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On modelling dispersion in microstructured solids

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Abstract

The Mindlin-type model is used for describing the longitudinal deformation waves in microstructured solids. A simplified hierarchical model is derived in one-dimensional setting which is a two-wave equation. In addition, the evolution equations (one-wave equations) are derived for both the full and simplified models. It is shown that the simplified model as well as evolution equations grasp main effects of dispersion in a wide range of physical parameters.

Keywords: Dispersion; Microstructure; Hierarchy of waves

1. Introduction

In contemporary materials science and structural mechanics much attention is given to microstructured materials possessing internal scales. Microstructured materials like alloys, crystallites, ceramics, functionally graded materials, etc have gained wide application in modern technology because combining the mechanical properties of different constituencies as in functionally graded materials or composites yields better (optimal) properties of solids. Very often they are used in severe loading conditions including impact, which means generation of stress/deformation waves. The modelling of wave propagation in such materials should be able to account for various scales of microstructure. The scale dependence involves dispersive effects and if in addition the material behaves nonlinearly then dispersive and nonlinear effects may be balanced. As widely known, in this case solitary waves may emerge as a result of such a balance.

Clearly the classical theory of continuous media is not able to describe the influence of microstructure which is needed for explain dispersive and dissipative effects. There are many studies in this field, starting from the papers of Mindlin [1] and Eringen [2] several decades ago. Now we have a solid theoretical background, see for example [3,4], but another problem has arisen: the governing equations tend to be rather complicated and the number of material parameters needed to describe the stress field, is rather high. Therefore there is an

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urgent need to find simplified governing equations but the physical effects should still be described with the needed accuracy.

The problem is not only in the mathematical complexity of governing equations but also in the number of waves. If in the linear theory, for example, longitudinal and shear waves can be easily separated then in the nonlinear theory the coupling can affect both waves considerably. In a general case of a complicated system of equations the main question is to understand to which wave which physical effects are related both quantitatively and qualitatively.

One of the possibilities to overcome such difficulties in contemporary mathematical physics is to introduce the notion of evolution equations governing just one single wave. Physically it means the separation (if possible) of a multi-wave process into separate waves. The waves are then governed by the so-called evolution equations every one of which describe the distortion of a single wave along a properly chosen characteristics (ray).

In this paper the attention is focused to the analysis of dispersion described by Mindlin-type models [1]. Engelbrecht et al. [5,6] have derived the one-dimensional mathematical model for longitudinal waves in microstructured materials. Based on the separation of macro- and microstructure of a material, this model is characterised by a clear physical structure of the governing equation. The analysis of the full-dispersion relation of this model compared with others is briefly presented in [5] (see also references therein). Our question here is the following: if we use asymptotic methods to simplify the model then can we describe still the physics with acceptable accuracy? We shall use two asymptotic approaches: (i) the slaving principle [7] in order to get a hierarchical asymptotic Whitham-type model from the basic one and (ii) the perturbative reduction method [8,9] in order to get evolution equations. Although nonlinearity is an important factor, here we deal only with dispersive effects and nonlinear waves will be analysed in our further publications.

The paper is organized as follows: the basic model following [5,6] is presented in Section 2. In Section 3, the asymptotic models are derived following two approaches resulting in a hierarchical simplified equation and in evolution equations. Section 4 is devoted to the dispersion analysis of the basic and the simplified models. In Section 5, final remarks are presented. It has been shown that the simplified model as well as evolution equations grasp main effects of dispersion in a wide range of physical parameters.

2. Basic model

The basic model is that of Mindlin [1] and we follow the presentation of that in [5,6]. The main idea is to distinguish between macro- and microdisplacements $u_i(x_i,t)$ and $u'_j(x'_i,t)$, respectively. Assuming that microdisplacement is defined in coordinates x'_k moving with microvolume, we define

$$u'_j = x'_k \varphi_{kj}(x_i,t), \tag{1}$$

where φ_{kj} is an arbitrary function. It is clear that actually it is microdeformation while

$$\partial u'_j/\partial x'_i = \partial'_i u'_j = \varphi_{ij}. \tag{2}$$

Further we consider the simplest 1D case and drop the indices i and j.

