DEFORMATION WAVES IN MICROSTRUCTURED SOLIDS AND DISPERSION

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The governing equations for a 1D case of the Mindlin model for microstructured materials are derived and analysed. These equations exhibit hierarchical properties assigning the wave operators to internal scales. The dispersion of waves is characterized by higher-order derivatives including also the mixed derivatives with respect to coordinate and time.

1. Introduction

Materials used in contemporary high technology are characterized often by their complex structure in order to satisfy many requirements in practice. This concerns polycrystalline solids, ceramic composites, alloys, functionally graded materials, granular materials, etc. Often one should also account for the damage effects when materials have microcracks. All that shows the existence of intrinsic space-scales in matter, like the lattice period, the size of a crystallite or a grain, and the distance between microcracks. This scale-dependence should also be taken into account in governing equa-

tions. The classical theory of the continuous media is built up using the assumption of the smoothness of continua. The microstructured materials, i.e. materials with irregularities have one or more internal scales and their complex dynamic behaviour cannot be explained by the classical theories.

Within the theories of continua the problems of irregularities of media have been foreseen long time ago by the Cosserats and Voigt, and more recently by Mindlin [1] and Eringen [2]. The elegant mathematical theories of continua with voids or with vector microstructure, or continua with spins of Cosserat continuum or micromorphic continuum, etc. have been elaborated since then, see overviews by Capriz [3] and Eringen [4]. The straight-forward modelling of microstructured solids leads to assigning all the physical properties to every volume element in a solid introducing so the dependence on material coordinates. This leads to an extremely complex system. Another probably much more effective way is to separate macro- and microstructure in continua. Then the conservation laws for both structures should be separately formulated (Mindlin [1], Eringen [2, 4]), or the microstructural quantities are separately taken into account in one set of conservation laws (Maugin [5]). Here we proceed according to the ideas of separating macro- and microstructures.

2. Governing equations

The governing equations are derived following Mindlin [1] who has interpreted the microstructure "as a molecule of a polymer, a crystallite of a polycrystal or a grain of a granular material". This microelement is taken as a deformable cell. Note that if this cell is rigid, then the Cosserat model follows. The displacement u of a material particle in terms of macrostructure is defined by its components $u_i \equiv x_i - X_i$, where $x_i, X_i\,(i = 1, 2, 3)$ are the components of the spatial and material position vectors, respectively. Within each material volume (particle) there is a microvolume and the microdisplacement u' is defined by its components $u'_i \equiv x'_i - X'_i$, where the origin of the coordinates x'_i moves with the displacement u. The displacement gradient is assumed to be small. This leads to the *basic assumption* of Mindlin [1] that "the microdisplacement can be expressed as a sum of products of specified functions of x'_i and arbitrary functions of the x_i and t". The first approximation is then

$$u'_j = x'_k \varphi_{kj}(x_i, t). \tag{1}$$

The *microdeformation* is

$$\frac{\partial u'_j}{\partial x'_i} = \partial'_i u'_j = \varphi_{ij}. \tag{2}$$

Further we consider the simplest 1D case and drop the indices i, j dealing with u and φ only. The indices t and x used in the sequel denote differentiation.

The fundamental balance laws for microstructured materials can be formulated separately for macroscopic and microscopic scales using the Lagrangian and the Euler-Lagrange equations (see Engelbrecht et al. [6]). For the basic *single-scaled* model we take the potential energy W in the form of a quadratic function

$$W = \frac{1}{2}\left(\alpha u_x^2 + 2A\varphi u_x + B\varphi^2 + C\varphi_x^2\right) \tag{3}$$

with α, A, B, C denoting material constants. Then the governing equations take the form

$$\rho u_{tt} = \alpha u_{xx} + A\varphi_x, \tag{4}$$

$$I\varphi_{tt} = C\varphi_{xx} - Au_x - B\varphi, \tag{5}$$

where ρ is the density and I the microinertia.

In the two-scale situation (scale within the scale), it is assumed that every deformable cell of the microstructure includes new deformable cells at a smaller scale. The displacements at the different scales are then (cf. (1)):

$$u_j = u_j(x_i, t), \quad u'_j = x'_k\ \varphi_{kj}\ (x_i, t), \quad u''_j = x''_k \psi_{kj}(x'_i, t), \tag{6}$$

respectively, where x'_k, x''_k correspond to the local coordinate within respective cells. As we are interested in motion on the macrolevel, it is assumed that

$$u''_j = x''_k \psi_{kj}(x_i, t). \tag{7}$$

Then we get

$$\frac{\partial u'_j}{\partial x'_i} = \varphi_{ij}, \quad \frac{\partial u''_j}{\partial x''_i} = \psi_{ij}. \tag{8}$$

As before, we drop the indices i, j, k for the 1D case. Now the potential energy function is taken

$$W = \frac{1}{2}(\alpha u_x^2 + 2A_1 u_x \varphi + B_1 \varphi^2 + C_1 \varphi_x^2 + 2A_2 u_x \psi + B_2 \psi^2 + C_2 \psi_x^2), \quad (9)$$

where α and A_i, B_i, C_i $(i = 1, 2)$ denote material constants. The governing equations are then the following:

$$\rho u_{tt} = \alpha u_{xx} + A_1 \varphi_x, \quad (10)$$

$$I_1 \varphi_{tt} = C_1 \varphi_{xx} - A_1 u_x - B_1 \varphi + A_2 \psi_x, \quad (11)$$

$$I_2 \psi_{tt} = C_2 \psi_{xx} - A_2 \varphi_x - B_2 \psi, \quad (12)$$

generalizing the system (4), (5). Here I_i $(i = 1, 2)$ denote microinertia of corresponding microstructures.

