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Inverse problems related to a coupled system of microstructure

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Abstract

Inverse problems to determine parameters of microstructured solids are studied. These parameters enter as coefficients to a coupled system of 1D wave propagation derived from the Mindlin model. Characteristics of harmonic waves and wave packets are used as data for the inverse problems. Uniqueness of the solution to the inverse problems in the case of normal dispersion is shown. In the case of anomalous dispersion the solution is double.

1. Introduction

Mathematical models for describing wave propagation in microstructured materials are derived on the basis of different physical considerations but the scale dependence is always accounted for. The general theories of Mindlin [1], Eringen [2], Capriz [3], and Maugin [4] *et al* explain the essence of various models and their accuracies in detail. Returning to the physical background, such mathematical models involve dispersive effects due to the scale dependence.

The straight-forward modeling of microstructured solids leads to the assignment of all the physical properties to every volume element dV in a solid. Then the governing equations implicitly include space-dependent parameters but the resulting system is too complex for analytical treatment and needs numerical simulation for analysis. If macro- and microstructure are separated then the conservation laws for both structures should also be separately formulated [1, 2]. In the 1D case the result is then two coupled equations involving d'Alembert-type wave operators for both macro- and microstructure.

Such mathematical models are now used widely to describe wave processes in polycrystalline solids, ceramic composites, alloys, functionally graded materials, etc. The deformation waves propagating in these materials are affected by dispersive effects.

In inverse problems given an input and a measured wave field, the material parameters are determined. This is a huge discipline with many technological applications, called non-destructive testing (NDT). However, most of the applications are based on measuring the velocities and attenuation of deformation waves. Traditional NDT methods measure these parameters by the conventional time-of-flight method and treat the recorded data on the level of the linear or linearized theories [5–7]. Numerous methods for the evaluation of mechanical properties of material, for the determination of flaws, inclusions, structural properties, prestress, etc are developed [8, 9]. These methods break down in complicated cases, for example, when the material has multiparametric properties, physical inhomogeneity, inhomogeneous prestress, when the wave velocity is frequency dependent, etc. This calls for the necessity to extract additional information from the wave velocity and attenuation measurement data. The accuracy of solving the inverse problems is directly related to the accuracy of the basic mathematical models.

Our earlier studies include the derivation of corresponding mathematical models [10], dispersion analysis [11], analysis of solitary waves [12] and solving the inverse problems for the Boussinesq-type approximate models [13, 14].

In the present paper our analysis is based on the system of equations which result directly from the continuum theory [1, 11]. We believe that such an approach reflects better the basic properties of a microstructured material. We will be limited to the linear case, but nonlinear problems will be dealt with in the future. The identification of material parameters by means of wavenumbers (or phase velocities) of harmonic waves and spectral analysis will be studied.

The plan for the paper is as follows. In section 2, the problem is formulated and in section 3, the dispersion relation is deduced. In section 4 we study an inverse problem for harmonic waves. Further, in section 5 we deduce formulae for more general linear waves and section 6 describes the reconstruction of coefficients by means of spectral analysis of these waves. In section 7, final remarks are presented. Section 8 contains a proof of a technical theorem.

The main attention is paid to the case with the constant coefficients, i.e. to the case when the microstructure is homogeneously distributed over the material. However, our approach can be extended to the piecewise homogeneous case, too. This will be described in subsection 6.3.

2. Model description and problem formulation

The 1D model governing the longitudinal waves in a microstructured material was derived [11] following the Mindlin theory [1]. Briefly the idea behind the model is the following. The microstructure has been interpreted 'as a molecule of a polymer, a crystallite of a polycrystal or a grain of a granular material'. This microelement is taken as a deformable cell. In a 1D setting a cell might be interpreted also as a layer. The displacement $\mathbf{u}$ of a material particle in terms of macrostructure is defined by its components $u_i \equiv x_i - X_i$, where $x_i, X_i (i = 1, 2, 3)$ are the components of the spatial and material position vectors, respectively. Within each material volume (particle) there is a microvolume and the microdisplacement $\mathbf{u}'$ is defined by its components $u'_i \equiv x'_i - X'_i$, where the origin of the coordinates x'_i moves with the displacement $\mathbf{u}$. The displacement gradient is assumed to be small. This leads to the basic assumption that the microdisplacement can be expressed as a sum of products of specified functions of x'_i and arbitrary functions of x_i and t (see [1], p 52). The first approximation is then $u'_j = x'_k \psi_{kj}(x_i, t)$ and consequently the microdeformation depends only on the macroscopic argument x_i and time t, i.e. $\frac{\partial u'_j}{\partial x'_i} = \partial'_i u'_j = \psi_{ij}(x_i, t)$.

Using this relation in one-dimensional macroscopic and microscopic momentum balance equations we get the system [11]

$$\rho u_{tt} = \sigma_x \qquad I\psi_{tt} = \eta_x - \tau, \tag{2.1}$$

where $u = u_{11}$ is the macrodisplacement, $\psi = \psi_{11}$ is the microdeformation, $\sigma = \sigma_{11}$ is the macrostress (Piola stress), $\eta = \eta_{11}$ is the microstress and $\tau = \tau_1$ is the interactive microforce; ρ is the macrodensity, I is the microinertia. The functions $u, \sigma, \psi, \eta, \tau$ involved in (2.1) depend on the macroscopic argument $x = x_1$ and time t. The equilibrium components of σ, η and τ are deduced from the free energy W [11]

$$\sigma = \frac{\partial W}{\partial u_x}, \qquad \eta = \frac{\partial W}{\partial \psi_x}, \qquad \tau = \frac{\partial W}{\partial \psi}. \tag{2.2}$$

We consider the simplified model, where the dissipation is neglected, hence non-equilibrium components of σ, η and τ vanish. The free energy function W, depending on u_x, ψ_x and ψ can be approximated by a Taylor polynomial. The simplest approximation describing the influence of a microstructure is the quadratic function

$$W = \tfrac{1}{2}au_x^2 + \tfrac{1}{2}B\psi^2 + \tfrac{1}{2}C\psi_x^2 + D\psi u_x \tag{2.3}$$

with a, B, C, D as constants. Then equations (2.1)–(2.3) yield the linear system of governing equations

$$\rho u_{tt} = au_{xx} + D\psi_x, \qquad I\psi_{tt} = C\psi_{xx} - Du_x - B\psi. \tag{2.4}$$

Due to the physical background, the material parameters satisfy the inequalities [11]

$$\rho, I, a, B, C > 0. \tag{2.5}$$

Let us rewrite (2.4) in dimensionless variables $X = \frac{x}{L}$, $T = \frac{t}{T_0}$, $U = \frac{u}{U_0}$ where L, T_0, U_0 are certain constant values. Introducing in addition the geometric parameters

$$\delta = \frac{l^2}{L^2}, \qquad \epsilon = \frac{U_0}{L}, \qquad \vartheta = \frac{T_0^2}{L^2}, \tag{2.6}$$

where l is the scale of the microstructure, this system reads

$$\rho U_{TT} = a\vartheta U_{XX} + \frac{D\vartheta}{\epsilon}\psi_X, \qquad \delta\frac{I}{\vartheta l^2}\psi_{TT} = \delta\frac{C}{l^2}\psi_{XX} - D\epsilon U_X - B\psi. \tag{2.7}$$

For the macrodeformation v we have the relation $v = u_x = \epsilon U_X$. Let us replace the first equation in (2.7) by the corresponding equation for v, namely $\rho v_{TT} = a\vartheta v_{XX} + \frac{N\vartheta}{2}(v^2)_{XX} + D\vartheta\psi_{XX}$, and rewrite the resulting system by means of the lower-case letters x and t. We obtain

$$\rho v_{tt} = a\vartheta v_{xx} + D\vartheta\psi_{xx}, \qquad \delta\frac{I}{\vartheta l^2}\psi_{tt} = \delta\frac{C}{l^2}\psi_{xx} - Dv - B\psi. \tag{2.8}$$

We are going to study inverse problems to determine the coefficients of this system. It has in total seven coefficients: ρ, $a\vartheta$, $D\vartheta$, $\frac{I}{\vartheta l^2}$, $\frac{C}{l^2}$, D and B. The geometric parameter δ is assumed to be known.

