*Acc. Sc. Torino*
*Memorie Sc. Fis. 35 (2011)*



# Non linear waves in complex microstructured solids

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presentata dal Socio Franco PASTRONE nell'adunanza del 18 maggio 2011

**Abstract.** *The problem of the propagation of non linear waves in solids with different internal structural scales is analysed. Making use of a general model of microstructured solids, two particular relevant models are studied in one dimensional setting: a solid with hierarchical microstructure and a solid with concurrent microstructures. In the first case the behaviour of internal structures with two different scales is studied and for a particular choice of the strain energy function a 6th order PDE is obtained with characteristic hierarchical structure. Using the same basic model, the case of two concurrent microstructures, as introduced in [2], is studied and again for suitable explicit form of the energy function we can obtain a 4th order PDE and actually prove the possibility of propagation of solitary and cnoidal waves.*

Keywords: wave propagation, microstructure, multiple scales, wave hierarchy.

**Riassunto.** *Viene affrontato il problema della propagazione di onde non lineari in solidi con differenti scale per strutture interne. A partire da un modello generale di solidi con microstrutture si studiano due casi unidimensionali interessanti: un solido con gerarchia di microstrutture ed un solido con microstrutture concorrenti. Nel primo caso si analizza il comportamento di strutture interne con due diverse scale e con una particolare scelta dell'energia di deformazione si ottiene una PDE del sesto ordine, con una struttura gerarchica caratteristica. A partire dallo stesso modello di base, si studia il caso di due microstrutture concorrenti, come introdotte da [2] e si ottiene ora una PDE del quarto ordine e si può infine dimostrare la possibilità di propagazione di onde solitarie e cnoidali.*

Parole chiave: propagazione di onde, microstrutture, scale multiple, gerarchie di onde.

*Mathematics Subject Classification 2000*: 74J30, 74J35, 74B20, 74M25.

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## 1. Introduction

A complex system as it is understood nowadays is composed by its constituents that interact with each other resulting in emergent properties of the system as a whole. Although the different meaning of the word "complexity" are still in use in informatics, the mainstream of studies focus on interaction and emerging processes while nonlinearity plays a decisive role. This in a sharp contrast to a common specialization of classical research that means to split-up general problems into their simpler components and then to study them as deep as possible, hence try (not very often) to put together the parts and reconstruct the whole [18]. In mechanics the concepts of complexity has been analysed by Engelbrecht [8] with a focus on wave propagation. Here we want to elaborate the idea that complex systems in Continuum Mechanics are strictly related to a more general theory, as briefly described in Maugin [12], that we can obviously call Generalized Continuum Theory. This theory is based on some cornerstones like the introduction of internal structures at different scales and the nonlinearity of the models which in other words means incorporating intrinsic microstructural and nonlinear effects. Another approach, equivalently valid, introduces the concept of internal variables [2]. Such variables are supposed to describe the internal structure, but they are not observable. Roughly speaking, they do not appear as kinematical (observable) variables, in most cases they do not have inertia, but they appear in the stress-strain relations, or, if there exists a free energy, this energy must depend also on internal variables. A typical case in continuum mechanics, among many, is the case of materials with certain microstructure. In this case complexity means that we have different scales, with several interaction processes which encompass many physically meaningful phenomena. The pioneering work of Mindlin [13] is a basic reference, another more recent general treatment can be found in Capriz [3], while many papers have appeared where different particular and less particular cases have been described (see the papers by Engelbrecht, Pastrone, Cermelli, Porubov quoted in the references). Usually a microstructured body, as we shall see, is modeled as a 3D solid with an internal structure at a different scale, which is apt to describe the mechanical behaviour of solids with dislocations, polycrystalline solids, ceramic composites, granular media, etc. One main aspect of such theories is that they always take into account the nonlinearity of the materials, the nonlocality and the interactions between micro- and macroscales. It is possible, and useful, to develop also models with a hierarchy of microstructures, i.e. a first level micro-structure which contains a second level micro-structure, and so on, as done in Casasso and Pastrone [4], Engelbrecht et al [11]. But it is meaningful also the case of concurrent micro-structures (see Berezovski et al [2]).

