Available online at www.sciencedirect.com
SCIENCE @ DIRECT•
ELSEVIER
Wave Motion 43 (2005) 1–11
WAVE MOTION
www.elsevier.com/locate/wavemoti

Waves in microstructured solids: Inverse problems ☆

J. Janno*, J. Engelbrecht

Department of Mechanics and Applied Mathematics, Institute of Cybernetics at Tallinn UT, Akadeemia Road 21, 12618 Tallinn, Estonia

Received 8 December 2004; received in revised form 12 April 2005; accepted 27 April 2005
Available online 20 June 2005

Abstract

Microstructured solids are characterized by their dispersive properties. Here inverse problems for such dispersive solids are analyzed based on the one-dimensional wave propagation. Governing equation of the Mindlin type model is considered which consists of higher order derivatives responsible for dispersive effects. Two cases are analyzed. The first deals with harmonic waves, the second – with a localized harmonic excitation. The dispersion relations are derived and then the corresponding inverse problems stated and solved. It is shown what information can be extracted from known (measured) phase and group velocities for determining the physical parameters. The results can be used in the nondestructive testing (NDT).

MSC: 74J25; 35R30; 74E15

Keywords: Microstructure; Dispersion; Phase velocity; Group velocity; Inverse problem

1. Introduction

Progress in materials science and technology needs also proper testing methods to evaluate the properties of materials [1]. This need is especially acute for microstructured materials where microstructural properties affect considerably the macrobehaviour of a compound material or a structure. In most general terms, microstructure means the existence of grains, inclusions, layers, block walls, etc., and the influence of anisotropy. There are powerful methods in continuum mechanics in order to describe the influence of such irregularities of media starting from early works of Cosserats and Voigt up to contemporary formulations [2–4]. Apart from the usual physical parameters characteristic to homogeneous materials, the theories of microstructured materials include a much larger number of parameters needed for describing the behaviour of such materials. Given the wide range of microstructure, see

☆ Work supported by Estonian Science Foundation (Grant No. 4706) and European Science Foundation programme NATEMIS.
* Corresponding author.
E-mail addresses: janno@ioc.ee (J. Janno); je@ioc.ee (J. Engelbrecht)

0165-2125/$ – see front matter
doi:10.1016/j.wavemoti.2005.04.006

for example [5–7], one needs also a wide range of experimental methods in order to determine those parameters. Among these methods [1], ultrasonic techniques play an important role. By measuring deformation wave velocities, times of flight of waves and/or amplitudes and attenuation of Fourier components for a given input (excitation), it is possible to solve the inverse problem where the unknowns are related to material parameters. In more general terms, an inverse problem is: given the external disturbances and the global behaviour, determine the properties of the material. Solutions to inverse problems form the backbone of the contemporary nondestructive testing (NDT). In this context, the mathematical models should be consistent, i.e. should reflect all the interesting physical phenomena with a needed accuracy [8]. On the other hand, experiments provide often only indirect data on physical phenomena and from the informational viewpoint the most important question is how to extract the information from measured signals.

In principle, wave fields are influenced by several properties of the material. Usually in homogeneous materials the attenuation plays an important role, especially in case of viscoelastic materials [9]. The distortion of wave profiles – amplitude decay – gives then sufficient information for solving inverse problems [10]. In case of microstructured materials one of basic phenomena influencing wave fields is dispersion of waves because every inclusion (irregularity, layer) becomes a scatterer [3,7,11]. In terms of NDT, the dispersion affects wave velocities which are then very informative and what is important, easily measured. Whether attenuation or dispersion is the governing effect or both should be taken into account, depends on concrete cases. In order to solve complicated problems, the model cases should be clearly understood. Here we concentrate our attention on dispersion but even in this case a consistent mathematical model is needed reflecting all the parameters necessary to describe this effect.

This paper deals with inverse problems for microstructured materials based on an early derived mathematical model [12,13] which is more general compared with others [14]. Unknown material parameters enter the governing displacement equation of the model as coefficients of lower and higher order derivatives. We will determine these parameters using measurements of wave velocities and amplitude changes or phase shifts. To the authors' knowledge, inverse problems for coefficients of higher order terms in dispersive wave equations have not been studied in the literature.

In Section 2, the physical problem is presented to describe leading dispersive effects in microstructured materials. Inverse problems are stated and solved in Sections 3 and 4. First, Section 3 deals with harmonic waves and then Section 4 is focused on a localized excitation which is more realistic in the NDT. A brief discussion is presented in Section 5.

