Acta Mechanica manuscript No.
(will be inserted by the editor)

Arkadi Berezovski · Jüri Engelbrecht · Mihhail Berezovski

Waves in microstructured solids: a unified viewpoint of modelling

Received: date / Accepted: date

Abstract The basic ideas for describing the dispersive wave motion in microstructured solids are discussed in the one-dimensional setting because then the differences between various microstructure models are clearly visible. An overview of models demonstrates a variety of approaches, but the consistent structure of the theory is best considered from the unified viewpoint of internal variables. It is shown that the unification of microstructure models can be achieved using the concept of dual internal variables.

Keywords Wave propagation · Microstructured solids · Internal variables

1 Introduction

Wave propagation in a homogeneous medium is a well known phenomenon in mechanics. The corresponding wave equation is a classical example of hyperbolic partial differential equations in textbooks. However, the situation is more complicated in inhomogeneous media due to dispersion caused by intrinsic microstructural effects [1].

The classical equation of linear elastic wave propagation in homogeneous solids in the one-dimensional case reads

$$u_{tt} = c^2 u_{xx}, \tag{1}$$

where u is the displacement, c is the elastic wave speed and subscripts denote derivatives. Considering a harmonic wave

$$u(x,t) = \hat{u} exp\,[i(kx - \omega t)] \tag{2}$$

A. Berezovski · J. Engelbrecht · M. Berezovski
Centre for Nonlinear Studies, Institute of Cybernetics at Tallinn University of Technology, Akadeemia tee 21, 12618 Tallinn, Estonia
Tel.: +372-62-04164, Fax: +372-62-04151, E-mail: Arkadi.Berezovski@cs.ioc.ee

with wave number k and frequency ω, we obtain the dispersion relation

$$\omega^2 = c^2 k^2. \tag{3}$$

It is easy to see that here the group velocity $\partial\omega/\partial k$ is equal to the phase velocity c, which means that no dispersion is present.

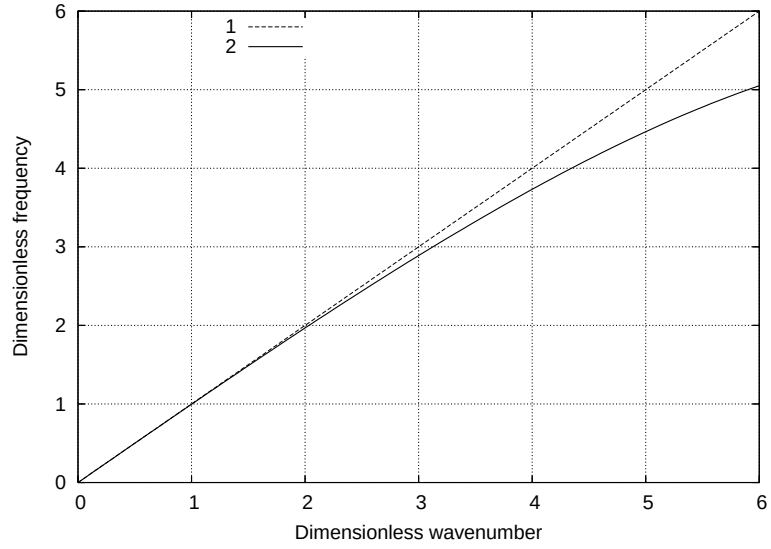

Fig. 1 Dispersion curves for $\gamma' = 0.3$: 1 – wave equation (1), 2 – Boussinesq-type equation (8).

To describe wave propagation in heterogeneous materials reflecting dispersion effects, several modifications of the wave equation are proposed. The simplest generalization of the wave equation is the linear version of the Boussinesq equation for elastic crystals (cf. [2])

$$u_{tt} = c^2 u_{xx} + c^2 l^2 A_{11} u_{xxxx}, \tag{4}$$

where l is an internal length parameter and A_{11} is a dimensionless coefficient. Similar equations were obtained by using the homogenization of a periodically layered medium [3–5] or using strain gradient theories [6]. The dispersion relation is obtained by using again the harmonic wave solution (2)

$$\omega^2 = c^2 k^2 - c^2 l^2 A_{11} k^4. \tag{5}$$

Introducing dimensionless frequency and wavenumber by

$$\eta = \frac{\omega}{\omega_0}, \quad \xi = \frac{ck}{\omega_0}, \tag{6}$$

where ω_0 is a characteristic frequency, and using the dimensionless parameter γ' defined by

$$\gamma'^4 = \frac{l^2\omega_0^2 A_{11}}{c^2}, \tag{7}$$

we can rewrite the dispersion relation (5) in the dimensionless form

$$\eta^2 = \xi^2 - \gamma'^4\xi^4. \tag{8}$$

The corresponding dispersion curve is shown in Fig. 1. Its deviation from the non-dispersive case (the straight line) increases for higher frequencies and wavenumbers.

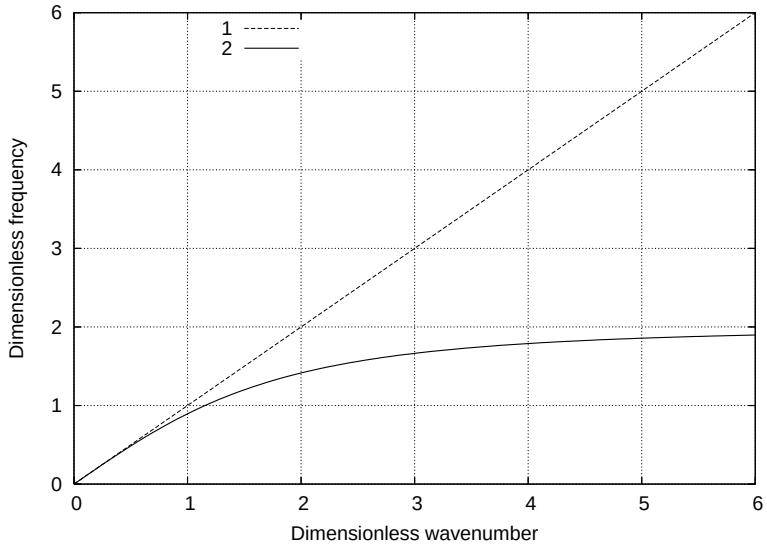

Fig. 2 Dispersion curves for $\gamma_1 = 0.5$: 1 – wave equation (1), 2 – Love-Rayleigh-type equation (11).