Evidently, it is not possible to recover all coefficients of the homogeneous system (2.8). Indeed, any vector of coefficients that fits to this system can be multiplied by an arbitrary constant to get another vector of coefficients that also fits to this system. The determination of all coefficients could be possible only in the case of non-homogeneous system containing mass forces. Therefore, we divide equations (2.8) by ρ and $\frac{I}{\vartheta l^2}$, respectively, to obtain the systems

$$v_{tt} = a_0 v_{xx} + \alpha\psi_{xx}, \tag{2.9}$$

$$\psi_{tt} = a_1\psi_{xx} - \frac{\beta}{\delta}v - \frac{\gamma}{\delta}\psi \tag{2.10}$$

that contains five coefficients to be determined

$$a_0 = \frac{a\vartheta}{\rho}, \qquad \alpha = \frac{D\vartheta}{\rho}, \qquad a_1 = \frac{C\vartheta}{I}, \qquad \beta = \frac{D\vartheta l^2}{I}, \qquad \gamma = \frac{B\vartheta l^2}{I}. \tag{2.11}$$

The geometric quantity δ is assumed to be known. Otherwise we have to pose a problem to determine the quotients $\frac{\beta}{\delta}$ and $\frac{\gamma}{\delta}$ instead of β and γ.

The coefficients to be determined satisfy the following *a priori* inequalities:

$$a_0, a_1, \gamma, \alpha\beta > 0 \tag{2.12}$$

that easily follow from (2.11) in view of the physical inequalities (2.5). Moreover, in the case when the scale of the microstructure is zero, i.e. $\delta = 0$, from (2.10) we get $v = -\frac{\gamma}{\beta}\psi$. Plugging this relation into (2.9) we reach the equation $v_{tt} = \left(a_0 - \frac{\alpha\beta}{\gamma}\right)v_{xx}$ for the macrodeformation. From this equation we infer the following necessary hyperbolicity condition for the coefficients:

$$a_0\gamma - \alpha\beta > 0. \tag{2.13}$$

3. Harmonic waves and the dispersion relation

Let us seek for the solutions (v, ψ) of the system (2.9), (2.10) such that their first component v has the form

$$v(x,t) = A\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}, \tag{3.1}$$

where $A > 0$, $k \in \mathbb{R}$ and $\omega \in \mathbb{R}$ are some constants (the amplitude, the wavenumber and the frequency of the macrodeformation, respectively). Plugging this formula into (2.9) we easily get the equation for ψ: $\alpha\psi_{xx} = -A(\omega^2 - a_0k^2)\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}$. The solution of this equation is

$$\psi(x,t) = \frac{A(\omega^2 - a_0k^2)}{\alpha k^2}\,\mathrm{e}^{\mathrm{i}(kx-\omega t)}. \tag{3.2}$$

Comparing (3.1) with (3.2) we see that v and ψ are synchronous. They relate each other as

$$\frac{\psi(x,t)}{v(x,t)} = \frac{\omega^2 - a_0k^2}{\alpha k^2}. \tag{3.3}$$

Plugging (3.1) and (3.2) into (2.10), dividing by $\mathrm{e}^{\mathrm{i}(kx-\omega t)}$ and simplifying we obtain the following quartic equation for k and ω:

$$\omega^4 + \varkappa_1\omega^2k^2 + \varkappa_2k^4 + \varkappa_3\omega^2 + \varkappa_4k^2 = 0, \tag{3.4}$$

where

$$\varkappa_1 = -(a_0 + a_1), \qquad \varkappa_2 = a_0a_1, \qquad \varkappa_3 = -\frac{\gamma}{\delta}, \qquad \varkappa_4 = \frac{a_0\gamma - \alpha\beta}{\delta}. \tag{3.5}$$

For given $\omega \in \mathbb{R}$, equation (3.4) has four solutions $k = \pm k(\omega)$ and $k = \pm k_2(\omega)$, where

$$k(\omega) = \omega\sqrt{\frac{1}{2a_0a_1}\left[a_0 + a_1 - \frac{a_0\gamma - \alpha\beta}{\delta\omega^2} + \sqrt{\left(a_0 - a_1 - \frac{a_0\gamma - \alpha\beta}{\delta\omega^2}\right)^2 + \frac{4a_1\alpha\beta}{\delta\omega^2}}\right]}, \tag{3.6}$$

$$k_2(\omega) = \omega\sqrt{\frac{1}{2a_0a_1}\left[a_0 + a_1 - \frac{a_0\gamma - \alpha\beta}{\delta\omega^2} - \sqrt{\left(a_0 - a_1 - \frac{a_0\gamma - \alpha\beta}{\delta\omega^2}\right)^2 + \frac{4a_1\alpha\beta}{\delta\omega^2}}\right]}. \tag{3.7}$$

Due to the inequalities (2.12) and (2.13), the argument of the inner radical in (3.6) and (3.7) is positive for any $\omega \in \mathbb{R}$. Similarly, for any $\omega \in \mathbb{R}$ the argument of the outer radical is positive in (3.6), but negative in (3.7). This implies that $k(\omega)$ is real valued and $k_2(\omega)$ is imaginary valued. The solutions with real ω and imaginary $k = \pm k_2(\omega)$ have no physical meaning. Only the solutions $k = +k(\omega)$ and $k = -k(\omega)$ are related to the harmonic waves (propagating to the right and left, respectively).

Some important properties of $k(\omega)$ in our model: it is strictly increasing, $k(\omega) \sim \omega\sqrt{\frac{\gamma}{a_0\gamma-\alpha\beta}}$ as $\omega \to 0$ and $k(\omega) \sim \omega \max\left\{\frac{1}{\sqrt{a_0}}; \frac{1}{\sqrt{a_1}}\right\}$ as $\omega^2 \to +\infty$.

We also have to establish the dependence of the type of dispersion on the coefficients, because, as we will see later on, this is crucial for the character of inverse problems. After tedious but elementary computations we come to the following result.

(1) In case $a_0\gamma - a_1\gamma - \alpha\beta > 0$ the group velocity is less than the phase velocity, i.e. $c_g(\omega) = \frac{1}{k'(\omega)} < c_{ph}(\omega) = \frac{\omega}{k(\omega)}$ for any $\omega \in \mathbb{R}$. This means that the model possesses *normal dispersion*.

