

## MICROSTRUCTURED CONTINUA AND SCALING FOR WAVE MOTION

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**ABSTRACT.** This paper deals with wave motion in microstructured solids. A short introduction explains how the basic mathematical models for description of microstructure(s) of solids are derived. Based on the Mindlin-type micromorphic theory, the governing equations for wave motion in such solids are presented in one-dimensional setting. The focus of the paper is in explaining the importance of internal scales in microstructured solids. It is shown that the proper scaling permits to construct the mathematical models which involve hierarchies of wave operators. Depending on the scale parameter (the ratio of an internal scale over the wave length), the various operators govern the wave propagation. The main case analysed here consists of the second-order operators but the first-order operators which are characteristic to evolution equations, are also briefly explained.

### 1. Introduction

The classical theories of continua are based on the assumption of the homogeneity of solids *resp.* materials although in reality all materials have a certain microstructure at various scales. The response of many materials of contemporary technological interest (e.g. metals, alloys, composites, etc.) to external loading is, however, greatly influenced by an embedded microstructure. The influence of the microstructure to macroscopic behaviour depends on scales and internal forces in the material. Dealing with impact loading and its consequence – wave motion in contemporary materials, it is clear that the classical theories become inadequate for describing wave motion properly in cases when the wave length becomes comparable with the characteristic size (length) of embedded microstructures. This is the reason why during the last decade or two, the modelling of microstructured continua has got a lot of attention.

In this paper the focus is on the influence of internal scales on the wave motion at the macroscale. In Section 2, a brief overview is presented on principles of mathematical modelling of waves in microstructured solids. Then the definite mathematical models are described in Section 3 based on micromorphic theory and the concept of internal variables. Section 4 is a central one in this paper. It is shown there how the proper scaling leads to the hierarchy of waves. In Section 5 several modifications of such an approach are discussed. Finally a general discussion is presented in Section 6.

## 2. Principles of mathematical modelling of microstructured continua

The character of a microstructure can be regular (like in laminated composites) or completely irregular (like in polycrystalline solids or functionally graded materials). In general terms, the starting point for describing a microstructure could be either discrete or continuum approach. In the first case the discrete elements are treated as point masses with certain interaction forces between them [1, 2]. Evidently, the system of governing equations of motion becomes very large which creates difficulties in numerical simulation. For a regular structure (like a laminated composite) the piece-wise homogeneous continuum models can be used [3].

The continuum approach for microstructured solids is elaborated by incorporating the intrinsic microstructural effects into governing equations [4, 5, 6]. A leading concept is to separate the macro- and microstructure in continua and to formulate the balance for both structures separately. Such an approach is very elegantly described by Mindlin [5]. There is also a possibility to introduce the microstructural quantities into one set of balance laws [7, 8]. In this case the interaction forces between the macro- and microstructure are explicitly described. Following Maugin [7, 8], the balance of the canonical (material) momentum on the material manifold  $\mathfrak{M}^3$  reads

$$\left. \frac{\partial \mathbf{P}}{\partial t} \right|_{\mathbf{x}} - \text{Div}_R \mathbf{b} = \mathbf{f}^{int} + \mathbf{f}^{ext} + \mathbf{f}^{inh}, \quad (1)$$

The corresponding dissipation inequality is

$$S\dot{\theta} + \mathbf{S} \cdot \nabla_R \theta \leq h^{int} + \nabla_R(\theta \mathbf{K}). \quad (2)$$

Here the following notation is used:  $\mathbf{P}$  is the material momentum (pseudomomentum),  $\mathbf{b}$  is the material Eshelby stress,  $\mathbf{f}^{int}$ ,  $\mathbf{f}^{ext}$ ,  $\mathbf{f}^{inh}$ , are the material inhomogeneity force, the material external (body) force, and the material internal force, respectively;  $S$  is the entropy density per unit reference volume,  $\mathbf{S}$  is the entropy flux,  $\theta$  is the absolute temperature,  $h^{int}$  is the source term (if any) and  $\mathbf{K}$  is the extra entropy flux (if any).

As seen from Eq(1), the forces are explicitly accounted for but the question how to construct the free energy function needed for their determination remains to be answered. Here the concept of internal variables can effectively be used in order to compensate the lack of a precise description of a microstructure. The formalism of internal variables has been introduced in [9, 10] but later the formalism for describing the dynamics of microstructured continua has been generalized [11, 12]. According to this generalization the internal variables are introduced to the free energy function, all forces can then be easily calculated but the governing equations for internal variables are also needed. These are obtained by satisfying the dissipation inequality (2). In this way only one balance law (Eq (1)) is used together with energy considerations. What is extremely important, in this way also the inertia of internal variables can be taken into account. Following such an approach, several mathematical models are elaborated ascribing internal variables to microdeformation and microtemperature [12, 13]. It has also been shown that the Mindlin model [5] can be derived by the approach using the material momentum [14] and by the concept of dual internal variables [12, 15].