In this paper we want to analyze the subject, recalling some main results in the theory of complex microstructures, developing new results in the case of multiple microstructures, exploiting hierarchical governing equations and analyzing nonlinear wave propagation, which is crucial to put in evidence the weight of the different scales and the interaction of micro- and macro-structures. To give a simple idea of the different scales we can use this figure:

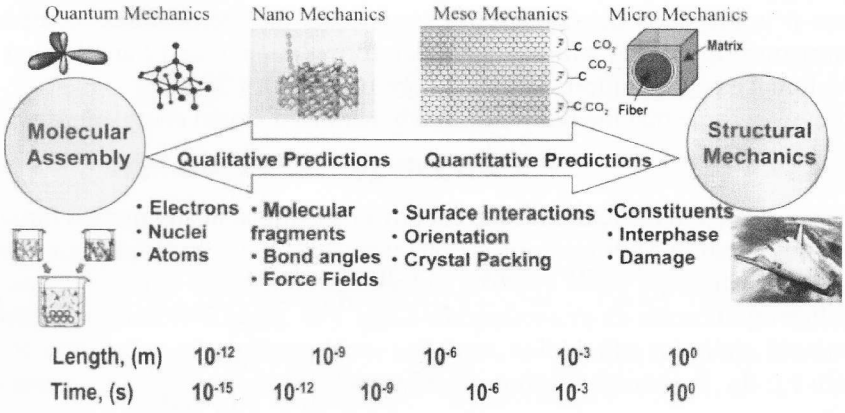


Figure 1: Reproduced from: T.S. Gates, G.M. Odegard, S.J.V. Frankland, T.C. Clancy. *Computational materials: Multi-scale modeling and simulation of nanostructured materials*, «Composites Science and Technology», 65, 2005, pp. 2416.

## 2. The field equations

The kinetic energy of a microstructured body is defined as a quadratic form in the velocities $\dot{\mathbf{r}}, \dot{\mathbf{d}}_H$, where $\mathbf{r}$ is the position vector describing the macrostructure, $\mathbf{d}_H$ is the director apt to provide a description of some properties of the microstructure as they act at the macroscopic level, dot means derivative with respect to time $t$ while the material point $\mathbf{X}$ is kept fixed, the notation $(\cdot)_{,j} = \partial(\cdot)/\partial X^j$ will mean derivatives with respect to the material coordinates $X^i$. Here and in the following $H, K, \cdots = 1, 2, 3, \ldots, n$, where $n$ depends on the type of microstructure we want to model; $i, j = 1, 2, 3$.

$$T = \frac{1}{2}\left(\rho \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} + 2\rho^H \dot{\mathbf{r}} \cdot \dot{\mathbf{d}}_H + \rho^{HK} \dot{\mathbf{d}}_H \cdot \dot{\mathbf{d}}_K\right) \tag{1}$$

where $\rho, \rho^H, \rho^{HK}$ (which represent inertia terms) are functions of $\mathbf{x}$ and $t$ if the Eulerian description is used, and functions of $\mathbf{X}$ only in the Lagrangian formulation. $T$ is a definite positive quadratic form. Without any loss of generality, we can reduce it to a diagonal form

$$T = \frac{1}{2}\left(\rho\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} + I^{HK}\dot{\mathbf{d}}_H\cdot\dot{\mathbf{d}}_K\right) \tag{2}$$

where $\rho$ is the mass density and $I^{HK}$ represents the inertia form of the microstructure. If we deal with a Lagrangian formulation, $\rho$ and $I$ are defined on the reference configuration, hence they are functions of $\mathbf{X}$ only.