(2) In case $a_0\gamma - a_1\gamma - \alpha\beta < 0$ we have $c_g(\omega) > c_{ph}(\omega)$ for any $\omega \in \mathbb{R}$, hence *anomalous dispersion* occurs.

(3) In case $a_0\gamma - a_1\gamma - \alpha\beta = 0$ we have $k(\omega) = \frac{1}{\sqrt{a_1}}\omega$ for any $\omega \in \mathbb{R}$. Thus, $c_g(\omega) = c_{ph}(\omega) = \sqrt{a_1}$ is constant. The model has *no dispersion*.

4. Inverse problem for harmonic waves

In this section we discuss the determination of the coefficients a_0, a_1, α, β and γ of the system (2.9), (2.10) by means of harmonic waves.

We remark that the harmonic waves in marcoscale do not contain enough information to separate α and β, because they occur in the form of the product in formulae (3.5) and (3.6). Therefore, we will be concerned in the determination of the product $\alpha\beta$ instead of α and β.

The mathematical problem under consideration is as follows: *given wave numbers $k_j = k(\omega_j)$ of four harmonic waves with frequencies ω_j, $j = 1, \ldots, 4$, such that ω_j^2, $j = 1, \ldots, 4$, are different, determine a_0, a_1, γ and the product $\alpha\beta$.*

This problem serves for two purposes. First, it can be used practically to determine the unknown coefficients in case the frequencies and phase velocities of harmonic waves can be measured (then the wave numbers can easily be computed). Several experimental techniques for phase velocity measurement are available, e.g., pulse-echo and the continuous wave resonance method [15]. Second, the inverse problem for harmonic waves is a step of a more general reconstruction method that will be discussed in section 6.

We decompose the inverse problem into two subproblems:

(1) determine the coefficients $\varkappa_1, \ldots, \varkappa_4$ of equation (3.4) by means of the given pairs (k_j, ω_j), $j = 1, \ldots, 4$;

(2) solve the system (3.5) for a_0, a_1, γ and the product $\alpha\beta$ by means of the computed values of $\varkappa_1, \ldots, \varkappa_4$.

The first subproblem is the following 4×4 linear system

$$\omega_j^2 k_j^2 \cdot \varkappa_1 + k_j^4 \cdot \varkappa_2 + \omega_j^2 \cdot \varkappa_3 + k_j^2 \cdot \varkappa_4 = -\omega_j^4, \qquad j = 1, \ldots, 4, \tag{4.1}$$

with unknowns $\varkappa_1, \ldots, \varkappa_4$. The solution of (4.1) is unique in case we *a priori* assume that the dispersion is present, i.e. when $a_0\gamma - a_1\gamma - \alpha\beta \neq 0$. In other words: the pairs (ω_j, k_j), $j = 1, \ldots, 4$ in a dispersive model always generate a regular matrix in the system (4.1). This a consequence of the following theorem, whose proof is shifted to section 2.

Theorem. *Assume that* $a_0, a_1, \gamma, \alpha, \beta$ *satisfy* $a_0\gamma - a_1\gamma - \alpha\beta \neq 0$ *and* $\varkappa_1, \ldots, \varkappa_4$ *are given by (3.5) in terms of* $a_0, a_1, \gamma, \alpha, \beta$. *Further, let* $\omega_j \in \mathbb{R}, \omega_j \neq 0, j = 1, \ldots, 4$, *be such that* $\omega_j^2, j = 1, \ldots, 4$ *are different and define* $k_j = k(\omega_j), j = 1, \ldots, 4$, *where the function* $k(\omega)$ *is given by (3.6) in terms of* $a_0, a_1, \gamma, \alpha, \beta$. *By this definition,* $\varkappa_1, \ldots, \varkappa_4$, *solve (4.1) with the data* $\omega_j, k_j, j = 1, \ldots, 4$. *If* $\widetilde{\varkappa}_1, \ldots, \widetilde{\varkappa}_4$ *solve (4.1) with the same data, then* $\widetilde{\varkappa}_j = \varkappa_j, j = 1, \ldots, 4$.

From practical point of view it is reasonable to use more than four pairs (ω_j, k_j), $j = 1, \ldots, J$ and instead of (4.1) solve the over-determined system

$$\omega_j^2 k_j^2 \cdot \varkappa_1 + k_j^4 \cdot \varkappa_2 + \omega_j^2 \cdot \varkappa_3 + k_j^2 \cdot \varkappa_4 = -\omega_j^4, \qquad j = 1, \ldots, J \tag{4.2}$$

by least squares. This diminishes the impact of stochastic noise of data.

The second subproblem, i.e. the system (3.5), has two mathematical solutions

$$a_0 = a_0^+, \quad a_1 = a_1^-, \quad \gamma = -\delta\varkappa_3, \quad \alpha\beta = (\alpha\beta)_1 := a_0^+\gamma - \delta\varkappa_4, \tag{4.3}$$

$$a_0 = a_0^-, \quad a_1 = a_1^+, \quad \gamma = -\delta\varkappa_3, \quad \alpha\beta = (\alpha\beta)_2 := a_0^-\gamma - \delta\varkappa_4, \tag{4.4}$$

where

$$a_0^\pm = a_1^\pm = \frac{-\varkappa_1 \pm \sqrt{\varkappa_1^2 - 4\varkappa_2}}{2}. \tag{4.5}$$

By means of elementary computations, we can check whether these solutions satisfy the physical inequalities (2.12) or not. It turns out that if we *a priori* assume the normal dispersion, i.e. $a_0\gamma - a_1\gamma - \alpha\beta > 0$, then the first solution (4.3) satisfies (2.12), but for the second solution (4.4) we have $(\alpha\beta)_2 < 0$, hence (2.12) fails. In case of anomalous dispersion, i.e. when $a_0\gamma - a_1\gamma - \alpha\beta < 0$, both (4.3) and (4.4) satisfy (2.12).

4.1. Summary

Taking the theorem and the latter discussion about the system (3.5) into account, we come to the following conclusions.

(1) In the case of normal dispersion the physical solution of the inverse problem for the harmonic waves is unique. It has the form (4.3) where $a_{0,1}, a_{1,1}$ are given by (4.5) and $\varkappa_1, \ldots, \varkappa_4$ solve (4.1).

(2) In the case of anomalous dispersion the physical solution of the inverse problem for the harmonic waves is double. It has the forms (4.3) and (4.4) where $a_{0,1}, a_{1,1}, a_{0,2}, a_{1,2}$ are given by (4.5) and $\varkappa_1, \ldots, \varkappa_4$ solve (4.1).

In practice, the sign of $a_0\gamma - a_1\gamma - \alpha\beta$ can be computed by means of the first solution (4.3) to clear up the type of dispersion and see whether the solution is unique or double.