3. Hierarchies of waves

Whitham [7] has described certain complicated wave systems where a scale parameter δ plays a crucial role. Depending on its limit values, $\delta \to \infty$ or $\delta \to 0$, one or another wave operator governs the process asymptotically. Thus, the full system includes a hierarchy of waves with certain stability conditions [7]. Here we show that waves in microstructured materials exhibit the hierarchical behaviour governed by a parameter which is the ratio of the characteristic scale of a microstructure and the wave length of the excitation.

First, the *single scale*. Let the scale of the microstructure be l and the excitation characterised by its amplitude U_0 and wavelength L. Then we can introduce the following dimensionless variables and parameters

$$U = u/U_0, \quad X = x/L, \quad T = c_0 t/L, \quad \delta = l^2/L^2, \quad \epsilon = U_0/L, \quad (13)$$

where $c_0^2 = \alpha/\rho$. We also suppose that $I = \rho l^2 I^*$, $C = l^2 C^*$, where I^* is dimensionless and C^* has the dimension of stress. The difference of densities is embedded in I^*. By means of series representation and the slaving principle (Christiansen et al. [8]) we get finally

$$U_{TT} = (1 - k)\, U_{XX} + \delta I^* m\, (U_{TT} - n U_{XX})_{XX}\,, \quad (14)$$

where $k = c_A^2/c_0^2$, $m = A^2/B^2$, $n = c_1^2/c_0^2$, $c_1^2 = C/I$, $c_A^2 = A^2/\rho B$.

In case of *multiple scales* we have to introduce

$$\delta_1 = l_1^2/L^2, \quad \delta_2 = l_2^2/L^2, \quad (15)$$

where l_1 and l_2 are the scales of microstructures φ and ψ, respectively. The coefficients I_i, C_i, A_i are scaled as above. The same technique as for the single scale yields the governing equation

$$U_{TT} = (1-k_1)\,U_{XX} + \delta_1 I_1^* m_1 \left(U_{TT} - n_1 U_{XX}\right)_{XX} + \\ + \delta_2 I_2^2 m_2 p_2 \left(U_{XX}\right)_{XX} - \delta_2^2 I_2^* m_2 \left(U_{TT} - n_2 U_{XX}\right)_{XXXX} \quad (16)$$

with $k_1 = A_1^2/\alpha B_1$, $m_1 = A_1^2/B_1^2$, $n_1 = C_1^*/\alpha I_1^*$, $m_2 = A_1^2(A_2^*)^2/B_1^2 B_2^2$, $p_2 = B_2/\alpha I_2^*$, $n_2 = C_2^*/\alpha I_2^*$.

4. Discussion

Equations (14) and (16) reflect clearly the hierarchical character of wave propagation in microstructured materials as indicated by Whitham [7] within general wave theory. Indeed, in case of eq (14):

(i) if δ is small, then the last two terms are negligible, if δ is large, then the first two terms are negligible and the properties are governed by properties of microstructure;
(ii) the wave speed in the compound material is affected by the microstructure (1 *versus* $k = c_A^2/c_o^2$) and only $A = 0$ (no coupling) excludes this dependence;
(iii) the influence of the microstructure is, as expected, characterized by dispersive terms; however, contrary to the idealized models, the double dispersion (different terms U_{TTXX} and U_{XXXX}) is of importance.

The multi-scale model (16) actually prolongs the hierarchical properties of the single-scale model (14). Indeed, the wave operators macro *versus* micro 1 and micro 1 *versus* micro 2 are related by similar sign convention, and the wave velocity in microstructure 1 is affected by the properties of microstructure 2 in a similar way as the wave velocity in macrostructure is affected by properties of microstructure 1. It is seen that higher-order dispersive terms U_{XXXX}, $U_{XXXXXX}, \ldots$ coincide with those derived from the lattice theory [9] but again mixed derivatives U_{TTXX}, $U_{TTXXXX}, \ldots$ reflecting the role of microinertias also enter the equations.

The dispersion analysis [6] shows explicitly how the dispersion curves tend asymptotically from one velocity (macrostructure) to another (microstructure). Although the eqs (14) and (16) are asymptotical, the corresponding dispersion curves are close to the exact ones derived from the system (4), (5) and (10), (11), (12), respectively.

In case of nonlinear waves, the potential energy W should include also the cubic terms. Such problems are discussed in [10, 11]. The balance of nonlinear and dispersive effects in microstructured solids may lead to the emergence of solitary waves. The nonlinearity at the macroscale together with dispersive effects results in a symmetric solitary wave while the nonlinearity at the microscale [10] causes the emergence of an asymmetric solitary wave [11]. The prospects of using special characteristics of waves in microstructured materials briefly described above for Nondestructive Testing (NDT) of materials are now extensively studied (cf also [11]).

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