Another generalization of the wave equation is the Love-Rayleigh equation for rods accounting for lateral inertia (cf. [7], p.428)

$$u_{tt} = c^2 u_{xx} + l^2 A_{12} u_{xxtt}, \tag{9}$$

where A_{12} is again a dimensionless constant. This equation is derived also in [8–11]. The corresponding dispersion equation has the form

$$\omega^2 = c^2k^2 - l^2A_{12}\omega^2k^2. \tag{10}$$

Its dimensionless version is written as

$$\eta^2 = \xi^2 - \gamma_1^2\eta^2\xi^2, \tag{11}$$

where the new dimensionless parameter γ_1 is introduced

$$\gamma_1^2 = \frac{l^2\omega_0^2 A_{12}}{c^2}. \tag{12}$$

The deviation of the dispersive curve from the non-dispersive case (straight line) is essentially larger than in the previous case, as one can see in Fig. 2.

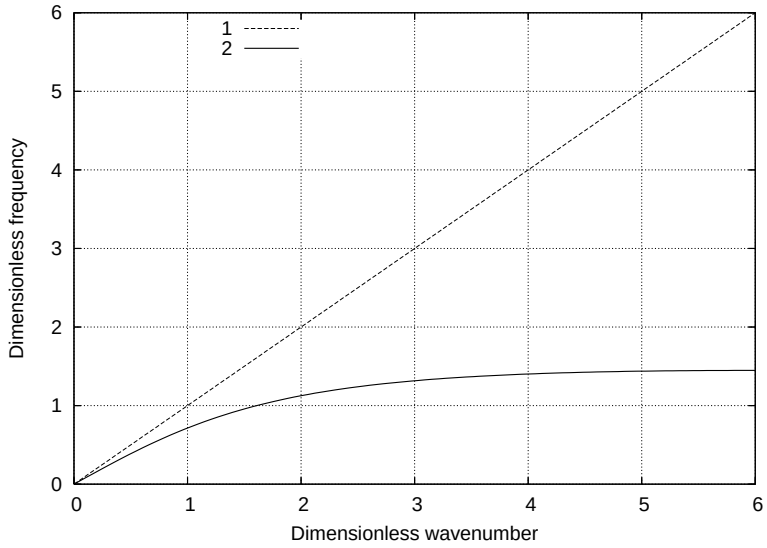

Fig. 3 Dispersion curves for $\gamma_1 = 0.5, \gamma_A = 0.6, \gamma' = 0.2$: 1 – wave equation (1), 2 – combined model (16).

A more general equation combining the two dispersion models gives [5,12, 13]

$$u_{tt} = c^2 u_{xx} + c^2 l^2 A_{11} u_{xxxx} + l^2 A_{12} u_{xxtt}. \tag{13}$$

Similar model proposed by Engelbrecht & Pastrone [14] introduces additionally a contribution of microstructure on slowing down of the propagation velocity c_A^2

$$u_{tt} = \left(c^2 - c_A^2\right) u_{xx} + c^2 l^2 A_{11} u_{xxxx} + l^2 A_{12} u_{xxtt}. \tag{14}$$

Accordingly, the dispersion relation

$$\omega^2 = (c^2 - c_A^2)k^2 - c^2 l^2 A_{11} k^4 - l^2 A_{12}\omega^2 k^2, \tag{15}$$

has dimensionless form

$$\eta^2 = (1 - \gamma_A^2)\xi^2 - \gamma_1^2 \eta^2 \xi^2 - \gamma'^4 \xi^4, \tag{16}$$

where $\gamma_A^2 = c^2/c_A^2$. Due to three additional terms combined, the last model demonstrates even a larger deviation from the non-dispersive case (Fig. 3).

In its turn, the Maxwell-Rayleigh model of anomalous dispersion [2] introduces in consideration the four-order time derivative

$$u_{tt} = c^2 u_{xx} + \frac{l^2 A_{22}}{c^2}\left(u_{tt} - c^2 u_{xx}\right)_{tt}. \tag{17}$$

However, there is no dispersion unless the velocities in both wave operators are not equal.

Four-order time derivatives are included also in the "causal" model for the dispersive wave propagation proposed by Metrikine [12]

$$u_{tt} = c^2 u_{xx} - c^2 l^2 A_{11} u_{xxxx} + l^2 A_{12} u_{xxtt} - \frac{l^2}{c^2} A_{22} u_{tttt}, \tag{18}$$

and in the model based on the Mindlin theory of microstructure [15] proposed by Engelbrecht et al [16] in the form

$$u_{tt} = \left(c^2 - c_A^2\right) u_{xx} - p^2 \left(u_{tt} - c^2\, u_{xx}\right)_{tt} + p^2\, c_1^2 \left(u_{tt} - c^2\, u_{xx}\right)_{xx}. \tag{19}$$

Here p and pc_1 determine time and length scales of the microstructure, respectively, c_1 can be associated with the wave propagation velocity in the microstructure itself.

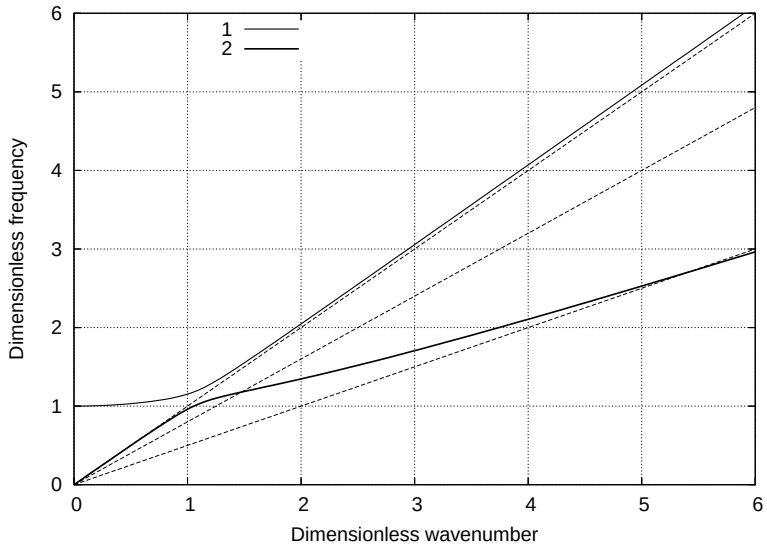

Fig. 4 Dispersion curves for the "causal" model (20) with $\gamma_1 = 0.5, \gamma' = 0.4$: 1 – optical branch, 2 – acoustical branch; dotted lines correspond to asymptotes to dispersion curves.

The last two equations differ from each other in two aspects: (i) the latter accounts for the slowing down of the propagation velocity in the microstructured

medium in comparison with that without microstructure and (ii) higher-order derivatives appear as derivatives of wave operators in the latter model in contrast to the former one. If the explicit expression for the slowing down of the propagation velocity can be an advantage of the latter model, then the appearance of the higher-order terms only as derivatives of the wave operator is not desirable.