In non-dispersive media, i.e. when $a_0\gamma - a_1\gamma - \alpha\beta = 0$, the theorem does not apply. One can easily see that in this case the inverse problem has infinitely many solutions. Indeed, as we saw at the end of section 3, the function $k(\omega)$ has then the degenerate form $k(\omega) = \frac{1}{\sqrt{a_1}}\omega$. This means that the data of the inverse problem contain information about the coefficient a_1 only, i.e. $a_1 = \frac{\omega_j^2}{k_j^2}$ for $j = 1, \ldots, 4$. The remaining coefficients a_0, γ and $\alpha\beta$ may be arbitrary. (However, they satisfy the equality $a_0\gamma - a_1\gamma - \alpha\beta = 0$ in this case.)

Although the non-dispersive case is rather theoretical, one should take into account that when the weak dispersion occurs, i.e. $a_0\gamma - a_1\gamma - \alpha\beta \approx 0$, then the matrix of (4.1) is ill-conditioned, i.e. close to singular. This may cause large computational errors in the solution.

Finally, we remark that it is possible to separate α and β from their product in case a certain information about the microdeformation is available, too. Namely, let we be given the amplitudes of the macro- and microdeformation A_{macro} and A_{micro} of the first wave with $\omega = \omega_1$ and $k = k_1$. Then, by (3.3) we have the equation $\frac{A_{\mathrm{micro}}}{A_{\mathrm{macro}}} = \frac{\omega_1^2 - a_0 k_1^2}{\alpha k_1^2}$. This yields the parameter α: $\alpha = \frac{A_{\mathrm{macro}}(\omega_1^2 - a_0 k_1^2)}{A_{\mathrm{micro}} k_1^2}$.

5. Wave packets and boundary value problems

In section 3 we deduced and studied harmonic solutions of the form $\mathrm{e}^{\mathrm{i}(\pm k(\omega)x - \omega t)}$ for the systems (2.9), (2.10). Moreover, we saw that there exist additional mathematical solutions of the form $\mathrm{e}^{\mathrm{i}(\pm k_2(\omega)x - \omega t)}$, where ω is real and $k_2(\omega)$ is imaginary. The latter ones represent a kind of synchronous oscillation of the whole medium (waves with infinite velocities). Therefore, they cannot be physical solutions.

Let us proceed to more general waves. The hyperbolic system with constant coefficients (2.9), (2.10) can be solved by standard techniques. Using the Fourier method with respect to the time, we obtain the general mathematical solution $v = v_+ + v_- + v_{2,+} + v_{2,-}$, $\psi = \psi_+ + \psi_- + \psi_{2,+} + \psi_{2,-}$, where

$$v_\pm(x,t) = \int_{-\infty}^{\infty} A_\pm(\omega)\,\mathrm{e}^{\mathrm{i}(\pm k(\omega)x - \omega t)}\,\mathrm{d}\omega, \qquad v_{2,\pm}(x,t) = \int_{-\infty}^{\infty} A_{2,\pm}(\omega)\,\mathrm{e}^{\mathrm{i}(\pm k_2(\omega)x - \omega t)}\,\mathrm{d}\omega,$$

$$\psi_\pm(x,t) = \int_{-\infty}^{\infty} A^m_\pm(\omega)\,\mathrm{e}^{\mathrm{i}(\pm k(\omega)x - \omega t)}\,\mathrm{d}\omega, \qquad \psi_{2,\pm}(x,t) = \int_{-\infty}^{\infty} A^m_{2,\pm}(\omega)\,\mathrm{e}^{\mathrm{i}(\pm k_2(\omega)x - \omega t)}\,\mathrm{d}\omega,$$

$$A^m_\pm(\omega) = \frac{A_\pm(\omega)[w^2 - a_0(k(\omega))^2]}{\alpha(k(\omega))^2}, \qquad A^m_{2,\pm}(\omega) = \frac{A_{2,\pm}(\omega)[w^2 - a_0(k(\omega))^2]}{\alpha(k(\omega))^2}$$

and $A_\pm$ and $A_{2,\pm}$ are arbitrary functions (may be singular distributions, too). However, the terms $v_{2,\pm}$ and $\psi_{2,\pm}$ cannot be contained in physical solutions, because they involve a certain information that propagates with infinite speed. Indeed, suppose that (v, ψ) is a solution of some boundary value problem for (2.9), (2.10) and either $A_{2,+}(\omega_0) \neq 0$ or $A_{2,-}(\omega_0) \neq 0$ for some ω_0. Then the component with frequency ω_0 in the given boundary excitation(s) propagate(s) with infinite speed.

After removal of the terms $v_{2,\pm}$ and $\psi_{2,\pm}$ from the formulae of v and ψ, the superposition of harmonic waves remains

$$\begin{aligned} v(x,t) &= \int_{-\infty}^{\infty} A_+(\omega)\,\mathrm{e}^{\mathrm{i}(k(\omega)x - \omega t)}\,\mathrm{d}\omega + \int_{-\infty}^{\infty} A_-(\omega)\,\mathrm{e}^{\mathrm{i}(-k(\omega)x - \omega t)}\,\mathrm{d}\omega, \\ \psi(x,t) &= \int_{-\infty}^{\infty} A^m_+(\omega)\,\mathrm{e}^{\mathrm{i}(k(\omega)x - \omega t)}\,\mathrm{d}\omega + \int_{-\infty}^{\infty} A^m_-(\omega)\,\mathrm{e}^{\mathrm{i}(-k(\omega)x - \omega t)}\,\mathrm{d}\omega, \end{aligned} \tag{5.1}$$

where the addends with A_+ and A_- represent the wave packets propagating to the right and left, respectively.

Let the symbol $\widehat{\ }$ denote the Fourier transform, i.e. $\widehat{f}(\omega) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}t\omega} f(t)\,\mathrm{d}t$. In the standard manner, we can extract from the general solution (5.1) solutions corresponding to any kind of boundary conditions. Here are some examples (we will write formulae for v, only):

(A) *Semi-infinite body I.* Let the macrodeformation be specified on the plane $x = 0$, i.e., $v(0,t) = g(t)$, and the packet contain only the waves propagating to the right. Then

$$v(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{g}(\omega)\,\mathrm{e}^{\mathrm{i}(k(\omega)x - \omega t)}\,\mathrm{d}\omega. \tag{5.2}$$

(B) *Semi-infinite body II*. Let the displacement be given at $x = 0$, i.e., $u(0, t) = \varphi(t)$ and the packet contain only the waves propagating to the right. Then

$$v(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{i}k(\omega)\widehat{\varphi}(\omega)\,\mathrm{e}^{\mathrm{i}(k(\omega)x-\omega t)}\,\mathrm{d}\omega. \tag{5.3}$$

(C) *Finite body*. Let us consider a body between the planes $x = 0$ and $x = l$. Let the deformations be given on the left and right surfaces, i.e., $v(0, t) = g_0(t)$ and $v(l, t) = g_l(t)$. (In case of free surface at $x = l$ the function g_l vanishes.) Then the solution is

$$v(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\widehat{g}_0(\omega)\sin[k(\omega)(l-x)] + \widehat{g}_l(\omega)\sin[k(\omega)x]}{\sin[k(\omega)l]}\,\mathrm{d}\omega. \tag{5.4}$$

We emphasize that it is possible to incorporate zero initial conditions in all these models assuming that either $g(t)$ or $\varphi(t)$ is zero for $t < 0$. Treatment of nonzero initial conditions is more complicated.