Let us assume that the body admits a generalized stored energy density

$$W = W(\mathbf{r}_{,h}, \mathbf{d}_H, \mathbf{d}_{H,j}, \mathbf{X}) \tag{3}$$

which is related to the total mechanical power expended in any motion through the equation

$$P_T = \frac{dW}{dt} = \frac{\partial W}{\partial \mathbf{r}_{,h}}\dot{\mathbf{r}}_{,h} + \frac{\partial W}{\partial \mathbf{d}_H}\dot{\mathbf{d}}_H + \frac{\partial W}{\partial \mathbf{d}_{H,i}}\dot{\mathbf{d}}_{H,i} \tag{4}$$

where $\dot{\mathbf{r}}_{,h}$, $\dot{\mathbf{d}}_H$, $\dot{\mathbf{d}}_{H,j}$ are the strain velocities and $\dfrac{\partial W}{\partial \mathbf{r}_{,h}}$, $\dfrac{\partial W}{\partial \mathbf{d}_H}$, $\dfrac{\partial W}{\partial \mathbf{d}_{H,j}}$ represent the generalized stresses.

We will take in account also conservative body forces such that there exists a potential

$$W_b = W_b(\mathbf{r}, \mathbf{d}_H, \mathbf{X})$$

and the power of the body forces is given by

$$P_T^b = \frac{\partial W_b}{\partial \mathbf{r}}\cdot\dot{\mathbf{r}} + \frac{\partial W_b}{\partial \mathbf{d}_H}\cdot\dot{\mathbf{d}}_H.$$

We can derive the field equations via the usual variational principle, namely requiring that the motions of the body in a certain interval of time $[t_0, t_1]$ will make the energy functional

$$\mathcal{E} = \int_{t_0}^{t_1}\int_{\mathcal{B}}(T - W - W_b)\,dV\,dt. \tag{5}$$

stationary in comparison with all admissible motions. The Euler-Lagrange equations read

$$\begin{cases} \left(\dfrac{\partial W}{\partial \mathbf{r}_{,i}}\right)_{,i} - \dfrac{\partial W_b}{\partial \mathbf{r}} = \dfrac{d}{dt}\dfrac{\partial T}{\partial \dot{\mathbf{r}}} \\ \left(\dfrac{\partial W}{\partial \mathbf{d}_{H,i}}\right)_{,i} - \dfrac{\partial W}{\partial \mathbf{d}_H} - \dfrac{\partial W_b}{\partial \mathbf{d}_H} = \dfrac{d}{dt}\dfrac{\partial T}{\partial \dot{\mathbf{d}}_H} \end{cases} \tag{6}$$

In general, from a physical point of view, the microbody forces are different from the macrobody forces, hence we can split $W_b$ in two parts:

$$W_b = W_b^{macro}(\mathbf{r}, \mathbf{X}) + W_b^{micro}(\mathbf{d}_H, \mathbf{X}) \tag{7}$$

From a formal point of view, we can recover the field equations in a classical form (Euler equations) introducing a set of forces, coupling moments, generalized forces and moments as follows:

$$\begin{aligned} \sigma^i &= \frac{\partial W}{\partial \mathbf{r}_{,i}}, \qquad \eta^{Hi} = \frac{\partial W}{\partial \mathbf{d}_{H,i}}, \qquad \mathbf{B} = -\frac{\partial W_b^{macro}}{\partial \mathbf{r}} \\ \tau^H &= -\frac{\partial W}{\partial \mathbf{d}_H}, \qquad \mathbf{b}_H^{micro} = -\frac{\partial W_b^{micro}}{\partial \mathbf{d}_H} \end{aligned} \tag{8}$$

The general equation , with particular specifications of the number of directors $\mathbf{d}_H$ and the energy density $W$, allowed us to study many particular models of solids with microstructures, in 1 and 2 dimensions, with interesting results in the determination of nonlinear wave solutions, soliton like solutions, hierarchy of waves, as one can see in references [5, 11, 6, 9, 10, 15, 16].

## 3. One dimensional solid with hierarchical microstructure

We consider a one-dimensional microstructured model with two different scale levels applied for the microstructure. Instead of the two-scale elastic system, containing both macro- and microstructures, we introduce a material, which is supposed to be a compound of a macrostructure, a first level microstructure and a second level microstructure at a much smaller scale. The last may be interpreted as a nanostructure, to some extent (see [4], [11],[14]).