It must be stressed that this brief overview above reflects the studies on modelling the dynamical processes in microstructured solids. In statics, one should also deal with averaging ideas of microstructural properties which are not in focus here.

### 3. Governing equations

Based on principles briefly described in Section 2: one balance law for the canonical momentum plus the dissipation inequality, the governing equations for wave motion are easily derived. Although the general theory is derived for the 3D case [16], we shall use here the 1D setting for transparency reasons. As far as our main aim is to focus on scaling, we restrict ourselves to the linear case. Omitting the details (see [13, 14]), the governing system is then the following:

$$\rho_0 u_{tt} - \alpha u_{xx} - A\psi_x = 0, \quad (3)$$

$$I \psi_{tt} - C\psi_{xx} + Au_x + B\psi = 0. \quad (4)$$

Here  $u$  denotes the longitudinal (macro)displacement and  $\psi$  – the microdeformation if we follow the Mindlin model or the internal variable if we follow the concept of internal variables. Indices here and further denote the differentiation, as usual. The constants  $\alpha, A, B, C$  are the material parameters specifying the free energy function [14] while  $\rho_0$  is the density of the microstructure and  $I$  – inertia of the microstructure.

There are two wave operators in the model:

$$L_{ma}(u) = \rho_0 u_{tt} - \alpha u_{xx}, \quad (5)$$

$$L_{mi}(\psi) = I \psi_{tt} - C\psi_{xx}. \quad (6)$$

Suppose the initial and boundary conditions are given

$$u(x, t = 0) = u_t(x, t = 0) = 0, \quad (7)$$

$$u(x = 0, t) = f(t), \quad (8)$$

$$\psi(x = 0, t) = 0, \quad (9)$$

$$\lim_{x \rightarrow 0} u(x, t) = \lim_{x \rightarrow 0} \Psi(x, t) = 0. \quad (10)$$

The wave operators are coupled in case  $A \neq 0$  which is of our primary interest. The question is which of the wave operators prevails or are they competing with each other in wave motion generated by (7) – (10). In order to answer this question we have to find a suitable scaling procedure.

### 4. Scaling procedure

On the one side, a characteristic scale of the microstructure (the size of an element) must be known, let us denote it by  $l$ . On the other side, let the excitation be characterized by its amplitude  $U_0$  and wavelength  $L$ . The dimensionless variables are then introduced by

$$U = u/U_0, \quad X = x/L, \quad T = c_o t/L, \quad (11)$$

where  $c_o^2 = \alpha/\rho_0$ . Two nondimensional parameters are introduced by

$$\delta = l^2/L^2, \quad \epsilon = U_0/L. \quad (12)$$

Concerning the coefficients of Eqs (3), (4), we suppose that  $I = \rho_0 l^2 I^*$ ,  $C = l^2 C^*$  where  $I^*$  is dimensionless and  $C^*$  has the dimension of stress. It must be noted that  $I$  is scaled against  $\rho_0$  so that the difference between the densities of the macro- and microstructure is embedded in  $I^*$ .

We rewrite the system (3), (4) in its dimensionless form and apply the slaving principle: the variable  $\psi$  related to the microstructure will be determined in terms of  $U$  using a series representation. The ideas of such an approach are envisaged by Whitham [17] and elaborated by Porubov [18], see also [19].

Two steps are needed for such a procedure. Firstly we consider that

$$\psi = \psi_0 + \delta\psi_1 + \dots, \quad (13)$$

and secondly, we determine  $\psi$  from Eq (4) in its dimensionless form

$$\psi = -\frac{\epsilon A}{B} U_X - \frac{\delta}{B} (\alpha I^* \psi_{TT} - C^* \psi_{XX}). \quad (14)$$

Then it is possible to determine

$$\psi_0 = -\frac{\epsilon A}{B} U_X, \quad (15)$$

$$\psi_1 = \epsilon \frac{A}{B^2} (\alpha I^* U_{XTT} - C^* U_{XXX}). \quad (16)$$