6. Inverse problems for wave packets

It is possible to determine the parameters a_0, a_1, γ and $\alpha\beta$ by means of more complex linear waves, too. The idea is to extract harmonic counterparts from the spectral decomposition of the wave and reduce the problem to the inverse problem for harmonic waves discussed in section 4.

6.1. Problem for model A

Let us consider the model A from the previous section. From the solution formula (5.2) we deduce the expression $\widehat{v}(x, \omega) = \widehat{g}(\omega)\,\mathrm{e}^{\mathrm{i}k(\omega)x}$ for the Fourier transform of v at some fixed point x_1. This implies

$$\cos[k(\omega)x_1] = \mathrm{Re}\,\frac{\widehat{v}(x_1,\omega)}{\widehat{g}(\omega)}. \tag{6.1}$$

This is the basic equation we will use in the frequency domain for the solution of inverse problem.

Suppose that in addition to the boundary condition $g(t)$ we know the deformation function $v(x, t)$ in some point $x_1 > 0$ over the time t. We have to solve equation (6.1) for $k(\omega)$ in order to determine pairs (ω_j, k_j) for the inverse problem for harmonic waves. This means that we have to invert cosine in a proper way. Here we can think as follows. Since $k(\omega)$ is strictly increasing, $k(0) = 0$ and $\lim_{w\to\infty} k(\omega) = \infty$, the function $\cos[k(\omega)x_1]$ oscillates between 1 and -1. Namely, $\cos[k(\omega)x_1]$ decreases for $\omega \in \mathcal{P}_1 = (0, \zeta_1)$, increases for $\omega \in \mathcal{P}_2 = (\zeta_1, \zeta_2)$ and so on, where $0 < \zeta_1 < \zeta_2 < \ldots$ are some numbers. Thus, for the right inversion of the cosine it is necessary to locate the intervals $\mathcal{P}_j$ using the known right-hand side $\mathrm{Re}\,\frac{\widehat{v}(x_1,\omega)}{\widehat{g}(\omega)}$ of (6.1). Then the desired function $k(\omega)$ can be computed by the formula

$$k(\omega) = \frac{1}{x_1}\left[(-1)^n \arccos \mathrm{Re}\,\frac{\widehat{v}(x_1,\omega)}{\widehat{g}(\omega)} + \pi(n+\theta_n)\right] \qquad \text{for} \quad \omega \in \mathcal{P}_n, \tag{6.2}$$

where $\theta_n = 0$ for odd n and $\theta_n = 1$ for even n.

Let us look how this method can be discretized. In practice, we have in our disposal a finite number of measured deformations $v_l^\epsilon \approx v(x_1, t_l), l = 1, \ldots, N$ in a time interval $[T, T_1]$, where $t_l = T + lh, h = \frac{T_1-T}{N}$ and ϵ is some noise level. To compute the Fourier transforms, several methods are available. We choose the rectangular rule for truncated

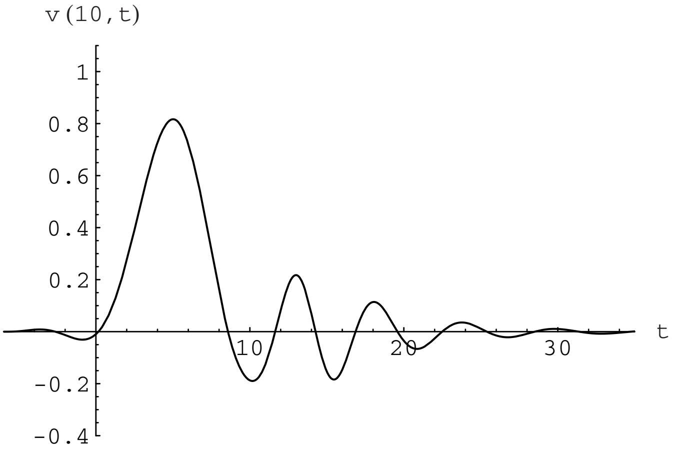

Figure 1. Function $v(10,t)$ in case $v(0,t) = e^{\frac{-t^2}{4}}$.

Fourier integrals because this leads to the standard discrete Fourier transform available in mathematical softwares. Then the discrete spectra of the data have the formulae

$$\widehat{g}(\omega_m) \approx \hat{g}_m = \frac{e^{iT\omega_m}}{N}\sum_{l=1}^{N} e^{\frac{2\pi ib(l-1)(m-1)}{N}} g_l, \qquad \widehat{v}(x_1,\omega_m) \approx \hat{v}_m^{\epsilon} = \frac{e^{iT\omega_m}}{N}\sum_{l=1}^{N} e^{\frac{2\pi ib(l-1)(m-1)}{N}} v_l^{\epsilon}$$

for $m = 1,\ldots,N$, where $\tau > 0$ is the stepsize in the frequency domain, $\omega_m = (m-1)\tau$, $g_l = g((l-1)h)$ and $b = \frac{h\tau}{2\pi}$. The discrete spectrum yields the oscillating sequence $z_m^{\epsilon} = \mathrm{Re}\,\frac{\hat{v}_m^{\epsilon}}{\hat{g}_m}$ that decreases for $m = s_0,\ldots,s_1$, increases for $m = s_1,\ldots,s_2$ and so on, where $1 = s_0 < s_1 < s_2 < \ldots$ are some integers. It is necessary to determine the critical numbers $s_1, s_2, \ldots$ of this sequence. They enable to use the following formula for wavenumbers that is derived from (6.2)

$$k_m = k(\omega_m) = \frac{1}{x_1}[(-1)^n \arccos z_m + \pi(n+\theta_n)] \qquad \text{for} \quad s_{n-1} < m < s_n. \tag{6.3}$$

Using this formula it is possible to compute the data (ω_j, k_j), $j = 1,\ldots,J$ for the system (4.2) and finish the solution of the inverse problem by the method described in section 4. Note that (6.3) does not apply to critical frequencies ω_{s_n}, because the discrete problem does not contain information about the intervals of monotonicity ω_{s_n} belong to.

Numerical example. We illustrate the described procedure by means of a numerical example constructed by means of synthetic data. The computations were performed using Mathematica 5.1.

We set $a_0 = 10$, $a_1 = \alpha = 1$ and $\gamma = \beta = \delta = 10^{-4}$ (the parameters γ, β and δ contain the small quantity l^2 (see (2.6), (2.11))). Note that this choice corresponds to the case of the normal dispersion.

We solved the problem for the local excitation $g(t) = e^{-\frac{t^2}{4}}$ at $x = 0$. Then $\widehat{g}(\omega) = 2\sqrt{\pi}\,e^{-\omega^2}$ and it is possible to compute v from (5.2). We chose $x_1 = 10$, truncated the improper integral in (5.2) and computed it by Simpson's rule to obtain discrete values of the wavefunction $v(10,t)$ for $t \in [-10, 50]$. The cubic-spline interpolation of the result is given in figure 1.