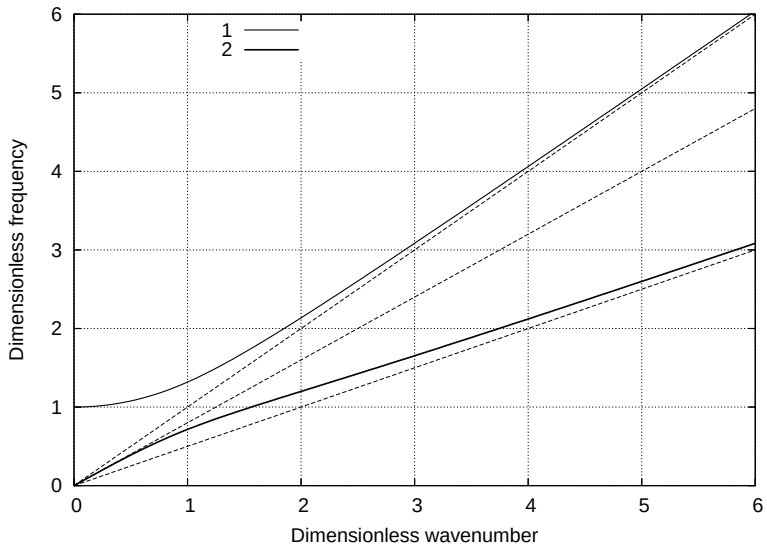

Fig. 5 Dispersion curves for the Mindlin-type model (21) with $\gamma_1 = 0.5, \gamma_A = 0.6$: 1 – optical branch, 2 – acoustical branch; dotted lines correspond to asymptotes to dispersion curves.

It is instructive to compare the dispersion properties of two last models. The corresponding dispersion equations can be represented as

$$\eta^2 = \xi^2 + \left(\eta^2 - \xi^2\right)\left(\eta^2 - \gamma_1^2\xi^2\right) - \gamma'^4\xi^4, \tag{20}$$

for the causal model (18) and

$$\eta^2 = \left(1 - \gamma_A^2\right)\xi^2 + \left(\eta^2 - \xi^2\right)\left(\eta^2 - \gamma_1^2\xi^2\right) \tag{21}$$

for the Mindlin-type model (19). Dispersion curves for both models have the acoustical as well as optical branches (Figs. 4,5), but dispersion curves for the so-called "causal" model [12] deviate from asymptotes with the increase of the parameter γ'.

All the models listed above are based either on homogenization [3,4,10], or on continualisation [5,9,12], or on generalized continuum theories [6,13,14,16], which means that all the models are of the mechanical origin. An alternative approach to the description of microstructural effects is provided by the internal variable theory, which is intimately related to thermodynamics. The use of internal variables in the description of the behavior of materials with microstructure has a long tradition and nowadays it is commonly accepted (cf. [17]).

The thermodynamic theory of internal variables presented by Coleman & Gurtin [18] had presupposed first-order evolution equations for the internal variables and did not include their gradients. Accounting for the gradients leads to the weakly nonlocal theory [9,19], which can be also enriched by the extra entropy flux [20]. The complete theory of the internal state variables is presented recently by Maugin [21]. Moreover, the limitation of evolution equations by only first-order ones is got over by the concept of dual internal variables [22].

Given the plethora of models, there is a clear need to understand their structure and characteristics from a unified viewpoint. In this paper, the internal variable approach is applied consecutively to the description of non-dissipative processes of linear dispersive wave propagation. We start in Section 2 with the governing equations in the material formulation of continuum mechanics [23]. After demonstrating the role of internal variables in Section 3, we introduce dual internal variables (Section 4) and derive evolution equations for them. As a result, we arrive at the known dispersive wave equations for micro- and macromotion (Sections 5 and 6) depending on the choice of the free energy function. The different dispersive wave equations are unified in Section 7. Some conclusions are given in the last Section.

2 Governing equations

In the linear case, the one-dimensional motion of the thermoelastic conductors of heat without body forces is governed by local balance laws for linear momentum and energy [24, e.g.]

$$\frac{\partial}{\partial t}(\rho v) - \frac{\partial \sigma}{\partial x} = 0, \tag{22}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + E\right) - \frac{\partial}{\partial x}(\sigma v - Q) = 0, \tag{23}$$

and by the second law of thermodynamics

$$\frac{\partial S}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q}{\theta} + K\right) \geq 0. \tag{24}$$

Here t is time, ρ is the matter density, $v = u_t$ is the physical velocity, u is the displacement, σ is the Cauchy stress, E is the internal energy per unit volume, S is the entropy per unit volume, θ is temperature, Q is the material heat flux, and the "extra entropy flux" K vanishes in most cases, but this is not a basic requirement.

Our main goal is the description of wave propagation in solids with microstructure. The existence of the microstructure generally means that the medium is inhomogeneous. The most consistent way to treat the inhomogeneities is the material formulation of continuum mechanics [23].

2.1 Canonical form of the energy conservation

To derive the canonical energy equation, the free energy per unit volume $W := E - S\theta$ is introduced into the energy balance (23), and the balance of linear momentum

(22) is multiplied by v. The canonical form of the energy conservation follows from the combination of the obtained equations:

$$\frac{\partial(S\theta)}{\partial t}+\frac{\partial Q}{\partial x}=h^{int},\quad h^{int}:=\sigma\varepsilon_t-\frac{\partial W}{\partial t}, \tag{25}$$

where the right-hand side of Eq. $(25)_1$ is formally an internal heat source [21]. The second law of thermodynamics (24) gives then

$$-\left(\frac{\partial W}{\partial t}+S\frac{\partial\theta}{\partial t}\right)+\sigma\varepsilon_t+\frac{\partial}{\partial x}(\theta K)-\left(\frac{Q}{\theta}+K\right)\frac{\partial\theta}{\partial x}\geq 0, \tag{26}$$

where $\varepsilon=u_x$ is the one-dimensional strain measure. The dissipation inequality (26) can be also represented as follows:

$$S\frac{\partial\theta}{\partial t}+\left(\frac{Q}{\theta}+K\right)\frac{\partial\theta}{\partial x}\leq h^{int}+\frac{\partial}{\partial x}(\theta K). \tag{27}$$

2.2 Canonical (material) momentum conservation

The canonical balance of momentum is derived by the multiplication of the balance of linear momentum (22) by u_x (cf. [23])

$$u_x\frac{\partial}{\partial t}(\rho v)-u_x\frac{\partial\sigma}{\partial x}=0. \tag{28}$$

Defining then the material momentum P, the material Eshelby stress b, the material inhomogeneity force f^{inh}, and the material internal force f^{int} by [23]

$$P:=-\rho u_t u_x,\quad b:=-\left(\frac{1}{2}\rho v^2-W+\sigma\varepsilon\right), \tag{29}$$

$$f^{inh}:=\left(\frac{1}{2}v^2\right)\frac{\partial\rho}{\partial x}-\left.\frac{\partial W}{\partial x}\right|_{expl},\quad f^{int}:=\sigma u_{xx}-\left.\frac{\partial W}{\partial x}\right|_{impl}, \tag{30}$$

we can represent the Eq. (28) in the canonical form [23]

$$\frac{\partial P}{\partial t}-\frac{\partial b}{\partial x}=f^{int}+f^{inh}. \tag{31}$$

Here the subscript notations *expl* and *impl* mean, respectively, the derivative keeping the fields fixed (and thus extracting the explicit dependence on x), and taking the derivative only through the fields present in the function.