Inserting (15), (16) into the Eq (3) in its dimensionless form, we get finally in terms of the macrodisplacement  $U$  the equation

$$U_{TT} - \left(1 - \frac{c_A^2}{c_0^2}\right) U_{XX} = \frac{c_A^2}{c_B^2} \left( U_{TT} - \frac{c_1^2}{c_0^2} U_{XX} \right)_{XX}. \quad (17)$$

Here  $c_1^2 = C/I$ ,  $c_A^2 = A^2/\rho_0 B$ ,  $c_B^2 = BL^2/I$ . A more close look to the velocities reveals that  $c_B^2$  includes the interaction effects between macro- and microstructure. It is possible to establish that

$$\frac{c_A^2}{c_B^2} = \delta I^* \frac{A^2}{B^2}. \quad (18)$$

A more compact presentation of Eq (17) together with expression (18) reads

$$U_{TT} - k_1 U_{XX} = \delta m_1 (U_{TT} - p_1 U_{XX})_{XX}, \quad (19)$$

where

$$k_1 = 1 - \frac{c_A^2}{c_0^2}, \quad m_1 = I^* \frac{A^2}{B^2}, \quad p_1 = \frac{c_1^2}{c_0^2}. \quad (20)$$

Equation (19) involves two wave operators

$$L_{ma}(U) = U_{TT} - k_1 U_{XX}, \quad (21)$$

$$L_{mi}(U) = U_{TT} - p_1 U_{XX}. \quad (22)$$

As far as they are weighted by the scale parameter  $\delta$  with other parameters being of order of  $O(1)$ , the hierarchical nature of wave propagation is clearly revealed. If  $\delta$  is small, then waves are governed by the properties of macrostructure. If, however,  $\delta$  is large, then waves “feel” more the properties of microstructure. It is in full accordance with the wave hierarchy principle described by Whitham [17]. As far as Eq (19) involves the second

derivative of the wave operator  $L_{mi}(U)$ , the microstructural effects are of the dispersive character governed by terms  $U_{TTXX}$  and  $U_{XXXX}$ .

### 5. Possible generations: multiscale and one-wave models

It is possible to generalize the scaling process to multiscale models. In the simplest case of two hierarchical microstructure (a scale within another scale) the models are derived in [20, 21]. By introducing microdeformations  $\psi$  (as before) and  $\varphi$  (the microstructure in  $\psi$ ), the governing equations are

$$\rho_0 u_{tt} - \alpha u_{xx} - A_1 \psi_x = 0, \quad (23)$$

$$I_1 \psi_{tt} - C_1 \psi_{xx} + A_1 u_x + B_1 \psi - A_2 \varphi_x = 0, \quad (24)$$

$$I_2 \varphi_{tt} - C_2 \varphi_{xx} + A_2 \psi_x + B_2 \varphi, \quad (25)$$

where  $I_1, I_2$  are the corresponding microinertia,  $A_i, C_i, B_i, i = 1, 2$  are coefficients [20]. Note that  $A_1$  expresses the coupling of  $u$  and  $\psi$  while  $A_2$  expresses the coupling of  $\psi$  and  $\varphi$ . In order to carry on scaling, the dimensionless variables  $U, X, T$  are introduced as before (see Eqs (11)) together with parameters

$$\delta_1 = l_1^2/L^2, \quad \delta_2 = l_2^2/L^2. \quad (26)$$

Here  $l_1$  and  $l_2$  denote the characteristic scales of both microstructures. With scaling of  $I_i, C_i, i = 1, 2$  like in Section 4 and using the series representation, we obtain the following governing equation of motion in terms of  $U$  [20]:

$$U_{TT} - k_{11}(A_1)U_{XX} = \delta_1 m_{11}(U_{TT} - p_{11}(A_2)U_{XX})_{XX} + \delta_2^2 m_{12}(U_{TT} - p_{22}U_{XX})_{XXXX}. \quad (27)$$

where  $k_{11}, m_{11}, m_{12}, p_{11}, p_{22}$  are coefficients related to the coefficients of Eqs (23), (24), (25) with proper scaling [20].

Now there are wave operators

$$L_{ma}(U) = U_{TT} - k_{11}(A_1)U_{XX}, \quad (28)$$

$$L_{mi(1)}(U) = U_{TT} - p_{11}(A_2)U_{XX}, \quad (29)$$

$$L_{mi(2)}(U) = U_{TT} - p_{22}U_{XX}, \quad (30)$$

which are scaled by  $\delta_1$  and  $\delta_2$ . In this way the operators describe the wave motion over many scales and the influence of each of them is regulated by  $\delta_1$  and  $\delta_2$ .