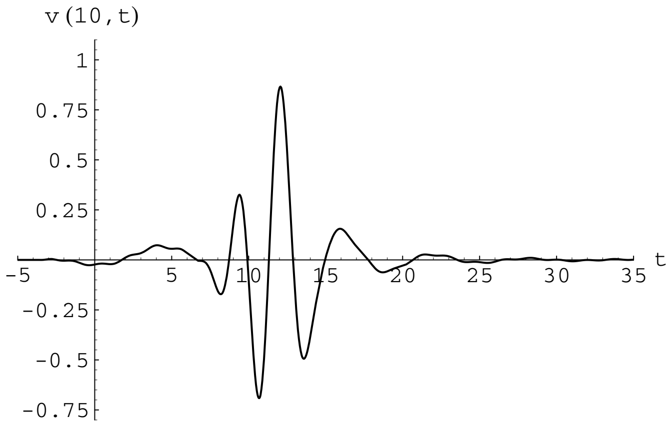

Figure 2. Function $v(10, t)$ in case $v(0, t) = \mathrm{e}^{\frac{-t^2}{4}} \cos 2t$.

Thereupon, we chose uniform mesh with $N = 200$ points in the time interval $[-5, 50]$ with the stepsize $h = \frac{55}{200}$. We disturbed the deformation values at the nodes by the formula $v_l^\epsilon = v(-5 + (l-1)h) + \epsilon R, l = 1, \ldots, N$, to deduce the *synthetic data* for the inverse problem. Here ϵ is a given noise level and R is the uniformly distributed random number in the interval $[-1, 1]$.

First, we solved the problem in case $\epsilon = 10^{-3}$. To get the first sight on the situation in the frequency domain, we took a larger stepsize $\tau = \frac{\pi}{55}$ and computed the sequence $z_m^\epsilon = \mathrm{Re}\, \frac{\hat{v}_m^\epsilon}{\hat{g}_m}$ (figure 3).

For comparison, exact $\cos[k(\omega)x_1]$ is given in figure 4. Clearly, z_m^ϵ is oscillating with critical numbers $s_0 = 0, s_1 = 16, s_2 = 20, s_3 = 24, \ldots$. Higher disturbance occurs from fourth period because of the very small denominator $\hat{g}_m$ ($\widehat{g}(\omega)$ is rapidly decreasing). To guarantee maximal accuracy, we truncated the subrange $\omega > 2$ and computed new values of the spectra with the smaller stepsize $\tau = \frac{1}{200}$ in order to remain with all computations inside the interval $\omega \in (0, 2)$. By means of formula (6.3) we constructed data for the system (4.2), solved it by least squares and computed the unknown coefficients by (4.3). The result was $a_0^\epsilon = 9.97, a_1^\epsilon = 1.002, \gamma^\epsilon = 0.995 \cdot 10^{-6}, (\alpha\beta)^\epsilon = 0.975 \cdot 10^{-6}$. We repeated the solution procedure 50 times taking different random numbers R with the same noise level $\epsilon = 10^{-3}$. The biggest relative errors in a_0, a_1, γ and $\alpha\beta$ were 0.6%, 0.4%, 1.0% and 5.2%, respectively. Moreover, we solved the problem for the noise level $\delta = 10^{-2}$ for 50 cases of random R. Then the biggest relative errors were 3.6%, 2.2%, 1.7% and 35%, respectively.

Finally, we changed the boundary condition taking $g(t) = \mathrm{e}^{-\frac{t^2}{4}} \cos 2t$. Then in case $\epsilon = 10^{-3}$ the biggest relative errors of 50 solutions for a_0, a_1, γ and $\alpha\beta$ were 0.5%, 0.2%, 1.2% and 5.7%, respectively, and in case $\epsilon = 10^{-2}$ they were 7.6%, 6.0%, 3.3% and 64%, respectively.

The numerical results show that the product $\alpha\beta$ is much more sensitive with respect to the noise than other coefficients. The lower accuracy in the example $g(t) = \mathrm{e}^{-\frac{t^2}{4}} \cos 2t$ for the bigger error $\epsilon = 10^{-2}$ is supposedly caused by the lower smoothness of the deformation function (figure 2).

6.2. Problems for other homogeneous models

The described solution procedure can be extended to other wave processes in homogeneous bodies. Let us consider the model C from section 5. The simplest inverse problem we get

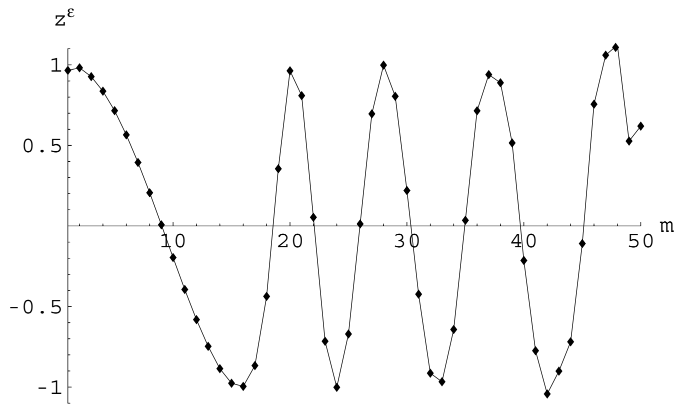

Figure 3. Sequence z_m^ϵ for $m = 1, \ldots, 50$.

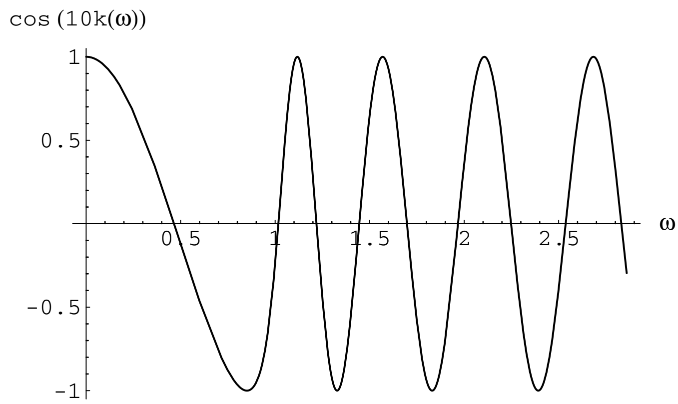

Figure 4. Function $\cos[k(\omega)x_1]$ in case $x_1 = 10$.

in case the deformation is measured at the center of the interval, i.e., when we have $v(x_1, t)$, where $x_1 = \frac{l}{2}$. Then from (5.4) we immediately obtain $\widehat{v}(x_1, \omega) = \frac{\widehat{g}_0(\omega)+\widehat{g}_l(\omega)}{2\cos[k(\omega)x_1]}$. This yields the relation

$$\cos[k(\omega)x_1] = \frac{\widehat{g}_0(\omega) + \widehat{g}_l(\omega)}{2\widehat{v}(x_1, \omega)}. \tag{6.4}$$

This is similar to equation (6.1) above. Only the right-hand side, that involves the data of the problem, is different. This means that we can continue the solution of the problem analogously to the case of model A extracting the function $k(\omega)$ from (6.4) and computing the data (ω_j, k_j) for the inverse problem for harmonic waves.