The canonical equations for energy and momentum (25) and (31) are the most general expressions we can write down without a postulate of the full dependency of the free energy W [21]. Together with the dissipation inequality, they provide a consistent framework for the introduction of internal variables.

3 Single internal variable

As it was mentioned, the introduction of an internal variable associated with the distributed effect of the microstructure is not a new idea. It is expected that internal variables extending the state space represent some microscopic material structural characteristics [18,19,29,30] which are not explicitly determined like in the Mindlin theory [15] based on the notion of a "deformable cell". The most comprehensive theory of thermomechanics with internal variables is presented recently by Maugin [21]. We remind here its one-dimensional version and focus on its consequences for wave propagation.

In the one-dimensional case, the free energy W is specified as a general sufficiently regular function of the strain, temperature, the internal variable, φ, and its space gradient [21]

$$W = \overline{W}(u_x, \theta, \varphi, \varphi_x). \tag{32}$$

Then the equations of state determine the macroscopic stress σ, the entropy S, the internal stress η, and interactive internal force τ by

$$\sigma = \frac{\partial \overline{W}}{\partial u_x}, \quad S = -\frac{\partial \overline{W}}{\partial \theta}, \quad \tau := -\frac{\partial \overline{W}}{\partial \varphi} \quad \eta := -\frac{\partial \overline{W}}{\partial \varphi_x}. \tag{33}$$

The non-zero extra entropy flux K is represented in the form

$$K = -\theta^{-1}\eta\varphi_t, \tag{34}$$

following the scheme originally developed in [20] for materials with diffusive dissipative processes described by means of internal variables of state.

The canonical equations of momentum and energy keep their form

$$\frac{\partial P}{\partial t} - \frac{\partial \widetilde{b}}{\partial x} = f^{th} + \widetilde{f}^{intr}, \tag{35}$$

$$\frac{\partial (S\theta)}{\partial t} + \frac{\partial \widetilde{Q}}{\partial x} = h^{th} + \widetilde{h}^{intr}, \tag{36}$$

provided the new definitions are introduced [21]:

$$\widetilde{\tau} \equiv -\frac{\delta \overline{W}}{\delta \varphi} := -\left(\frac{\partial \overline{W}}{\partial \varphi} - \frac{\partial}{\partial x}\left(\frac{\partial \overline{W}}{\partial \varphi_x}\right)\right) = \tau - \eta_x, \tag{37}$$

$$\widetilde{b} := -\left(\frac{1}{2}\rho v^2 - W + \sigma u_x - \eta\varphi_x\right). \tag{38}$$

In this case, the "internal" material force and heat source each are split in two terms according to

$$f^{int} = f^{th} + \widetilde{f}^{intr}, \quad h^{int} = h^{th} + \widetilde{h}^{intr}, \tag{39}$$

where the *thermal sources* and the *"intrinsic" sources* are given by [21]

$$f^{th} := S\theta_x, \quad h^{th} := S\theta_t, \tag{40}$$

$$\widetilde{f}^{intr} := \widetilde{\tau}\varphi_x, \quad \widetilde{h}^{intr} := \widetilde{\tau}\varphi_t, \tag{41}$$

so the dissipation inequality reads

$$\Phi = \widetilde{h}^{intr} - \left(\frac{Q-\eta\dot{\varphi}}{\theta}\right)\frac{\partial\theta}{\partial x} \geq 0. \tag{42}$$

The dissipation inequality (42) is automatically satisfied in the isothermal case under choice

$$\widetilde{\tau} = k\varphi_t, \quad k \geq 0, \tag{43}$$

since

$$\Phi = k\varphi_t^2 \geq 0. \tag{44}$$

The fully non-dissipative case corresponds to $k = 0$.

3.1 Dispersive wave equation I

Now we have to prescribe the free energy function to be more specific. The simplest free energy dependence is a quadratic function (cf. [16])

$$\overline{W} = \frac{\rho c^2}{2}u_x^2 + A\varphi u_x + \frac{1}{2}B\varphi^2 + \frac{1}{2}C\varphi_x^2, \tag{45}$$

where coefficients A, B, and C depend on the material.

The corresponding stresses $(33)_{1,4}$ are calculated as follows:

$$\sigma = \frac{\partial\overline{W}}{\partial u_x} = \rho c^2 u_x + A\varphi, \quad \eta = -\frac{\partial\overline{W}}{\partial\varphi_x} = -C\varphi_x, \tag{46}$$

and the interactive internal force τ is, respectively,

$$\tau = -\frac{\partial\overline{W}}{\partial\varphi} = -Au_x - B\varphi. \tag{47}$$

The balance of linear momentum (22) takes the form

$$\rho_0 u_{tt} = \rho_0 c^2 u_{xx} + A\varphi_x, \tag{48}$$

and the evolution equation for the internal variable (43) in the fully non-dissipative case (with $k = 0$) reduces to

$$\widetilde{\tau} = \tau - \eta_x = C\varphi_{xx} - Au_x - B\varphi = 0. \tag{49}$$

Evaluating the first space derivative of the internal variable from the last equation

$$\varphi_x = \frac{C}{B}\varphi_{xxx} - \frac{A}{B}u_{xx}, \tag{50}$$

and its third space derivative from Eq. (48)

$$\frac{A}{\rho_0}\varphi_{xxx} = \left(u_{tt} - c^2 u_{xx}\right)_{xx}, \tag{51}$$

we will have, inserting the results into the balance of linear momentum (48)

$$u_{tt} = c^2 u_{xx} + \frac{C}{B}\left(u_{tt} - c^2 u_{xx}\right)_{xx} - \frac{A^2}{\rho B} u_{xx}. \tag{52}$$

Obtained equation is similar to that for the microstructure model (13) derived in [14]. It should be noted that higher-order derivatives appear in the dispersive wave equation (52) "en bloc", i.e., as derivatives of the wave operator, and cannot be eliminated separately. This means that Eq. (52) cannot be reduced either to Eq. (4) or to Eq. (9). The difference in the models is related to distinct free energy dependencies.