2. Governing equation

The 1D model governing the longitudinal waves in a microstructured material was derived earlier [12] following the Mindlin theory [15]. Briefly the idea behind the model is following. The microstructure has been interpreted [15] "as a molecule of a polymer, a crystallite of a polycrystal or a grain of a granular material". This microelement is taken as a deformable cell. In 1D setting a cell might be interpreted also as a layer. The displacement $\mathbf{u}$ of a material particle in terms of macrostructure is defined by its components $u_i \equiv x_i - X_i$, where $x_i, X_i (i = 1, 2, 3)$ are the components of the spatial and material position vectors, respectively. Within each material volume (particle) there is a microvolume and the microdisplacement $\mathbf{u}'$ is defined by its components $u'_i \equiv x'_i - X'_i$, where the origin of the coordinates x'_i moves with the displacement $\mathbf{u}$. The displacement gradient is assumed to be small. This assumption permits to represent

$$u'_j = x'_k \varphi_{kj}(x_i, t) \tag{1}$$

and consequently

$$\frac{\partial u'_j}{\partial x'_i} = \partial'_i u'_j = \varphi_{ij}. \tag{2}$$

According to [2,15], the fundamental balance laws for microstructured materials should be formulated separately for macroscopic and microscopic scales. For the simplest 1D model we get then [12]:

$$\rho u_{tt} = \sigma_x, \tag{3}$$

$$I\varphi_{tt} = \eta_x - \tau, \tag{4}$$

where $u = u_{11}$ is the macrodisplacement, $\varphi = \varphi_{11}$ the microdeformation, $\sigma = \sigma_{11}$ the macrostress (Piola stress), $\eta = \eta_{11}$ the microstress, $\tau = \tau_1$ is the interactive microforce, ρ the macrodensity, and I is the microinertia and indices x and t denote the differentiation. Sign convention shows the direction of the microforce. As usual, we have

$$\sigma = \frac{\partial W}{\partial u_x}, \quad \eta = \frac{\partial W}{\partial \varphi_x}, \quad \tau = \frac{\partial W}{\partial \varphi}, \tag{5}$$

where W is the free energy. At this moment, we neglect the dissipation.

The simplest free energy function describing the influence of a microstructure is a quadratic function:

$$W = \frac{1}{2}\alpha u_x^2 + A\varphi u_x + \frac{1}{2}B\varphi^2 + \frac{1}{2}C\varphi_x^2, \tag{6}$$

with α, A, B, C are the constants.

For further analysis we introduce dimensionless variables and use the slaving principle for eliminating φ. Omitting the details (for those see [12]), the final governing equation in terms of the nondimensional macrodisplacement U reads:

$$U_{TT} = \left(1 - \frac{A^2}{\alpha B}\right) U_{XX} + \delta \frac{A^2}{B^2} \left(I^* U_{TT} - \frac{C^*}{\alpha} U_{XX}\right)_{XX}. \tag{7}$$

Here $\delta = l^2 L^{-2}$, l is the scale of the microstructure and L, for example, is the wavelength of the excitation, while X and T are nondimensional space and time respectively. The quantities I^* and C^* are determined by $I = \rho l^2 I^*$ and $C = l^2 C^*$.

Eq. (7) is a comparatively simple model but surprisingly rich. It reflects the following physical phenomena:

(i) it describes the wave hierarchy in Whitham's sense [16] including two wave operators – one for macrostructure, another for microstructure; if the scale parameter δ is small then the last two terms, i.e. influence of microstructure can be neglected; if δ is large then on contrary, the influence of first two terms, i.e. influence of macrostructure is weaker and the wave process is governed by the properties of the microstructure; clearly the intermediate case includes both effects;

(ii) the wave speed in the compound material is affected by the microstructure (1 versus $1 - A^2\alpha^{-1}B^{-1}$) and clearly only $A = 0$ excludes this dependence;

(iii) the influence of the microstructure on wave motion is, as expected, characterized by dispersive terms; however, the double dispersion occurs due to the different higher order terms (U_{TTXX} and U_{XXXX}).

In short, Eq. (7) is a continuum model of a conventional homogeneous material reflecting the properties of a heterogeneous material in sense of [15]. It has been shown [17] how this model can be compared with the full Mindlin type model and how this full model stands in terms of Euler–Lagrange equations and the pseudomomentum theory. The dispersion curves of Eq. (7) are compared with those of the initial system (3), (4) with (5) and (6) taken into account and the matching is good [17]. These results are not repeated here.

There are several other models based on specific assumptions in lattice theory ([14] and references therein), periodic structures [18,19], etc. Actually the full dispersion relations for periodic composites were first derived in [20]. In many cases, however, either only U_{XXXX} type [14,18] or only U_{TTXX} type [19] terms appear in the governing equation. Here clearly the inertia of the microstructure (term U_{TTXX}) and elastic properties of the microstructure (term U_{XXXX}) are both taken into account. The similar effect – double dispersion – is also important for describing strain waves in rods [21].