Further, let us consider a model that is mathematically the same as C, but physically a bit different. We consider a linear wave that propagates in a body between the planes $x = 0$ and $x = l$, but we either do not know the boundary conditions at 0 and l (e.g. boundaries are not accessible) or neglect them. Instead we measure the deformation in three points x_0, x_1 and x_2 inside the interval $(0, l)$ over the time t. Assume that $0 < x_0 < x_2 < l$ and $x_1 = \frac{x_0+x_2}{2}$. Clearly, in this case we can formulate a new boundary value problem in the subinterval (x_0, x_2) with given values of v at x_0 and x_2. Therefore, as in (6.4), we have the following equation for $k(\omega)$:

$$\cos[k(\omega)x_1] = \frac{\widehat{v}(x_0,\omega) + \widehat{v}(x_2,\omega)}{2\widehat{v}(x_1,\omega)} \tag{6.5}$$

that can be solved as in model A.

In some exceptional cases the equation for $k(\omega)$ is non-periodic. Let us consider the model B from the previous section. Suppose again that v is measured at some point x_1 over the time. Then from (5.3) we have $\mathrm{i}k(\omega)\,\mathrm{e}^{\mathrm{i}k(\omega)x_1} = \frac{\widehat{v}(x_1,\omega)}{\widehat{\varphi}(\omega)}$. Since $|\mathrm{e}^{\mathrm{i}z}| = 1$ for any $z \in \mathbb{R}$ and $k(\omega) > 0$ for $\omega > 0$, this yields the explicit formula for $k(\omega)$

$$k(\omega) = \frac{1}{x_1}\left|\frac{\widehat{v}(x,\omega)}{\widehat{\varphi}(\omega)}\right| \qquad \text{for} \quad \omega > 0. \tag{6.6}$$

No inversion of a periodic function is needed.

6.3. Generalization to piecewise homogeneous case

Let a body consist of homogeneous layers located between planes $x = l^0, x = l^1, x = l^2 \ldots, x = l^M$, where $0 = l^0 < l^1 < l^2 < \ldots < l^M = l$. In every layer $(l^{m-1}, l^m), m = 1, \ldots, m$, the material parameters are $a_0^m, a_1^m, \gamma^m, \alpha^m, \beta^m$ and δ^m. The dispersion function corresponding to a layer (l^{m-1}, l^m) is denoted by $k^m(\omega)$. We are interested in the identification of the parameters a_0^m, a_1^m, γ^m and the products $\alpha^m\beta^m$. In this problem we have $4M$ unknowns, hence additional information is necessary.

It seems natural to measure the deformation in single points of every layer over the time, i.e to provide $v(x_1^m, t)$ where $x_1^m \in (l^{m-1}, l^m), m = 1, \ldots, M$. Then, we have to specify additionally boundary conditions at $x = 0, x = l$ and transmission conditions at planes $x = l_m, m = 1, \ldots, M-1$ (i.e continuity of displacement and stress). This leads to a $M \times M$ system for functions $k^m(\omega)$ that contains periodic composite functions and has a structure that is much more complicated than equations (6.1) and (6.4) before. Therefore, from the practical viewpoint it makes sense to find easier methods. Moreover, the usage of transmission conditions is complicated in case the material has no rapid jumps but changes continuously in neighborhoods of the planes $x = l^m, m = 1, \ldots, M-1$.

Another, easier method comes form the problem with neglected boundary conditions described in subsection 6.2. This method requires measurements in three points of every layer. Namely, let we be given $v\left(x_0^m, t\right), v\left(x_1^m, t\right)$ and $v\left(x_2^m, t\right)$, where $l^{m-1} < x_0^m < x_2^m < l^m, x_1^m = \frac{x_0^m + x_2^m}{2}$ and $m = 1, \ldots, M$. We must not specify any boundary or transmission conditions. By (6.5) the following independent equations for the functions $k^m(\omega)$ are valid:

$$\cos\left[k^m(\omega)x_1^m\right] = \frac{\widehat{v}\left(x_0^m,\omega\right) + \widehat{v}\left(x_2^m,\omega\right)}{2\widehat{v}\left(x_1^m,\omega\right)}, \qquad m = 1, \ldots, M. \tag{6.7}$$

We extract the functions $k^m(\omega)$ from (6.7) as in the case of model A and compute the data $\left(\omega_j^m, k_j^m\right)$ for the inverse problems for harmonic waves in every layer.

7. Final remarks

The practical applications of the NDT need always a strong theoretical basis in order to avoid possible inaccuracies and misinterpretations. Here we have studied the inverse problems for microstructured solids based on the Mindlin model. We have demonstrated the solvability of inverse problems for harmonic waves and for wave packets. The physical restrictions imposed to the unknown parameters play decisive role. For example, in case of weak dispersion, the governing matrix is ill-conditioned and it may cause large computational errors in the solution. It has been shown how any additional information on the microstructure (i.e. on its parameters) could enhance the solvability of the inverse problem. Clearly, the concept of harmonic waves is very instructive theoretically, but in the NDT not easily realized. In this sense the idea of the spectral decomposition permits to use wave packets for the measurements.

The presented results can be immediately extended to the Mindlin-Hermann model for rods with governing equations [16] $u_{tt} - c_1^2 u_{xx} - l w_x = 0$, $w_{tt} - c_2^2 w_{xx} + m w + n u_x = 0$ where c_1, c_2, l, m, n are constants. Clearly, these equations are similar to (2.7).

Moreover, we guess that the method of the paper can be adjusted to various inverse coefficient problems of dispersive media provided the involved dispersion equation for k and ω is a polynomial. Indeed, the algorithm is quite general: extraction of the pairs (ω_j, k_j) from the spectrum of a wave, solution of a linear system for coefficients $\varkappa_m$ of the dispersion equation, and the computation of the coefficients of governing PDE(s) by means of $\varkappa_m$.

Acknowledgments

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Appendix. Proof of theorem

For any $\omega \in \mathbb{C}$ we define the following set of maximally four elements: $K(\omega) = \{k \in \mathbb{C} : \text{ k solves (3.4) for given } \omega\}$. Note that $K(\omega)$ depends on the coefficients $\varkappa_1, \ldots, \varkappa_4$ of equation (3.4), and in turn on $a_0, a_1, \gamma, \alpha, \beta$.

Let us start by proving a lemma.

Lemma 1. *Assume that $a_0, a_1, \gamma, \alpha, \beta$ satisfy $a_0\gamma - a_1\gamma - \alpha\beta \neq 0$ and $\varkappa_1, \ldots, \varkappa_4$ are given by (3.4) in terms of $a_0, a_1, \gamma, \alpha, \beta$. Moreover, let $\omega_1, \omega_2 \in \mathbb{C}, \omega_1, \omega_2 \neq 0$, and $k_j \in K(\omega_j)$, $j = 1, 2$. If $\omega_1^2 \neq \omega_2^2$ then the slownesses $s_j = \frac{k_j}{\omega_j}$ satisfy $s_1^2 \neq s_2^2$.*

Proof. According to the choice of k_j, the equations

$$\omega_j^4 + \varkappa_1 \omega_j^2 k_j^2 + \varkappa_2 k_j^4 + \varkappa_3 \omega_j^2 + \varkappa_4 k_j^2 = 0 \tag{A.1}$$

hold for $j = 1, 2$. Let $\omega_1^2 \neq \omega_2^2$. Suppose contrary that $s_1^2 = s_2^2 =: s^2$. Then, dividing (A.1) by ω_j^4 we get

$$1 + \varkappa_1 s^2 + \varkappa_2 s^4 + \frac{1}{\omega_j^2}(\varkappa_3 + \varkappa_4 s^2) = 0, \qquad j = 1, 2. \tag{A.2}$$