3.2 Dispersive wave equation II

In fact, choosing the free energy in the quadratic form

$$\overline{W} = \frac{\rho c^2}{2} u_x^2 + A' \varphi_x u_x + \frac{1}{2} B \varphi^2 + \frac{1}{2} C \varphi_x^2, \tag{53}$$

we have for the corresponding stresses

$$\sigma = \frac{\partial \overline{W}}{\partial u_x} = \rho c^2 u_x + A' \varphi_x, \quad \eta = -\frac{\partial \overline{W}}{\partial \varphi_x} = -A' u_x - C \varphi_x. \tag{54}$$

Note that there the coupling is described differently compared with Eqs. (45), (46). Therefore, the balance of linear momentum is rewritten as follows:

$$\rho_0 u_{tt} = \rho_0 c^2 u_{xx} + A' \varphi_{xx}, \tag{55}$$

and the evolution equation for the internal variable (43) in the fully non-dissipative case (with $k = 0$) reduces to

$$\widetilde{\tau} = \tau - \eta_x = C \varphi_{xx} + A' u_{xx} - B \varphi = 0. \tag{56}$$

By means of Eq. (55) the latter relation can be represented in the form

$$\varphi = \frac{C}{B}\left(\rho_0 u_{tt} - \rho_0 c^2 u_{xx}\right) + \frac{A'}{B} u_{xx}. \tag{57}$$

If coefficient C vanishes then we arrive at the strain-gradient model

$$\varphi = \frac{A'}{B} u_{xx}, \tag{58}$$

which results in the equation of motion of the form (4)

$$\rho_0 u_{tt} = \rho_0 c^2 u_{xx} + \frac{A'^2}{B} u_{xxxx}. \tag{59}$$

It should be noted that in the terms of stresses the first-order strain-gradient model (58) coincides with the second-order strain-gradient model in the spirit of Aifantis [25,26], since, following $(54)_1$ and (58),

$$\sigma = \rho c^2 u_x + \frac{A'^2}{B} u_{xxx}. \tag{60}$$

Accordingly, in the case of a non-zero value of the coefficient C the more general model (13) is obtained

$$u_{tt} = c^2 u_{xx} + \frac{C}{B}\left(u_{tt} - c^2 u_{xx}\right)_{xx} + \frac{A'^2}{\rho_0 B} u_{xxxx}, \tag{61}$$

but without explicit slowing down of the propagation velocity.

As one can see, the material formulation of continuum mechanics provides a thermodynamically consistent framework for the derivation of equations of motion in the medium with microstructure described by internal variables. However, the considered dispersion effects correspond to higher-order space derivatives only. Remaining dispersive wave equations with higher-order time derivatives (17), (18), and (19) require further consideration. We are able to go on following the recent generalization of the internal variables theory [22].

4 Dual internal variables

Let us consider the free energy W as a (sufficiently smooth) function of two internal variables φ, ψ and their space derivatives

$$W = \overline{W}(u_x, \theta, \varphi, \varphi_x, \psi, \psi_x). \tag{62}$$

In this case the equations of state are given by

$$\sigma := \frac{\partial \overline{W}}{\partial u_x}, \quad S := -\frac{\partial \overline{W}}{\partial \theta}, \quad \tau := -\frac{\partial \overline{W}}{\partial \varphi}, \quad \eta := -\frac{\partial \overline{W}}{\partial \varphi_x}, \tag{63}$$

$$\xi := -\frac{\partial \overline{W}}{\partial \psi}, \quad \zeta := -\frac{\partial \overline{W}}{\partial \psi_x}. \tag{64}$$

The non-zero extra entropy flux is included into consideration similarly to the case of one internal variable

$$K = -\theta^{-1}\eta\varphi_t - \theta^{-1}\zeta\xi_t. \tag{65}$$

The canonical equations of momentum and energy keep their form

$$\frac{\partial P}{\partial t} - \frac{\partial \widetilde{b}}{\partial x} = f^{th} + \widetilde{f}^{intr}, \tag{66}$$

$$\frac{\partial (S\theta)}{\partial t} + \frac{\partial \widetilde{Q}}{\partial x} = h^{th} + \widetilde{h}^{intr}, \tag{67}$$

with the modified Eshelby stress tensor

$$\widetilde{b} = -\left(\frac{1}{2}\rho v^2 - W + \sigma u_x - \eta\varphi_x - \zeta\psi_x\right) \tag{68}$$

and intrinsic source terms

$$\widetilde{f}^{intr} := \widetilde{\tau}\varphi_x + \widetilde{\xi}\psi_x, \quad \widetilde{h}^{intr} := \widetilde{\tau}\varphi_t + \widetilde{\xi}\psi_t. \tag{69}$$

In the above equations the following definitions are used

$$\widetilde{\tau} \equiv -\frac{\delta\overline{W}}{\delta\varphi} := -\left(\frac{\partial\overline{W}}{\partial\varphi} - \frac{\partial}{\partial x}\left(\frac{\partial\overline{W}}{\partial\varphi_x}\right)\right) = \tau - \eta_x, \tag{70}$$

$$\widetilde{\xi} \equiv -\frac{\delta\overline{W}}{\delta\psi} := -\left(\frac{\partial\overline{W}}{\partial\psi} - \frac{\partial}{\partial x}\left(\frac{\partial\overline{W}}{\partial\psi_x}\right)\right) = \xi - \zeta_x, \tag{71}$$

$$\widetilde{S} = \theta^{-1}\widetilde{Q}, \quad \widetilde{Q} = Q - \eta\dot{\varphi} - \zeta\dot{\psi}, \tag{72}$$

which are similar to those in the case of one internal variable.

The corresponding dissipation is determined by

$$\Phi = \widetilde{h}^{intr} - \widetilde{S}\theta_x = \widetilde{\tau}\varphi_t + \widetilde{\xi}\psi_t - \widetilde{S}\theta_x \geq 0. \tag{73}$$

In the isothermal case the dissipation inequality reduces to the intrinsic part depending only on internal variables

$$\Phi = \widetilde{h}^{intr} = \widetilde{\tau}\varphi_t + \widetilde{\xi}\psi_t = (\tau - \eta_x)\varphi_t + (\xi - \zeta_x)\psi_t \geq 0. \tag{74}$$

It is easy to see that the choice

$$\varphi_t = R(\xi - \zeta_x), \qquad \psi_t = -R(\tau - \eta_x), \tag{75}$$

where R is an appropriate constant, leads to zero dissipation. Therefore, the dissipation inequality (74) is satisfied automatically with the choice (75). The latter two evolution equations express the duality between internal variables: one internal variable is driven by another one and vice versa.