Now the fundamental balance laws can be formulated separately for macroscopic and microscopic scales. Introducing the Lagrangian $L = K - W$, formed from the kinetic and potential energies

$$\begin{cases} K = \frac{1}{2}\rho u_t^2 + \frac{1}{2}I\varphi_t^2 \\ W = W(u_x, \varphi, \varphi_x), \end{cases} \tag{3}$$

where ρ and I denote the macroscopic density and the microinertia, respectively, we can derive the corresponding Euler–Lagrange equations:

$$\begin{cases} \left(\frac{\partial L}{\partial u_t}\right)_t + \left(\frac{\partial L}{\partial u_x}\right)_x - \left(\frac{\partial L}{\partial u}\right) = 0 \\ \left(\frac{\partial L}{\partial \varphi_t}\right)_t + \left(\frac{\partial L}{\partial \varphi_x}\right)_x - \left(\frac{\partial L}{\partial \varphi}\right) = 0. \end{cases} \tag{4}$$

Here and further, the indices x and t denote differentiation.

The partial derivatives

$$\sigma = \partial W/\partial u_x, \quad \eta = \partial W/\partial \varphi_x, \quad F = \partial W/\partial \varphi \tag{5}$$

are recognized as the macrostress, the microstress and the interactive force, respectively.

The equations of motion are now

$$\rho u_{tt} = \sigma_x, \quad I\varphi_{tt} = \eta_x - F. \tag{6}$$

The simplest potential energy function describing the influence of a microstructure is a quadratic function

$$W = \frac{1}{2}au_x^2 + A\varphi u_x + \frac{1}{2}B\varphi^2 + \frac{1}{2}C\varphi_x^2, \tag{7}$$

where a, A, B, C denote material constants. Introducing Eq. (7) into Eq. (5) we get finally

$$\begin{cases} \rho u_{tt} = au_{xx} + A\varphi_x \\ I\varphi_{tt} = C\varphi_{xx} - Au_x - B\varphi. \end{cases} \tag{8}$$

This is the governing system of two second-order equations that can also be represented in the form of one fourth-order equation

$$u_{tt} = (c_0^2 - c_A^2)u_{xx} - p^2(u_{tt} - c_0^2 u_{xx})_{tt} + p^2c_1^2(u_{tt} - c_0^2 u_{xx})_{xx}, \tag{9}$$

where material parameters

$$c_0^2 = a/\rho, \quad c_1^2 = C/I, \quad c_A^2 = A^2/\rho B, \quad p^2 = I/B \tag{10}$$

are introduced. The parameters c_0, c_1, c_A are velocities while p is a time parameter. This is the basic linear equation governing 1D longitudinal waves in microstructured solids.

3. Approximations

3.1. Slaving principle

This idea (see [7]) is used in [5,9] for deriving a hierarchical asymptotic model starting from Eq. (9). It is supposed that the inherent length-scale l is small compared with the wavelength L of the excitation. The following dimensionless variables and parameters are introduced

$$U = u/U_0, \quad X = x/L, \quad T = c_0t/L, \quad \delta = (l/L)^2, \quad \varepsilon = U_0/L, \tag{11}$$

where U_0 is the amplitude of the excitation. In addition, it is assumed that $I = \rho l^2 I^*$ and $C = l^2 C^*$, where I^* is dimensionless and C^* has the dimension of stress.

Next, the system Eq. (8) is rewritten in its dimensionless form and the slaving principle [7] is applied. It is supposed that

$$\varphi = \varphi_0 + \delta\varphi_1 + \delta^2\varphi_2 + \cdots. \tag{12}$$

The dimensionless form of Eq. (8b) yields

$$\varphi = -\varepsilon\frac{A}{B}U_X - \frac{\delta}{B}(aI^*\varphi_{TT} - C^*\varphi_{XX}) \tag{13}$$

from which the successive terms

$$\varphi_0 = -\varepsilon\frac{A}{B}U_X, \quad \varphi_1 = \varepsilon\frac{A}{B^2}(aI^*U_{XTT} - C^*U_{XXX}), \ldots \tag{14}$$

of the expansion Eq. (12) are obtained. Inserting them into Eq. (8a) in its dimensionless form, we finally get

$$U_{TT} = \left(1 - \frac{c_A^2}{c_0^2}\right)U_{XX} + \frac{c_A^2}{c_B^2}\left(U_{TT} - \frac{c_1^2}{c_0^2}U_{XX}\right)_{XX}, \tag{15}$$

where $c_B^2 = L^2/p^2 = BL^2/I$. Note that c_B involves the scales L and l and c_A includes the interaction effects through the parameter A. Eq. (15) is valid up to O(δ) because higher order terms are neglected. In addition, in general $\varepsilon \gg \delta^2$.