Subtracting these equations for $j = 1$ and $j = 2$ and observing that $\omega_1^2 \neq \omega_2^2$ we obtain $\varkappa_3 + \varkappa_4 s^2 = 0$. This together with (A.2) yields the equations $\varkappa_3 + \varkappa_4 s^2 = 0$ and $1 + \varkappa_1 s^2 + \varkappa_2 s^4 = 0$. Expressing s^2 from the first equation and substituting into the second one we get $1 - \varkappa_1 \frac{\varkappa_3}{\varkappa_4} + \varkappa_2 \left(\frac{\varkappa_3}{\varkappa_4}\right)^2 = 0$. Using here formulae (3.5) for $\varkappa_1, \ldots, \varkappa_4$ and simplifying we reach the relation $\frac{\alpha\beta}{(a_0\gamma - \alpha\beta)^2}(a_0\gamma - a_1\gamma - \alpha\beta) = 0$. But this cannot hold, because $\alpha\beta > 0$ and $a_0\gamma - a_1\gamma - \alpha\beta \neq 0$. Thus, the supposition $s_1^2 = s_2^2$ was not right. We have $s_1^2 \neq s_2^2$. Lemma is proved. □

It is convenient to prove the theorem in the following slightly more general form.

Lemma 2. *Let the assumptions of the theorem be valid for $a_0, a_1, \gamma, \alpha, \beta, \varkappa_1, \ldots, \varkappa_4$ and $\omega_j \in \mathbb{C}, \omega_j \neq 0$, $j = 1, \ldots, 4$, be such that ω_j^2, $j = 1, \ldots, 4$, are different. Let us choose* *some* $k_j \in K(\omega_j)$, $j = 1, \ldots, 4$. *By this construction,* $\varkappa_1, \ldots, \varkappa_4$ *solve* (4.1) *with the data* ω_j, k_j, $j = 1, \ldots, 4$. *If* $\widetilde{\varkappa}_1, \ldots, \widetilde{\varkappa}_4$ *solve* (4.1) *with the same data, then* $\widetilde{\varkappa}_j = \varkappa_j$, $j = 1, \ldots, 4$.

Proof. Dividing the systems (4.1) for the vectors $\varkappa_1, \ldots, \varkappa_4$ and $\widetilde{\varkappa}_1, \ldots, \widetilde{\varkappa}_4$ by ω_j^4 we get the following relations containing the slownesses $s_j = \frac{k_j}{\omega_j}$:

$$1 + \varkappa_1 s_j^2 + \varkappa_2 s_j^4 + \frac{1}{\omega_j^2}\left(\varkappa_3 + \varkappa_4 s_j^2\right) = 0, \qquad 1 + \widetilde{\varkappa}_1 s_j^2 + \widetilde{\varkappa}_2 s_j^4 + \frac{1}{\omega_j^2}\left(\widetilde{\varkappa}_3 + \widetilde{\varkappa}_4 s_j^2\right) = 0,$$

where $j = 1, \ldots, 4$. Let us eliminate ω_j from these expressions. To this end we multiply the left relations by $\widetilde{\varkappa}_3 + \widetilde{\varkappa}_4 s_j^2$ and the right relations by $\varkappa_3 + \varkappa_4 s_j^2$ and subtract. This results in the relations

$$(\varkappa_4\widetilde{\varkappa}_2 - \widetilde{\varkappa}_4\varkappa_2)s_j^6 + (\varkappa_3\widetilde{\varkappa}_2 - \widetilde{\varkappa}_3\varkappa_2 + \varkappa_4\widetilde{\varkappa}_1 - \widetilde{\varkappa}_4\varkappa_1)s_j^4 + (\varkappa_4 - \widetilde{\varkappa}_4 + \varkappa_3\widetilde{\varkappa}_1 - \widetilde{\varkappa}_3\varkappa_1)s_j^2 + \varkappa_3 - \widetilde{\varkappa}_3 = 0, \quad j = 1, \ldots, 4. \tag{A.3}$$

They show that s_j^2, $j = 1, \ldots, 4$, are the roots of the following cubic function:

$$f(\sigma) = (\varkappa_4\widetilde{\varkappa}_2 - \widetilde{\varkappa}_4\varkappa_2)\sigma^3 + (\varkappa_3\widetilde{\varkappa}_2 - \widetilde{\varkappa}_3\varkappa_2 + \varkappa_4\widetilde{\varkappa}_1 - \widetilde{\varkappa}_4\varkappa_1)\sigma^2 + (\varkappa_4 - \widetilde{\varkappa}_4 + \varkappa_3\widetilde{\varkappa}_1 - \widetilde{\varkappa}_3\varkappa_1)\sigma + \varkappa_3 - \widetilde{\varkappa}_3. \tag{A.4}$$

Since ω_j^2, $j = 1, \ldots, 4$, are different, lemma 1 implies that s_j^2, $j = 1, \ldots, 4$, are also different. Consequently, the cubic function (A.4) has four different roots, hence it is trivial. Setting the coefficients of (A.4) equal to zero, we after some transformations get the following 4×4 system for the vector $(\widetilde{\varkappa}_1 - \varkappa_1, \widetilde{\varkappa}_2 - \varkappa_2, \widetilde{\varkappa}_3 - \varkappa_3, \widetilde{\varkappa}_4 - \varkappa_4)$:

$$\begin{aligned}
&\widetilde{\varkappa}_3 - \varkappa_3 = 0\\
&\varkappa_3(\widetilde{\varkappa}_1 - \varkappa_1) - \varkappa_1(\widetilde{\varkappa}_3 - \varkappa_3) - (\widetilde{\varkappa}_4 - \varkappa_4) = 0\\
&\varkappa_4(\widetilde{\varkappa}_1 - \varkappa_1) + \varkappa_3(\widetilde{\varkappa}_2 - \varkappa_2) - \varkappa_2(\widetilde{\varkappa}_3 - \varkappa_3) - \varkappa_1(\widetilde{\varkappa}_4 - \varkappa_4) = 0\\
&\varkappa_4(\widetilde{\varkappa}_2 - \varkappa_2) - \varkappa_2(\widetilde{\varkappa}_4 - \varkappa_4) = 0.
\end{aligned}$$

The determinant of this system is

$$\Delta = -\varkappa_2\varkappa_3^2 - \varkappa_4^2 + \varkappa_1\varkappa_3\varkappa_4 = \frac{(a_0\gamma - a_1\gamma - \alpha\beta)\alpha\beta}{\delta^2} \neq 0,$$

because $\alpha\beta > 0$ and $a_0\gamma - a_1\gamma - \alpha\beta \neq 0$. This implies that the system has only the trivial solution. Consequently, $\widetilde{\varkappa}_1 = \varkappa_1$, $\widetilde{\varkappa}_2 = \varkappa_2$, $\widetilde{\varkappa}_3 = \varkappa_3$, $\widetilde{\varkappa}_4 = \varkappa_4$. This proves lemma. □

The theorem follows from lemma 2, because $k(\omega_j) \in K(\omega_j)$, $j = 1, \ldots, 4$.

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