For the further analysis we rewrite Eq. (7) as

$$u_{tt} = (1-b)u_{xx} + \delta(\beta u_{ttxx} - \gamma u_{xxxx}), \tag{8}$$

where

$$b = \frac{A^2}{\alpha B}, \quad \beta = \frac{A^2}{B^2} I^*, \quad \gamma = C^* A^2 (\alpha B^2)^{-1} \tag{9}$$

and $u(x, t)$, x, t are dimensionless exactly as in Eq. (7). Obviously (see Eq. (6)) b, δ, β, γ are positive and $b < 1$. Eq. (8) takes into account dispersion due to the microstructure and it is caused by inertia of the microstructure (i.e. microelement, see above) by term u_{ttxx} and velocity in the microstructure by term u_{xxxx}. Assuming the parameters b, δ, β, γ are known, then solving Eq. (8) with proper initial and boundary conditions gives us the wavefield $u(x, t)$ depending on dispersive effects. On the contrary, if the wavefield $u(x, t)$ is given (i.e. measured) for a certain excitation then the determining the parameters b, δ, β, γ means solving an inverse problem. The crucial question is whether this is possible. If we consider the scale parameter δ to be known, then the number of unknown parameters reduces to three.

3. Inverse problem for harmonic waves

Assume that Eq. (8) has a solution in the form of harmonic waves

$$u(x, t) = \exp[\mathrm{i}(kx - \omega t)], \tag{10}$$

where $k \in \mathbf{R}$, $\omega \in \mathbf{R}$ are the wave number and frequency, respectively. In this case the dispersion relation is $\omega = \pm k c_{\mathrm{ph}}$ and the phase velocity c_{ph} is determined by

$$c_{\mathrm{ph}}(k) = \left(\frac{\delta\gamma k^2 + 1 - b}{\delta\beta k^2 + 1}\right)^{1/2}. \tag{11}$$

The waves propagating to the right (u_k^+) and to the left (u_k^-) are then

$$u_k^+ = \exp[\mathrm{i}k(x - c_{\mathrm{ph}}(k)t)], \tag{12}$$

$$u_k^- = \exp[\mathrm{i}k(x + c_{\mathrm{ph}}(k)t)]. \tag{13}$$

It is easily seen that

$$c_{\mathrm{ph}}(0) = (1-b)^{1/2}, \quad \lim_{k \to \pm\infty} c_{\mathrm{ph}}(k) = \left(\frac{\gamma}{\beta}\right)^{1/2}, \tag{14}$$

and $c_{\mathrm{ph}}(k)$ is an even function, i.e. $c_{\mathrm{ph}}(-k) = c_{\mathrm{ph}}(k)$.

In addition, c_{ph} is monotone in intervals $(-\infty, 0)$ and $(0, \infty)$. The following subcases can be distinguished:

(i) Let $1 - b > \gamma/\beta$. Then $c_{\mathrm{ph}}(k)$ is increasing if $k \in (-\infty, 0)$, has a maximum at $k = 0$ and is decreasing if $k \in (0, \infty)$.
(ii) Let $1 - b < \gamma/\beta$. Then $c_{\mathrm{ph}}(k)$ is decreasing if $k \in (-\infty, 0)$, has a minimum at $k = 0$ and is increasing if $k \in (0, \infty)$.
(iii) Let $1 - b = \gamma/\beta$. Then $c_{\mathrm{ph}}(k)$ is constant, and $c_{\mathrm{ph}} = c = (1-b)^{1/2} = (\gamma/\beta)^{1/2}$.

Let us pose the following *inverse problem*: given three phase velocities $c_{\rm ph}(k_1)$, $c_{\rm ph}(k_2)$ and $c_{\rm ph}(k_3)$ which correspond to wave numbers k_1, k_2 and k_3 such that $k_1^2 \neq k_2^2$, $k_1^2 \neq k_3^2$, $k_2^2 \neq k_3^2$, determine the parameters b, β, and γ.