Now it is possible to restore the dimensions in order to compare the result with Eq. (9). Eq. (15) yields

$$u_{tt} = (c_0^2 - c_A^2)u_{xx} + p^2 c_A^2 (u_{tt} - c_1^2 u_{xx})_{xx}. \tag{16}$$

This is an example of the Whitham-type [10] hierarchical equation.

The dimensionless form of the basic linear Eq. (9) is

$$U_{TT} = \left(1 - \frac{c_A^2}{c_0^2}\right) U_{XX} - \frac{c_0^2}{c_A^2}\delta\beta U_{TTTT} + \left(\frac{c_0^2}{c_A^2} + \frac{c_1^2}{c_A^2}\right)\delta\beta U_{XXTT} - \frac{c_1^2}{c_A^2}\delta\beta U_{XXXX}, \tag{17}$$

where $\delta\beta = c_A^2/c_B^2$.

3.2. Evolution equations

Another idea to simplify the model is to use instead of the two-wave equation (16) an evolution equation that describes just one wave [8,9].

Here we follow [9] and apply the asymptotic (reductive perturbation) method. We can represent Eq. (15) in the matrix form

$$I\frac{\partial \mathbf{V}}{\partial T} + \widetilde{A}\frac{\partial \mathbf{V}}{\partial X} + \widetilde{B}\frac{\partial^3 \mathbf{V}}{\partial T \partial X^2} + \widetilde{C}\frac{\partial^3 \mathbf{V}}{\partial X^3} = 0, \tag{18}$$

where

$$\mathbf{V} = \begin{pmatrix} \partial U/\partial T \\ \partial U/\partial X \end{pmatrix} \tag{19}$$

and $I, \widetilde{A}, \widetilde{B}$ and $\widetilde{C}$ are following matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \widetilde{A} = \begin{pmatrix} 0 & -(1-n^2) \\ -1 & 0 \end{pmatrix},$$
$$\widetilde{B} = \begin{pmatrix} -\delta\beta & 0 \\ 0 & -1 \end{pmatrix}, \quad \widetilde{C} = \begin{pmatrix} 0 & \delta\beta m^2 \\ 1 & 0 \end{pmatrix},$$

where

$$n^2 = c_A^2/c_0^2 \neq 1, \qquad m^2 = c_1^2/c_0^2. \tag{20}$$

It is possible to develop vector $\mathbf{V}$ into the power series in a small parameter

$$\mathbf{V} = \mathbf{V}_0 + \varepsilon\mathbf{V}_1 + \varepsilon^2\mathbf{V}_2 + \cdots = \sum_{i=0} \varepsilon^i \mathbf{V}_i. \tag{21}$$

The space-space transformation is used:

$$\begin{cases} \xi = cT - X \\ \tau = \varepsilon X, \end{cases} \tag{22}$$

i.e.

$$\{X, T\} \rightarrow \{\xi, \tau\}, \tag{23}$$

where $c = \left(1 - \frac{A^2}{aB}\right)^{1/2} = \left(1 - \frac{c_A^2}{c_0^2}\right)^{1/2}$.

According to the asymptotic method [9] we get the sequence of equations of various powers in ε. Assuming that ε and δ are small parameters of the same order, we get finally the approximate linear evolution equation in the form

$$\frac{\partial \alpha}{\partial \tau} + \frac{\delta(\gamma - \beta c^2)}{2\varepsilon c^2} \frac{\partial^3 \alpha}{\partial \xi^3} = 0, \tag{24}$$

where $\beta = \frac{A^2 I^*}{B^2}, \gamma = \frac{A^2 C^*}{aB^2}$ and $\alpha = \frac{\partial U}{\partial T} = -c\frac{\partial U}{\partial X}$ is the unknown amplitude factor.