In Figure 2 we give a sketch of the possible configuration of a solid with two levels microstructures.

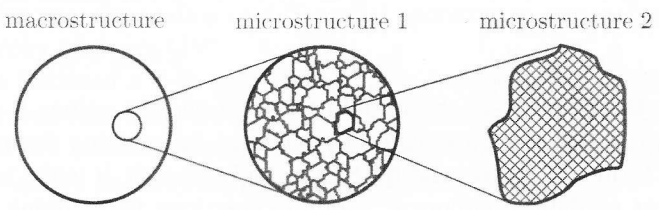


Figure 2: Two-levels microstructure

Therefore, following the model, we deal with three different scalar functions: the one for the macrostructure and two for the microstructures, one for each scale level. The model of a material is the one-dimensional manifold, and we consider the material coordinates in space $x$ and in time $t$; and the functions $v = v(x,t)$ for the macrostructure, $\varphi = \varphi(x,t)$ and $\psi = \psi(x,t)$ respectively for the first and the second scale level in microscale. The macro body is supposed to be purely elastic, and both the first and second level microstructures satisfy the same generalized elasticity hypothesis as well, therefore the existence of an internal strain energy is assumed.

A particular choice of the strain energy function $W$ defines different nonlinear models, see [4]; in this paper we consider it in the following form:

$$\begin{aligned} W = \; & 1/2\alpha\nu_x^2 + 1/2\beta\nu_x^3 + 1/2B_1\phi^2 + 1/2B_2\psi^2 + 1/2C_1\phi_x^2 + \\ & + 1/2C_2\psi_x^2 + \nu_x(A_1\phi + A_2\psi) + A\phi\psi + B\phi_x\psi \end{aligned} \tag{9}$$

which encompass both cases (two-levels and concurrent microstructures). This function is the generalization of the strain energy function for nonlinear elastic solids with one microstructure level to our case, where the introduction of the cubic term $v_x^3$ represents the nonlinear behaviour of the matrix. If we consider in (9) $A = A_2 = 0$ and $B \neq 0$ we have the hierarchical coupling.

The field equations can be derived as in [4] via a variational principle:

$$\begin{cases} \rho v_{tt} &= \alpha v_{xx} + \left(\beta v_x^2\right)_x + A_1\varphi_x \\ I_1\varphi_{tt} &= C_1\varphi_{xx} - A_1 v_x - B_1\varphi - B\psi_x \\ I_2\psi_{tt} &= C_2\psi_{xx} - B\varphi_x - B_2\psi \end{cases} \tag{10}$$

where $\alpha$, $\beta$ and $A_i$, $B$, $B_i$, $C_i$ $(i = 1,2)$ denote material constants.

To obtain the governing equation in dimensionless form, it is necessary to introduce some suitable parameters and constants (see [5]) and two different parameters $\delta_i$, $i = 1,2$, characterizing the ratio between the microstructure and the wave length $L$, and $\varepsilon$, accounting for small but finite elastic strain magnitude:

$$\delta_1 = (l_1/L)^2, \qquad \delta_2 = (l_2/L)^2, \qquad \varepsilon = v_0 << 1 \tag{11}$$

where $v_0$ is the intensity of the initial excitation and the values $l_1$ and $l_2$ represent the size of the microstructural elements. Introducing the macrostrain $\nu = v_x$ (the term "strain" is used for brevity only; in fact, it is the longitudinal displacement gradient component, while expressions for genuine strains are nonlinear with respect to $\nu$) and the dimensionless variables

$$u = \nu/v_0, \quad X = x/L, \quad T = (c_0/L)t$$

and substituting them into the previous system, we obtain the following coupled dimensionless equations:

$$\begin{cases} u_{TT} = \dfrac{\alpha}{\rho c_0^2} u_{XX} + \dfrac{\beta\varepsilon}{\rho c_0^2} \left(u^2\right)_{XX} - \dfrac{A_1^* l_1}{\varepsilon \rho c_0^2} \varphi_{XX} \\ \varphi = -\dfrac{A_1^* l_1 v_0}{B_1} u - \dfrac{B^* \sqrt{\delta_2}}{B_1} \psi_X + \dfrac{\delta_1}{B_1} \left[C_1^* \varphi_{XX} - \rho I_1^* c_0^2 \varphi_{TT}\right] \\ \psi = -\dfrac{B^* \sqrt{\delta_2}}{B_2} \varphi_X + \dfrac{\delta_2}{B_2} \left[C_2^* \psi_{XX} - \rho I_2^* c_0^2 \psi_{TT}\right] \end{cases} \tag{12}$$

