and/or wave packets (Figures 4, 7–9). These entities can be called solitonic structures or quasi-solitons following Champneys et al. [43], if the leading part of such a structure reveals the properties of solitons. The second typical class includes a special soliton ensembles or more exactly 'plaited' solitons (Figures 2 and 5). The plaited solitons propagate with a constant speed like the single solitons and restore their amplitude after interacting with each other [19,31]. It is remarkable that in both classes of such entities, the emergence of solitary solutions follows the classical pattern of the KdV solitons provided the energy of the initial excitation is large enough. If, however, the initial excitation is small, then for example Equation (2) reveals irregular solution due to weak interplay between dispersive and nonlinear effects provided both d and b are small.

It must be stressed that the evolution equations including those analysed in this article are derived by certain asymptotic expansions. The higher order terms usually destroy the integrability compared to classical soliton equations but nevertheless, the solutions reveal solitonic characteristics. Besides the interesting case studies, the general stability of solutions needs further analysis together with deeper studies of interaction processes of such solitonic entities.

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