Similarly, applying the asymptotic method [9] for the basic linear Eq. (17) we first represent it in the matrix form

$$I\frac{\partial \mathbf{V}}{\partial T} + \widetilde{A}\frac{\partial \mathbf{V}}{\partial X} + \widetilde{D}\frac{\partial^3 \mathbf{V}}{\partial T^3} + \widetilde{E}\frac{\partial^3 \mathbf{V}}{\partial T \partial X^2} + \widetilde{F}\frac{\partial^3 \mathbf{V}}{\partial X^3} = 0, \tag{25}$$

where $I, \widetilde{A}, \widetilde{D}, \widetilde{E}$ and $\widetilde{F}$ are following matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \widetilde{A} = \begin{pmatrix} 0 & -(1-n^2) \\ -1 & 0 \end{pmatrix},$$

$$\widetilde{D} = \begin{pmatrix} \frac{1}{n^2}\delta\beta & 0 \\ 0 & 0 \end{pmatrix}, \quad \widetilde{E} = \begin{pmatrix} -(\frac{1}{n^2}\delta\beta + n_1^2\delta\beta) & 0 \\ 0 & -1 \end{pmatrix},$$

$$\widetilde{F} = \begin{pmatrix} 0 & n_1^2\delta\beta \\ 1 & 0 \end{pmatrix},$$

where

$$n_1^2 = c_1^2/c_A^2, \tag{26}$$

and write the evolution equation in the form

$$\frac{\partial \alpha}{\partial \tau} + \frac{\delta(\gamma - \beta c^2)}{2\varepsilon c^2} \frac{\partial^3 \alpha}{\partial \xi^3} = 0. \tag{27}$$

This means, that the approximate Eq. (15) and the basic Eq. (17) yield the evolution equations in the same form, see Eqs. (24) and (27). Consequently, using the idea of evolution equations there is no difference whether we begin the derivation from the basic Eq. (17) with the addition term U_{TTTT} or from the approximate Eq. (15) with terms U_{XXTT} and U_{XXXX}. However, note that the parameters of Eqs. (15) and (16) are different.

The character of dispersion in the case of microstructured materials is analysed in [5] on the basis of the approximate Eq. (15). It has been shown that both of the effects – inertia of the microstructure (described by term U_{TTXX}) and elasticity of the microstructure (described by term U_{XXXX}) have influence on dispersive relations and corresponding dispersion curves. If only inertia of the microstructure (term U_{TTXX}) is taken into account then the dispersion curve is concave, if only elasticity of the microstructure (term U_{XXXX}) is taken into account then the dispersion curve is convex. With both terms (double dispersion) the curve tends from one asymptote to another.

In the case of the evolution equation these two effects are described by a single term (term $\alpha_{\xi\xi\xi}$) but the sign of this term (the sign of its coefficient) depends on the ratio of the double dispersion effects.

It is possible to conclude that in case of $\gamma > \beta c^2$ (elastic effects prevailing) the dispersion curve is convex and in case of $\gamma < \beta c^2$ (inertial effects prevailing) the dispersion curve is concave. So the evolution equation keeps the main characteristics of the process. In case of $\gamma - \beta c^2 = 0$ there is no microstructure and the dispersion curve is linear, as expected.

4. Dispersion analysis

4.1. Dispersion relations

Internal scales of microstructured solids lead to dispersive effects. This is also quite clear from the governing equations derived in previous sections. The presence of higher-order derivatives in the governing equations indicates dispersion.

In order to derive dispersion relations, we assume the solution in the form of a wave

$$u(x,t) = \hat{u}\exp[\mathrm{i}(kx - \omega t)], \tag{28}$$

with wave number k, frequency ω and amplitude û.

Introducing Eq. (28) into Eq. (9) the dispersion relation

$$\omega^2 = (c_0^2 - c_A^2)k^2 + p^2(\omega^2 - c_0^2k^2)(\omega^2 - c_1^2k^2) \tag{29}$$

is obtained.The parameters involved are a time constant p and three characteristic velocities c_0,c_1,c_A. Instead of c_A the velocity $c_R^2 = c_0^2 - c_A^2$ could be introduced as a parameter, since it has an obvious meaning for the given wave process. Waves of very low frequencies ($\omega \ll p^{-1}$) propagate at the velocity c_R. The auxiliary velocity c_A does not occur explicitly as a limit velocity. The phase speed of the wave is defined as $c_p = \omega/k$ and can be obtained directly from the dispersion relation.

In order to reduce the number of independent variables we normalise the wave number, the frequency and the relative propagation speeds defining

$$\kappa = pc_0k, \quad \eta = p\omega, \quad n = c_A/c_0, \quad m = c_1/c_0. \tag{30}$$

Using these new quantities the full-dispersion relation (29) assumes the form

$$\eta^2 = (1 - n^2)\kappa^2 + (\eta^2 - \kappa^2)(\eta^2 - m^2\kappa^2). \tag{31}$$

The dimensionless phase speed is defined as $\gamma_p = c_p/c_0 = \eta/\kappa$.