In formulas (12) the starred coefficients can be easily obtaind from the unstarred ones of eqs. (10). The slaving principle [11] can now be used for further transformations. This procedure allows us to write one function in terms of the other; on this way we can obtain the governing equation for the function $u(x,t)$ only. To this end, we determine the variable $\psi$ in terms of $\varphi$ and its derivatives from $(12)_3$. Then the equation $(12)_3$ can be used to express $\varphi$ in terms of derivatives of $u$. This expression will eventually be substituted into equation $(12)_1$ to obtain the one differential equation for $u$.

The resulting equation can be written as:

$$u_{TT} + \alpha_1 u_{XX} + \alpha_2 (u^2)_{XX} + (\alpha_3 u_{XX} + \alpha_4 u_{TT})_{XX} + (\alpha_5 u_{4X} + \alpha_6 u_{TTXX} + \alpha_7 u_{4T})_{XX} = 0 \tag{13}$$

where the $\alpha_i$ are constant coefficients explicitely defined in [5] and in the final remark of this section.

The equation (13) above may be considered as a hierarchical equation in terms of $u$, where two different levels of microstructure are expressed in five different dispersive terms, and the higher order terms contain the parameters of the second level of microstructure.

This scaling scheme that will be used also in the analysis below, follows [7] where the strength of effects, not the rate of changes (slow and fast variables) is scaled. In this way, the scaling of differential operators is constructed resulting in the hierarchy of waves [19].

We have obtained a 6-th order PDE that is hardly to be solved explicitly in general case. However, we will find some exact travelling wave solutions of the PDE (13), when the equation can be reformulated in terms of the phase variable $z = x \pm Vt$ in the corresponding ODE, as follows:

$$(V^2 + \alpha_1) u^{(\mathrm{II})} + \alpha_2 (u^2)^{(\mathrm{II})} + \left(\alpha_3 + V^2 \alpha_4\right) u^{(\mathrm{IV})} + \left(\alpha_5 + V^2 \alpha_6 + V^4 \alpha_7\right) u^{(\mathrm{VI})} = 0 \tag{14}$$

where $V$ is the velocity of wave propagation.

Following the method introduced by Samsonov in [17], upon the introduction of $z$ and integration twice with corresponding conditions at infinity $|z| \to \infty \Rightarrow u, u' \to 0$ the equation (14) may be rewritten as the nonlinear ODE of the 4th order:

$$u^{(\mathrm{IV})} + a u^{(\mathrm{II})} + b\, u^2 + c u = 0 \tag{15}$$

where obviously:

$$a = (\alpha_3 + V^2\alpha_4)/\chi;\ b = \alpha_2/\chi;\ c = (\alpha_1 + V^2)/\chi;\ \chi = \alpha_5 + V^2\alpha_6 + V^4\alpha_7 \tag{16}$$

Following the method described in [17], the exact solution to the ODE (15) in terms of elliptic functions, containing only poles as the critical singularities, can be found in the following form:

$$u = M\wp^2(x; g_2, g_3) + S\wp(x; g_2, g_3) + K \tag{17}$$

where the coefficients $M$, $S$, $K$ and invariants $g_i$ of the Weierstrass elliptic function $\wp$ are defined in [5].