5 Microstructure model I

Having the evolution equations for internal variables in the non-dissipative case, we can derive a microstructure model. We keep a quadratic free energy dependence

$$\overline{W} = \frac{\rho c^2}{2}u_x^2 + A\varphi u_x + \frac{1}{2}B\varphi^2 + \frac{1}{2}C\varphi_x^2 + \frac{1}{2}D\psi^2, \tag{76}$$

where, as before, c is the elastic wave speed in the medium without microstructure, A, B, C, and D are material parameters characterizing microstructure influence.

Here we include for simplicity only the contribution of the second internal variable itself. In this case, the stress components are calculated as follows:

$$\sigma = \frac{\partial\overline{W}}{\partial u_x} = \rho c^2 u_x + A\varphi, \quad \eta = -\frac{\partial\overline{W}}{\partial\varphi_x} = -C\varphi_x, \quad \zeta = -\frac{\partial\overline{W}}{\partial\psi_x} = 0, \tag{77}$$

and the expression for the interactive internal force τ is not changed

$$\tau = -\frac{\partial \overline{W}}{\partial \varphi} = -Au_x - B\varphi. \tag{78}$$

The derivative of the free energy with respect to the dual internal variable gives

$$\xi = -\frac{\partial \overline{W}}{\partial \psi} = -D\psi. \tag{79}$$

Therefore, the evolution equation for the primary internal variable φ $(75)_1$ can be rewritten as

$$\dot{\varphi} = -RD\psi. \tag{80}$$

Time differentiation of Eq. (80) and the evolution equation for the dual internal variable $(75)_2$ lead to the hyperbolic equation for the primary internal variable

$$\ddot{\varphi} = R^2 D(\tau - \eta_x). \tag{81}$$

This allows us to represent the equations of motion both for macro- and microstructure in the form, which includes only the primary internal variable

$$\rho_0 u_{tt} = \rho_0 c^2 u_{xx} + A\varphi_x, \tag{82}$$

$$I\varphi_{tt} = C\varphi_{xx} - Au_x - B\varphi, \tag{83}$$

where $I = 1/(R^2 D)$. In terms of stresses introduced by Eq. (77), the same system of equations is represented as

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x}, \tag{84}$$

$$I\frac{\partial^2 \varphi}{\partial t^2} = -\frac{\partial \eta}{\partial x} + \tau. \tag{85}$$

It is worth to note that the same equations are derived in [27] based on different considerations.

As in the case of single internal variable, the constructed model describing the influence of microstructure by means of dual internal variables is non-dissipative. Equations of motion at both macro- and microlevels are hyperbolic. The hyperbolicity of the equation of motion at the microlevel is a direct consequence of the non-dissipativity requirement. The thermodynamic consistency of the model is provided, as before, due to the use of the canonical framework of continuum mechanics.

5.1 Single wave equation

To derive the single wave equation, we can determine the first space derivative of the internal variable from Eq. (83)

$$\varphi_x = -\frac{I}{B}\varphi_{ttx} + \frac{C}{B}\varphi_{xxx} - \frac{A}{B}u_{xx}, \tag{86}$$

and its third derivatives from Eq. (82)

$$\frac{A}{\rho_0}\varphi_{xxx} = \left(u_{tt} - c^2 u_{xx}\right)_{xx}, \quad \frac{A}{\rho_0}\varphi_{ttx} = \left(u_{tt} - c^2 u_{xx}\right)_{tt}. \tag{87}$$

Inserting the results into the balance of linear momentum (82), we obtain a more general equation [28]

$$u_{tt} = c^2 u_{xx} + \frac{C}{B}\left(u_{tt} - c^2 u_{xx}\right)_{xx} - \frac{I}{B}\left(u_{tt} - c^2 u_{xx}\right)_{tt} - \frac{A^2}{\rho B}u_{xx}. \tag{88}$$

Identifying $A^2 = c_A^2 B\rho, C = Ic_1^2, B = I/p^2$, we see that the obtained equation is nothing else but the general model of the dispersive wave propagation (19). The Maxwell-Rayleigh model of anomalous dispersion (17) corresponds to a special case of the latter equation with $C = 0$.

The dispersion analysis of the dispersive wave equation (88) is given in [16].

6 Microstructure model II

It may be instructive to construct another microstructure model based on distinct free energy dependence similarly to the case of a single internal variable (cf. Eq. (53)). Here we apply the free energy in the form

$$\overline{W} = \frac{\rho c^2}{2}u_x^2 + A'\varphi_x u_x + \frac{1}{2}B\varphi^2 + \frac{1}{2}C\varphi_x^2 + \frac{1}{2}D\psi^2. \tag{89}$$

Note that the coupling between macromotion and microstructure is described by the term $A'\varphi_x u_x$ while in the microstructure model I this coupling is different, described by the term $A\varphi u_x$ (cf. also Eqs. (45), (53)).

In this case, the stress components are calculated as follows:

$$\sigma = \frac{\partial \overline{W}}{\partial u_x} = \rho c^2 u_x + A'\varphi_x, \quad \eta = -\frac{\partial \overline{W}}{\partial \varphi_x} = -A'u_x - C\varphi_x, \quad \zeta = -\frac{\partial \overline{W}}{\partial \psi_x} = 0, \tag{90}$$

while the interactive internal force τ is reduced to

$$\tau = -\frac{\partial \overline{W}}{\partial \varphi} = -B\varphi. \tag{91}$$

The evolution equation for the primary internal variable φ is the same as previously

$$\ddot{\varphi} = R^2 D(\tau - \eta_x), \tag{92}$$

and the equations of motion both for macro- and microstructure include only the primary internal variable

$$\rho_0 u_{tt} = \rho_0 c^2 u_{xx} + A' \varphi_{xx}, \tag{93}$$

$$I\varphi_{tt} = C\varphi_{xx} + A' u_{xx} - B\varphi. \tag{94}$$

In terms of stresses introduced by Eq. (90), the same system of equations is still represented as previously by

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x}, \tag{95}$$

$$I\frac{\partial^2 \varphi}{\partial t^2} = -\frac{\partial \eta}{\partial x} + \tau. \tag{96}$$

6.1 Single wave equation

To obtain a single wave equation from Eqs. (93) and (94), we determine the second space derivative of the internal variable from Eq. (94)

$$\varphi_{xx} = -\frac{I}{B}\varphi_{ttxx} + \frac{C}{B}\varphi_{xxxx} + \frac{A'}{B}u_{xxxx}, \tag{97}$$

and its fourth derivatives from Eq. (82)

$$\frac{A'}{\rho_0}\varphi_{xxxx} = \left(u_{tt} - c^2 u_{xx}\right)_{xx}, \quad \frac{A'}{\rho_0}\varphi_{ttxx} = \left(u_{tt} - c^2 u_{xx}\right)_{tt}. \tag{98}$$

Inserting the results into the balance of linear momentum (93), we obtain the fourth-order equation

$$u_{tt} = c^2 u_{xx} + \frac{C}{B}\left(u_{tt} - c^2 u_{xx}\right)_{xx} - \frac{I}{B}\left(u_{tt} - c^2 u_{xx}\right)_{tt} + \frac{A'^2}{\rho B}u_{xxxx}. \tag{99}$$

The higher-order dispersive wave equations (88) and (99) generalize the dispersive wave equations derived in Sec. 3.1 and 3.2, respectively. These equations differ from each other only by the last term in the right hand side. However, this difference is essential, because the second-order space derivative in Eq. (88) exhibits the slowing down the velocity of propagation, whereas the fourth-order derivative in Eq. (99) does not. At the same time, derivatives of the wave operator in Eq. (88) cannot be rearranged, whereas it is possible in Eq. (99) due to the additional fourth-order space derivative.