For convenience we also use the parameter $c = c_R/c_0 = (1 - n^2)^{1/2}$ (see Eq. (22)).

In the same way, the approximate differential equation (16) yields the dispersion relation

$$\omega^2 = (c_0^2 - c_A^2) - p^2c_A^2(\omega^2 - c_1^2k^2)k^2. \tag{32}$$

Introducing Eq. (30) into Eq. (32) we obtain

$$\eta^2 = (1 - n^2)\kappa^2 - n^2(\eta^2 - m^2\kappa^2)\kappa^2. \tag{33}$$

4.2. The range of parameters

The numerical simulation is done with the dimensionless Eqs. (31) and (33) and with the dimensionless parameters n and m. Since $c^2 = 1 - n^2$ then $n < 1$, which makes physically sense because the velocity c_0 is interpreted as the maximum possible velocity. Therefore also $m < 1$.

We also assume that $n \neq m \neq 0$. If $n = 0$ then also $A = 0$ and then the governing Eq. (8) will have the form where there is no interaction between the macro- and the microstructure.

Therefore we will consider the parameters in the following ranges

$$0 < n < 1, \qquad 0 < m < 1. \tag{34}$$

4.3. The results

The characteristic dispersion curves are shown in Fig. 1 from which the following can be concluded. The full-dispersion relation (31), which is represented by the continuous lines, represents two branches which in general are distinct. The upper, or 'optical' branch starts at $\eta = 1$ with zero slope and in the short wave limit the branch asymptotically approaches to the line $\eta = \kappa$. Lower, or 'acoustical', branch starts at the origin with a slope $\eta = c\kappa$ and in the short wave limit the branch approaches to the asymptotic line $\eta = m\kappa$. Here the dotted lines show asymptotic values.

The approximate dispersion relation (33), which is represented by the dashed line, provides an approximation of the acoustical branch only.

It is clear that the dispersion relations (31) and (33) differ and our intention is to analyse the ranges of parameters where the results coincide. This is dictated by the values of parameters n and m. Fig. 2 illustrates the ranges of the parameters where the values obtained from the both relations agree within 5% error (the area between the dashed lines) and within 10% error (the area between the continuous lines) at the point $\kappa = 1.5$.

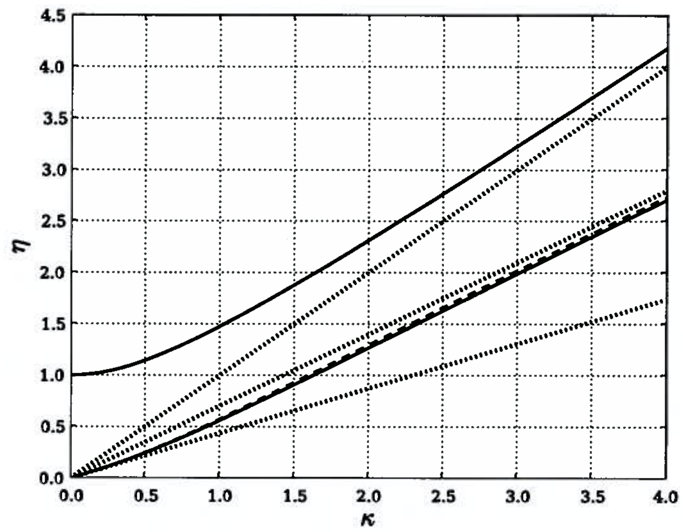

Fig. 1. The characteristic dispersion curves ($n = 0.9$, $m = 0.7$). See explanation in the text.

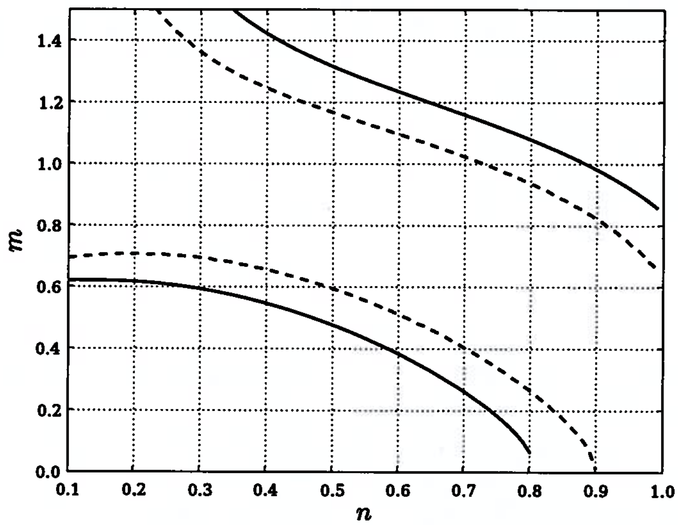

Fig. 2. The ranges of parameters. See explanation in the text.