In the appropriate limit the Weierstrass elliptic function $\wp$ may be further reduced to the elliptic Jacobi $cn-$ function and, in due course, to the bounded solution $u_0$ in terms of $\cosh^{-2}$ function, i.e., to *the solitary wave solution*, as follows:

$$\begin{aligned} &u_0 = s\cosh^{-4}(x) + q\cosh^{-2}(x) + p; \\ &p = -c/b = -\frac{-18928 + 3640a - 31a^2}{507b};\ q = \frac{140(52+a)}{13b};\ s = -840/b, \end{aligned} \tag{18}$$

which has a form of the so called "mexican hat". Figure 3 provides two graphical examples of the solutions for different values of the parameter b.

***Remark***

The approach used to obtain these solutions is similar to that introduced and grounded in [17], and can be applied to explicitly solve different higher order ODE, e.g., the 5th order KdV and the 5th order mKdV equations.

In the first step from $(12)_3$ the expansion

$$\psi = -\frac{A_2^*\sqrt{\delta_2}}{B_2}\varphi_x - \frac{A_2^*\sqrt{\delta_2}\delta_2}{B_2^2}\left[C_2^*\varphi_{xxx} - \rho I_2^* c_0^4 \varphi_{xtt}\right]$$

is obtained. Upon substitution into $(12)_2$, which also is to be expanded, this yields

$$\varphi = -\frac{A_1 v_0}{B_1}u - \frac{\delta_1 A_1 v_0}{B_1^2}\left[C_1^* u_{xx} - \rho I_1^* c_0^4 u_{tt}\right] - \frac{\delta_2^2 A_1 (A_2^*)^2 v_0}{B_1^2 B_2^2}\left[C_2^* u_{xxxx} - \rho I_2^* c_0^4 u_{xxtt}\right]$$

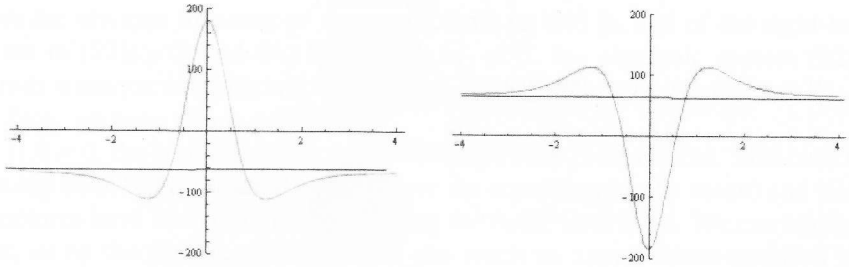


Figure 3: $a = -8$, $b = -1.5$ for the figure on the left, $a = -8$, $b = +1.5$ for the figure on the right

Finally this expression is inserted in $(12)_1$ resulting in the partial differential equation

$$u_{TT} = \left(\frac{\alpha\varepsilon L B_1 + A_1^2 v_0}{\varepsilon L \rho c_0^4 B_1}\right) u_{XX} + \frac{\beta\varepsilon L}{\rho c_0^4}(u^2)_{XX} - \frac{\delta_1 A_1^2 v_0}{\varepsilon L \rho c_0^4 B_1^2}\left[C_1^* u_{XX} - \rho I_1^* c_0^4 u_{TT}\right]_{XX} +$$

$$+ \frac{\delta_2^2 A_1^2 (B^*)^2 v_0}{\varepsilon L \rho c_0^4 B_1^2 B_2^2}\left[C_2^* u_{XX} - \rho I_2^* c_0^4 u_{TT}\right]_{XXXX}$$

## 4. Concurrent microstructures

Instead of a hierarchy of microstructures, one can be interested in concurrent microstructures, as introduced in [2], namely in two, or more, microstructures which act at the same scale level and interact with the macrostructure as well, as illustrated in Figure 4.

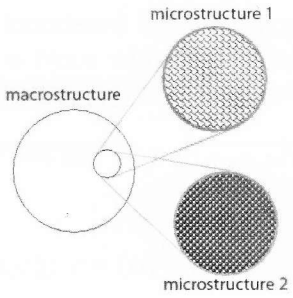


Figure 4: Concurrent microstructures

We can obtain the fields equations as done before considering $B = 0, A \neq 0$. Hence the field equations read:

$$\begin{cases} \rho v_{tt} &= \alpha v_{xx} + \beta (v_x^2)_x - A_1 \varphi_x - A_2 \psi_x \\ I_1 \varphi_{tt} &= C_1 \varphi_{xx} - A_1 v_x - B_1 \varphi - A \psi \\ I_2 \psi_{tt} &= C_2 \psi_{xx-} - A_2 v_x - B_2 \psi - A \varphi \end{cases} \tag{19}$$

where $\alpha$, $A_i$, $B_i$, $C_i$, $A_{12}$ $(i = 1,2)$ denote material constants.