Another generalization of system (3), (4) is to include nonlinearities in both macro- and microscale, then instead of system (3), (4) we obtain [20]

$$\rho_0 u_{tt} - \alpha u_{xx} - N u_x u_{xx} - A \psi_x = 0, \quad (31)$$

$$I \psi_{tt} - C_1 \psi_{xx} - M \psi_x \psi_{xx} + A u_x + B \psi = 0, \quad (32)$$

where  $N$  and  $M$  are coefficients of cubic terms in the free energy function. Following the scaling procedure like in Section 4, we arrive to the hierarchy of waves where instead of operators (21), (22) we obtain

$$L_{ma}(U) = U_{TT} - k_1 U_{XX} - \frac{\mu}{2} (U_X^2)_X, \quad (33)$$

$$L_{mi}(U) = U_{TT} - p_1 U_{XX} - \frac{\lambda}{2} \delta^{1/2} U_{XX}^2, \quad (34)$$

where  $\mu(N)$  and  $\lambda(M)$  are the coefficients. In this case wave operators are nonlinear and open the possibilities to emergence of solitary waves [22].

The basic idea on hierarchies of waves introduced by Whitham [17] is related to operators of the first order. Here the models described above involve the full second-order wave operators and enlarge the concept of wave motion in a homogeneous medium to the case of a microstructured medium. Like the classical wave equation, the derived equations are as a matter of fact two-wave equations. If the one-wave models (evolution equations) are used then the hierarchy could also be described a sequence of operators of the first order. Oliveri [23] has derived such a hierarchy for nonlinear waves in bubbly liquids. In this case the evolution equation reads

$$u_\tau + uu_\xi + \gamma\alpha u_{\xi\xi} + \delta\alpha u_{\xi\xi\xi} + \gamma\beta(u_\tau + uu_\xi)_\xi + \delta\beta(u_\tau + uu_\xi)_{\xi\xi} = 0, \quad (35)$$

where  $\tau, \xi$  is a moving frame as usually taken for evolution equations and  $\alpha, \beta, \gamma, \delta$  are constants [23]. If  $\beta \rightarrow 0$  then the classical Korteweg-de Vries-Burgers equation is recovered. In a general case the governing model is constituted by the hierarchy of nonlinear one-wave operators  $(u_\tau + uu_\xi)$ .

Another interesting case is described by Giovine and Oliveri [24] for waves in dilatant granular materials. Here the evolution equation takes the form

$$u_\tau + uu_\xi + \alpha_1 u_{\xi\xi\xi} + \beta(u_\tau + uu_\xi + \alpha_2 u_{\xi\xi\xi})_{\xi\xi} = 0, \quad (36)$$

where  $\alpha_1$  and  $\alpha_2$  are the dispersion parameters and  $\beta$  involves the ratio of the grain size and the wavelength. Equation (36) like Eq (35) above is written in a moving frame  $\tau, \xi$ . Here the model involves two Korteweg-de Vries (KdV) operators – one for motion in the macrostructure, another – in the microstructure while  $\beta$  regulates the weight of two operators.

## 6. Final remarks

Waves in microstructured (complex) materials can be modelled by hierarchical equations like described by Whitham [17]. Here we have shown that in the case of a Mindlin-type micromorphic model the resulting hierarchy involves the second-order wave operators (more about such hierarchies in [14, 20]). It is possible to derive hierarchies also for one-wave models, i.e. for evolution equations. Then the operators are of the first order like shown originally by Whitham [17] or derived by Oliveri [23] for the case of waves in bubbly liquids. The operators can certainly be more complicated like those constructed by Giovine and Oliveri [24] for waves in dilatant granular materials. In this case the operators are of the KdV-type. Whatever the structure of operators in a hierarchical model is, the scale parameter  $\delta$  (or  $\delta_i, i = 1, 2$ ) rules the behaviour of waves taking into account the ratio of a characteristic length of the microstructural element to the characteristic length of an excitation.

Widely known are also the scaling principles for the systems of ordinary differential equations. Here the Haken's subordination principle is used for distinguishing the fast and slow modes in such systems [25]. It means grouping of variables by their time rate of change and distinguishing so faster and slower changing variables. Then the slower

ones are represented as constants and the faster ones – as algebraic terms. In this way the system may asymptotically have a substantially lower dimension which is easier to solve. Mostly such an approach is used for reaction-diffusion systems. In contrast to this approach, the scaling procedure for wave motion yields the hierarchical wave operators and not the constants or functions in a governing equation.

It will be interesting to generalize the scaling principles described above down to nano-scales and additional fields.

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