7 Unification

The both approaches to derive the dispersive wave equations can be united by choosing the free energy function in the form

$$\overline{W} = \frac{\rho c^2}{2} u_x^2 + A u_x \varphi + A' u_x \varphi_x + \frac{1}{2} B \varphi^2 + \frac{1}{2} C \varphi_x^2 + \frac{1}{2} D \psi^2. \tag{100}$$

The corresponding stresses combine contributions from both cases mentioned above

$$\sigma = \frac{\partial \overline{W}}{\partial u_x} = \rho c^2 u_x + A\varphi + A' \varphi_x, \quad \eta = -\frac{\partial \overline{W}}{\partial \varphi_x} = -A' u_x - C\varphi_x, \tag{101}$$

and the interactive internal force is the same as in the first case

$$\tau = -\frac{\partial \overline{W}}{\partial \varphi} = -A u_x - B\varphi. \tag{102}$$

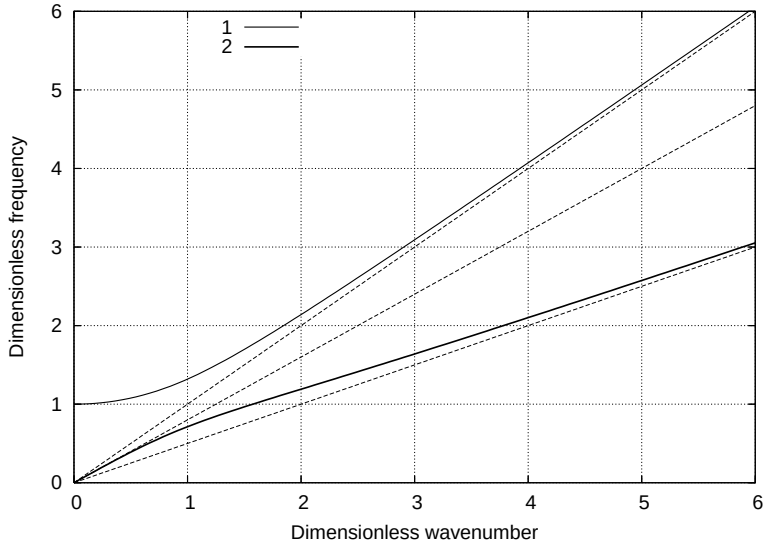

Fig. 6 Dispersion curves for the unified model (110) with $\gamma_1 = 0.5, \gamma_A = 0.6, \gamma' = 0.25$: 1 – optical branch, 2 – acoustical branch; dotted lines correspond to asymptotes to dispersion curves.

Accordingly, the balance of linear momentum results in

$$\rho_0 u_{tt} = \rho_0 c^2 u_{xx} + A\varphi_x + A' \varphi_{xx}, \tag{103}$$

and the evolution equation for the primary internal variable gives

$$I\varphi_{tt} = C\varphi_{xx} + A' u_{xx} - A u_x - B\varphi. \tag{104}$$

The first derivative of the internal variable can be determined from Eq. (104)

$$B\varphi_x = -I\varphi_{ttx} + C\varphi_{xxx} + A'u_{xxx} - Au_{xx}. \tag{105}$$

The third mixed derivative φ_{ttx} follows from Eq. (103)

$$A\varphi_{ttx} = \left(\rho_0 u_{tt} - \rho_0 c^2 u_{xx}\right)_{tt} - A'\varphi_{ttxx}. \tag{106}$$

The appeared fourth-order mixed derivative the internal variable is calculated by means Eq. (104)

$$I\varphi_{ttxx} = C\varphi_{xxxx} + A'u_{xxxx} - Au_{xxx} - B\varphi_{xx}, \tag{107}$$

and, in its turn, the fourth-order space derivative is determined again from Eq. (103)

$$A'\varphi_{xxxx} = \left(\rho_0 u_{tt} - \rho_0 c^2 u_{xx}\right)_{xx} - A\varphi_{xxx}. \tag{108}$$

Collecting all the results (105) - (108) and substituting them into Eq. (103) we arrive at the dispersive wave equation

$$u_{tt} = c^2 u_{xx} + \frac{C}{B}\left(u_{tt} - c^2 u_{xx}\right)_{xx} - \frac{I}{B}\left(u_{tt} - c^2 u_{xx}\right)_{tt} + \frac{A'^2}{\rho B}u_{xxxx} - \frac{A^2}{\rho B}u_{xx}, \tag{109}$$

that unifies and generalizes both approaches.

The dispersion equation for the unified model (109) reads

$$\eta^2 = \left(1 - \gamma_A^2\right)\xi^2 + \left(\eta^2 - \xi^2\right)\left(\eta^2 - \gamma_1^2\xi^2\right) - \gamma'^4\xi^4. \tag{110}$$

The dispersion curves for the unified model represented in Fig. 6 are similar to those in the Mindlin-type model [16]. However, they start do deviate from asymptotes with increasing of the parameter γ'.

8 Conclusions

The general 3D theory of microstructured materials in terms of internal variables is presented in [32]. Here the 1D setting is used in order to demonstrate explicitly how the structure of the governing equation depends on constitutive free energy function.

The earlier analysis of the Mindlin-type equation [13] has explicitly demonstrated the need to involve micro-elastic and micro-inertial characteristics into the physically acceptable models. The consistent analysis of gradient-type theories is given in [5,31]. Here we demonstrated that several well-known models (see [9], for example) together with generalization of the Mindlin-type models [14,16] can be derived by using internal variables. The adopted phenomenological approach is based on the material formulation of continuum mechanics [23] and provides the full thermodynamic consistency due to the dual internal variables concept [22]. So, Eq. (109) is rather general and well-grounded. Its typical feature is that besides the fourth-order derivatives it includes also the changes in the velocity of wave propagation at macroscale due to the coupling - a slowing down effect. This effect is also demonstrated by direct numerical computations [16,24] for regular and random microstructure distribution. The fourth-order dispersive terms in Eq.