With the substitution $v_0 u = v_x$, from $(19)_1$ we derive:

$$\rho v_0 u_{tt} = \alpha v_0 u_{xx} + \beta v_0^2 (u^2)_{xx} - A_1 \varphi_{xx} - A_2 \psi_{xx}. \tag{20}$$

Using the change of parameters already recalled in the previous Section we can obtain the dimensionless equations

$$\begin{cases} \dfrac{\rho v_0 c_0^2}{L} u_{TT} = \alpha \dfrac{v_0}{L^2} u_{XX} + \dfrac{\beta v_0^2}{L^2} \left(u^2\right)_{XX} - \dfrac{A_1^* l_1}{L^2} \varphi_{XX} - \dfrac{A_2^* l_2}{L^2} \psi_{XX} \\ B_1 \varphi + A \psi = -A_1^* l_1 v_0 u + \dfrac{l_1^2}{L^2} \left(C_1^* \varphi_{XX} - \rho I_1^* c_0^2 \varphi_{TT}\right) \\ A \varphi + B_2 \psi = -A_2^* l_2 v_0 u + \dfrac{l_2^2}{L^2} \left(C_2^* \psi_{XX} - \rho I_2^* c_0^2 \psi_{TT}\right) \end{cases} \tag{21}$$

This system can be re-written in the simpler formal way:

$$\begin{cases} u_{TT} = \alpha_1 u_{XX} + \beta_1 \left(u^2\right)_{XX} + \alpha_2 \varphi_{XX} + \alpha_3 \psi_{XX} \\ B_1 \varphi + A \psi = \Phi \\ A \varphi + B_2 \psi = \Psi \end{cases} \tag{22}$$

with the obvious meaning of the coefficients $\alpha_1$ and $\beta_1$ and of the right-hand terms in $(22)_{2,3}$ $\Phi$ and $\Psi$. Iff $B_1B_2 - A_{12}^2 \neq 0$, the algebraic system $(22)_{2,3}$ admits a unique solution and $\varphi$ and $\psi$ are linear combinations of $\Phi$ and $\Psi$.

Now we have two possibilities:

*i)* $A = 0$, the concurrent microstructures are fully independent. This case has already been briefly studied in [2], where the equations for the macro and micro structures have been obtained neglecting the cubic term in $W$. We can add here that, using the slaving principle, one can reach an approximate equation in $u$ only:

$$\frac{\rho v_0 c_0^2}{L} u_{TT} = \left[\alpha \frac{v_0}{L^2} - \frac{(A_1^* l_1)^2}{L^2} v_0 - \frac{(A_2^* l_2)^2}{L^2}\right] u_{XX} + \frac{\beta v_0^2}{L^2} \left(u^2\right)_{XX} \tag{23}$$

which is a well known nonlinear PDE widely studied elsewhere.

*ii)* $A \neq 0$, the microstructures are coupled. In a very similar way, we obtain a leading equation of the type:

$$u_{TT} - \mathcal{A} u_{XX} = \frac{\beta v_0^2}{L^2} \left(u^2\right)_{XX} \tag{24}$$

where $\mathcal{A}$ briefly denotes the set of coefficients of $u_{XX}$ analogous to the square brackets in (23), but it contains $A_{12}$, namely the coupling constant.