The ranges for other values of κ behave similarly , only for $\kappa > 1.5$ the area of good agreement is narrower and for $\kappa < 1.5$ the area is wider.

Fig. 3 shows an example where the approximate dispersion relation (33) agrees very well with the full-dispersion relation (31). In Fig. 3a, $c_R < c_1$ and in Fig. 3b, $c_R > c_1$. The continuous lines correspond to the full and the dashed lines to the approximate dispersion relation.

Figs. 4 and 5 are examples of the combination of the parameters where the approximate Eq. (33) and the full-dispersion relation (31) do not coincide well. Fig. 4 is an example of $m < 1$, but not in a good approximation range (see Fig. 2). The continuous line corresponds again to the full and the dashed line to the approximate dispersion relation.

This result can be understood by examining the approximate dispersion relation (33). The strength of the second term in the approximate dispersion relation depends on the parameter n and if parameter n is close to 0 then the influence of the second term is diminished.

Fig. 5 is an example of the situation when c_1 becomes larger than c_0 ($m > 1$). Now the behaviour of the dispersion curves is changed. The full-dispersion relation (31) (represented by the continuous lines) still represents

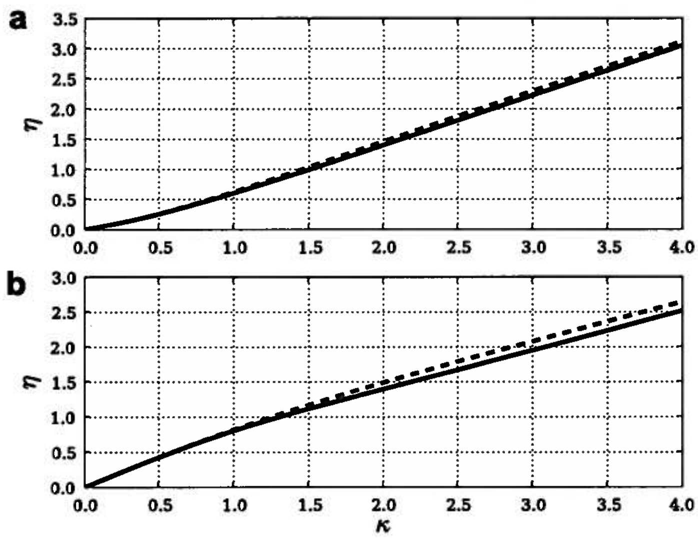

Fig. 3. The behaviour of the acoustic branches (a) $n = 0.9$, $m = 0.8$, (b) $n = 0.5$, $m = 0.6$. See explanation in the text.

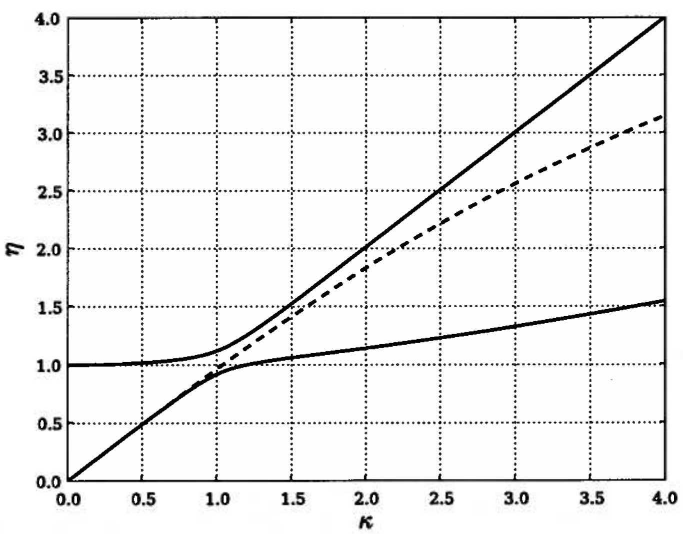

Fig. 4. The behaviour of the dispersion curves ($n = 0.2$, $m = 0.3$). See explanation in the text.

two branches but now the upper branch approaches to the asymptotic line $\eta = m\kappa$. Lower branch starts with a slope $\eta = c\kappa$ and in the short wave limit it approaches to the asymptotic line $\eta = \kappa$.