(109) are explicitly related to various terms in free energy function and reflect the effects of micro-elasticity and microinertia. The coupling between macromotion and microstructure deformation is taken into account by last two terms of Eq. (109).

Equation (109), first time derived in this paper, could be used as a basis for further generalizations. First, even more higher-order derivatives can appear in the model (cf. [13,31]). Second, using perturbation technique, Eq. (109) can be reduced to a "hierarchical equation" which includes one wave operator for the macromotion and another for the microstructure [16]. Third, the generalization to multiple microstructures is also possible [33]. At last, introduction of nonlinear terms, i.e. cubic terms in a free energy function, is described elsewhere for simpler models [14,34].

Acknowledgements Support of the Estonian Science Foundation is gratefully acknowledged.

References

1. Brillouin, L.: Wave propagation and group velocity. Academic Press, New York (1960)
2. Maugin, G.A.: On some generalizations of Boussinesq and KdV systems. Proc. Estonian Acad. Sci. Phys. Mat. 44, 40–55 (1995)
3. Santosa, F. and Symes, W.W.: A dispersive effective medium for wave propagation in periodic composites. SIAM J. Appl. Math. 51, 984–1005 (1991)
4. Fish, J., Chen, W. and Nagai, G.: Non-local dispersive model for wave propagation in heterogeneous media: one-dimensional case. Int. J. Numer. Meth. Engng. 54, 331–346 (2002)
5. Askes, H. and Metrikine, A.V.: One-dimensional dynamically consistent gradient elasticity models derived from a discrete microstructure Part 1: Generic formulation. Eur. J. Mech. A/Solids 21, 555–572 (2002)
6. Erofeyev, V.I.: Wave Processes in Solids with Microstructure. World Scientific, Singapore (2003)
7. Love, A.E.H.: Mathematical Theory of Elasticity. Dover, New York (1944)
8. Graff, K.F.: Wave Motion in Elastic Solids. Clarendon Press, Oxford (1975)
9. Maugin, G.A.: Nonlinear Waves in Elastic Crystals. Oxford University Press (1999)
10. Wang, Z.-P. and Sun, C.T.: Modeling micro-inertia in heterogeneous materials under dynamic loading. Wave Motion 36, 473–485 (2002)
11. Wang, L.-L.: Foundations of Stress Waves. Elsevier, Amsterdam (2007)
12. Metrikine, A.V.: On causality of the gradient elasticity models. J. Sound Vibr. 297, 727–742 (2006)
13. Papargyri-Beskou, S., Polyzos, D. and Beskos, D.E.: Wave dispersion in gradient elastic solids and structures: A unified treatment. Int. J. Solids Struct. 46, 3751–3759 (2009)
14. Engelbrecht, J. and Pastrone, F.: Waves in microstructured solids with nonlinearities in microscale. Proc. Estonian Acad. Sci. Phys. Mat. 52, 12–20 (2003)
15. Mindlin, R.D.: Microstructure in linear elasticity. Arch. Rat. Mech. Anal. 16, 51–78 (1964)
16. Engelbrecht, J., Berezovski, A., Pastrone, F. and Braun, M.: Waves in microstructured materials and dispersion. Phil. Mag. 85, 4127–4141 (2005)
17. Horstemeyer, M.F. and Bammann, D.J.: Historical review of internal state variable theory for inelasticity. Int. J. Plasticity 26, 1310–1334 (2010)
18. Coleman, B.D. and Gurtin, M.E.: Thermodynamics with internal state variables. J. Chem. Phys. 47, 597–613 (1967)
19. Maugin, G.A. and Muschik, W.: Thermodynamics with internal variables. J. Non-Equilib. Thermodyn. 19, 217–249 (1994)
20. Maugin, G.A.: Internal variables and dissipative structures. J. Non-Equilib. Thermodyn. 15, 173–192 (1990)
21. Maugin, G.A.: On the thermomechanics of continuous media with diffusion and/or weak nonlocality. Arch. Appl. Mech. 75, 723–738 (2006)

22. Ván, P., Berezovski, A. and Engelbrecht, J.: Internal variables and dynamic degrees of freedom. J. Non-Equilib. Thermodyn. 33, 235–254 (2008)
23. Maugin, G.A.: Material Inhomogeneities in Elasticity. Chapman and Hall, London (1993)
24. Berezovski, A., Engelbrecht, J. and Maugin, G.A.: Numerical simulation of waves and fronts in inhomogeneous solids. World Scientific, Singapore (2008)
25. Askes, H. and Aifantis, E.C.: Gradient elasticity theories in statics and dynamics - a unification of approaches. Int. J. Fract. 139, 297–304 (2006)
26. Askes, H., Metrikine, A. V., Pichugin, A. V. and Bennett, T.: Four simplified gradient elasticity models for the simulation of dispersive wave propagation. Phil. Mag, 88/28, 3415–3443 (2008)
27. Engelbrecht, J., Cermelli, P. and Pastrone, F.: Wave hierarchy in microstructured solids. In: Maugin, G.A. (ed.) Geometry, Continua and Microstructure. pp. 99–111. Hermann Publ., Paris (1999)
28. Berezovski, A., Engelbrecht, J. and Maugin, G.A.: One-dimensional microstructure dynamics, In: Ganghoffer, J.-F.; Pastrone, F. (eds.) Mechanics of microstructured solids: cellular materials, fibre reinforced solids and soft tissues. pp. 21 - 28, Springer, Series: Lecture Notes in Applied and Computational Mechanics, Vol. 46 (2009)
29. Rice, J.R.: Inelastic constitutive relations for solids: an internal-variable theory and its application to metal plasticity. J. Mech. Phys. Solids 19, 433–455 (1971)
30. Muschik, W., *Aspects of Non-Equilibrium Thermodynamics*, World Scientific, Singapore, 1990.
31. Askes, H. and Metrikine, A.V.: Higher-order continua derived from discrete media: continualisation aspects and boundary conditions. Int. J. Solids Struct. 42, 187–202 (2005)
32. Berezovski, A., Engelbrecht, J. and Maugin, G.A.: Generalized thermomechanics with internal variables. Arch. Appl. Mech. 81, 229–240 (2011)
33. Berezovski, A., Engelbrecht, J. and Peets, T.: Multiscale modelling of microstructured solids. Mech. Res. Commun. 37, 531–534 (2010)
34. Pastrone, F., Cermelli, P. and Porubov, A.: Nonlinear waves in 1-D solids with microstructure. Mater. Phys. Mech. 7, 9–16 (2004)