The approximation used to obtain (23) and (24) is very sharp, we can call it *zero order approximation*. Using the slaving principle at the first order approximation, namely, as done before, setting

$$\begin{aligned} \varphi &= \varphi_0 + \delta_1 \varphi_1 + \dots \\ \psi &= \psi_0 + \delta_1 \psi_2 + \dots \end{aligned} \tag{25}$$

we obtain from (21), with longer calculations, the 4th order PDE:

$$u_{TT} + \alpha_1 u_{XX} + \beta_1 (u^2)_{XX} + \delta_1 \left(\alpha_3 u_{XX} + \alpha_4 u_{TT}\right)_{XX} + \delta_2 \left(\alpha_5 u_{XX} + \alpha_6 u_{TT}\right)_{XX} = 0 \tag{26}$$

where the coefficients $\alpha_i$ can be explicitly evaluated in terms of the material constants appearing in (21).

Equation (26) is clearly different from (13), since here we have two concurrent microstructures acting at the same level, one is responsible of the term $\delta_1(\dots)$, the second one of the term $\delta_2(\dots)$, but in both cases the order of the derivatives is the 4th. In (13), as remarked, we have a hierarchy of microstructures, at the first level corresponding to 4th order derivatives, the second level to 6th order derivatives. The two terms $\delta_1(\dots)$ and $\delta_2(\dots)$ are coupled through

the coefficient $A_{12}$ which appears in both terms. Obviously, if $A_{12} = 0$, the microstructures are independent and the two terms are uncoupled.

Technically, the reason is that in the actual strain energy function (**??**) does not appear the term $-A_2\varphi_x\psi$ which is in (9), related to the hierarchy of the microstructures in that case, while here we have a "peer" coupling term $A_{12}\varphi\psi$.

Equation (26) is very similar with the DDE (3.16) in Samsonov [17], the meaning of the coefficients being obviously different. Hence we can follow the same procedure. Introducing the phase variable $z = x \pm Vt$ we obtain the 4th order PDE

$$(V^2+\alpha_1)u^{(\mathrm{II})} + \beta_1(u^2)^{(\mathrm{II})} + \delta_1\left(\alpha_3 + V^2\alpha_4\right)u^{(\mathrm{IV})} + \delta_2\left(\alpha_5 + V^2\alpha_6\right)u^{(\mathrm{IV})} = 0 \tag{27}$$

By double integration and regrouping the last two terms we obtain the 2nd order ODE

$$\delta_1\left(\alpha_3 + V^2\alpha_4\right) + \delta_2\left(\alpha_5 + V^2\alpha_6\right))u'' + \beta_1(u^2) + (V^2+\alpha_1)u + c_1z + c_2 = 0 \tag{28}$$

This equation can be formally written, with obvious meaning of the coefficients, as

$$u'' + b(u^2) + cu + c_1z + c_2 = 0 \tag{29}$$

Multiplying by $u'$ and integrating once more we have

$$\frac{1}{2}(u')^2 = -\frac{1}{3}bu^3 - \frac{1}{2}cu^2 - c_2u - c_1(z + \frac{1}{2}z^2)u + d \tag{30}$$

In Samsonov [17], Chapter 3 one can find an extensive analysis of equations of this kind and we can apply here his conclusions too, namely that by means of an appropriate choice of the invariants of the elliptic P-functions appearing in the Weierstrass equation associated to (30) and appropriate P-function limits in real axis, equation (30) can admit a discontinuous general travelling wave solution that can be reduced to solitary wave and to *cnoidal* wave solution.

Indeed one can imagine higher order coupling terms introducing in $W$ products of derivatives of $\varphi$ and $\psi$, namely terms containing $\varphi_x\psi$, $\psi_x\varphi$, $\varphi_x\psi_x$, but for need of brevity we do not go further in this direction.

## 5. Conclusions

The problem of the propagation of non linear waves in solids with different internal structural scales is studied. The general model developed in [4] and [14] has been used. In case of one microstructure only, a 6th order PDE is

obtained and the hierarchy of waves is clearly obtained. Using the same basic model, the case of two concurrent microstructures is studied and by means of the slaving principle one can reach meaningful approximate equations.

Such a general model is very fruitful because we can obtain many particular cases and study the solution of equations which encompass different physical situations. Moreover we can use simulation and numerical techniques and we can study also 2D cases, as done in [16], where a good approximation of reality has been obtained.

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