The approximate relation (33) (represented by the dashed line) also starts with a slope $\eta = c\kappa$, but in the short wave limit it approaches the asymptotic line $\eta = m\kappa$ and does not approach the acoustical branch.

5. Final remarks

Mindlin [1] has derived the dispersion relations for long wave-length (and very long wave-length) approximation and shown a similarity of dispersive effects with those in plates. While Mindlin [1] has used a concept of unit cells embedded in a surrounding medium, then many materials, especially composites, have clearly a defined layered structure. Sun et al. [12,13] have shown that an effective stiffness theory can be derived for describing waves in layered media. Actually, their result is a continuum [13] that bears clear resemblance to Mindlin's material, especially in a 1D case. It has also been shown that gradient elasticity theories [14] need both elastic and inertial effects to be taken into account. This shows again validity of the Mindlin idea. In addi-

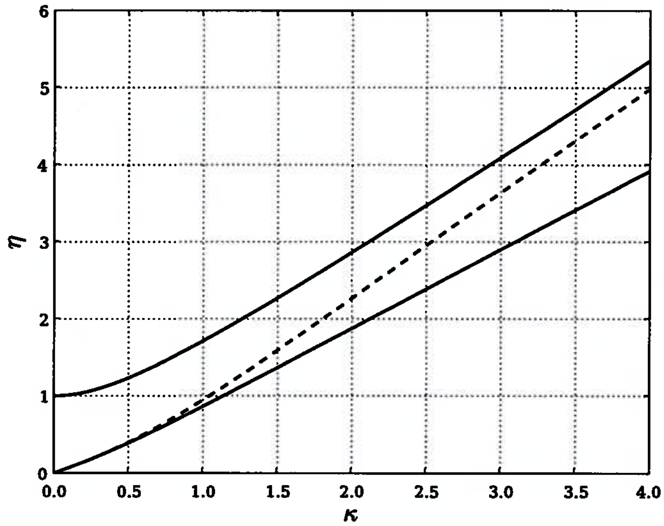

Fig. 5. The behaviour of the dispersion curves ($n = 0.7$, $m = 1.3$). See explanation in the text.

tion, the functionally graded materials (FGMs) which are widely used in contemporary technology [15], can be described by the Mindlin theory and the corresponding models presented above. The straight-forward numerical calculation of wave fields in FGMs [16] has shown explicitly the influence of microstructure for velocities as predicted by Mindlin-type models.

Here, we have derived hierarchical Mindlin-type models (Eqs. (9) and (16)) which describe well dispersive effects. In the wide range of parameters (see Fig. 2), the hierarchical asymptotic model is sufficient for grasping the real behaviour. The hierarchical model itself is certainly simpler and well-grounded physically. In addition, its similarity to discrete models [11] permits to bridge both types (continuous and discrete) models although some deeper analysis is needed in order to clarify the relations of model parameters. The full model (Eq. (9)) and its approximation (Eq. (16)) yield the same type of the evolution equation (cf. Eqs. (24) and (27)). This is not surprising because the proper scaling should lead to a result where the leading properties are accounted for. Even more, the evolution equation obtained in such a way shows clearly that for a homogeneous material (no microstructure) the dispersive effects disappear (here $\gamma = \beta c^2$, i.e. $c_1 = c$). In addition the convexity or concavity of the dispersion curve derived for cases $\gamma \neq \beta c^2$ depends clearly upon the influence of the material parameters. When in the microstructure elastic effects are stronger then $\gamma > \beta c^2$ and the dispersion curve is convex. When however the inertial effects in the microstructure are prevailing then $\gamma < \beta c^2$ and the dispersion curve is concave. The same effect follows from the analysis of full models.

This result is important even qualitatively for Nondestructive Testing (NDT). The concavity/convexity of the dispersion curve shows explicitly the influence of the basic material properties.

The main results of this paper shows that the asymptotic models, both hierarchical two-wave equation (15) (or (17)) and evolution equation (24) (or (27)) are able to grasp dispersive effects in microstructured solids within the wide range of parameters. As said in Section 1, the further studies should introduce nonlinearities like Pastrone [17].

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