Based on Eq. (11) this problem may be written in the form of system of nonlinear equations with three unknowns

$$c_{\rm ph}(k_j) = \left(\frac{\delta\gamma k_j^2 + 1 - b}{\delta\beta k_j^2 + 1}\right)^{1/2}, \qquad j = 1, 2, 3. \tag{15}$$

Suppose system (15) has two solutions (b_1, β_1, γ_1) and (b_2, β_2, γ_2). Then

$$\frac{\delta\gamma_1 k_j^2 + 1 - b_1}{\delta\beta_1 k_j^2 + 1} = \frac{\delta\gamma_2 k_j^2 + 1 - b_2}{\delta\beta_2 k_j^2 + 1}, \qquad j = 1, 2, 3. \tag{16}$$

After some algebra we can observe that k_j^2, $j = 1, 2, 3$ should satisfy the following quadratic equation

$$\delta^2(\gamma_1\beta_2 - \gamma_2\beta_1)k_j^4 + \delta[\beta_2(1 - b_1) - \beta_1(1 - b_2) + \gamma_1 - \gamma_2]k_j^2 + b_2 - b_1 = 0. \tag{17}$$

If at least one coefficient in this equation is nonzero, then Eq. (17) has maximally two different solutions. On the other hand, our problem has as a precondition three different solutions k_j^2, $j = 1, 2, 3$. Consequently, the coefficients of Eq. (17) must be zero

$$\gamma_1\beta_2 - \gamma_2\beta_1 = 0, \qquad \beta_2(1 - b_1) - \beta_1(1 - b_2) + \gamma_1 - \gamma_2 = 0, \qquad b_2 - b_1 = 0. \tag{18}$$

Immediately we have

$$b = b_1 = b_2. \tag{19}$$

In addition, we have

$$\frac{\gamma_1}{\beta_1} = \frac{\gamma_2}{\beta_2} =: z \tag{20}$$

and after some algebra:

$$\left(\frac{1 - b}{z} - 1\right)\left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right) = 0. \tag{21}$$

Equality (21) is satisfied either at

$$\frac{1}{\gamma_1} = \frac{1}{\gamma_2} \tag{22}$$

or at

$$\frac{1 - b}{z} - 1 = 0. \tag{23}$$

Equality (22) implies $\gamma_1 = \gamma_2$ and by Eq. (20) also $\beta_1 = \beta_2$. Consequently, in this case the coefficients b, γ, β are uniquely determined. Equality (23) implies $1 - b = z = \gamma_1/\beta_1 = \gamma_2/\beta_2$. This is the case (iii) above.

Summing up, we have proved the following statement:

Let (b_1, β_1, γ_1) and (b_2, β_2, γ_2) be two solutions of Eq. (15). Then either

$$b_1 = b_2 = b, \beta_1 = \beta_2 = \beta, \gamma_1 = \gamma_2 = \gamma$$

or

$$b_1 = b_2 = b, 1 - b = \frac{\gamma_1}{\beta_1} = \frac{\gamma_2}{\beta_2}$$

holds.

The realization of possibilities depends upon the given $c_{\mathrm{ph}}(k_j)$'s. By (i)–(iii) the function $c_{\mathrm{ph}}(k)$ is either strictly monotone for $k > 0$ or constant. The latter case occurs only when $1 - b = \gamma/\beta$. Consequently, if the data of the inverse problem satisfy the inequalities $c_{\mathrm{ph}}(k_1) \neq c_{\mathrm{ph}}(k_2), c_{\mathrm{ph}}(k_2) \neq c_{\mathrm{ph}}(k_3), c_{\mathrm{ph}}(k_1) \neq c_{\mathrm{ph}}(k_3)$, then $1 - b \neq \gamma/\beta$ and all three coefficients b, β, γ are uniquely determined. If all $c_{\mathrm{ph}}(k_j)$'s are equal then the second possibility is realized with $1 - b = \gamma/\beta$. Coefficient b and the ratio γ/β are uniquely determined. The cases when $c_{\mathrm{ph}}(k_j)$'s are pair-wise equal, are impossible.

Finally we note that system (15) can be transformed to the linear system

$$b + \delta k_j^2 c_{\mathrm{ph}}^2(k_j)\beta - \delta k_j^2 \gamma = 1 - c_{\mathrm{ph}}^2(k_j), \quad j = 1, 2, 3 \tag{24}$$

for b, β, γ which is more suitable for practical solution.

4. Inverse problem for a localized boundary condition

4.1. General solution

We solve Eq. (8) for $x > 0, -\infty < t < \infty$, satisfying the boundary condition $u(0, t) = g(t)$. A special solution of Eq. (8) in the Fourier space has the form

$$\hat{u}(x, \omega) = \exp(\mathrm{i}kx), \tag{25}$$

where k must satisfy the characteristic equation

$$-\omega^2 + (1 - b)k^2 - \delta(\beta\omega^2 k^2 - \gamma k^4) = 0 \tag{26}$$

This equation has two real and two imaginary solutions. We neglect the imaginary solutions and pick up only one real solution corresponding to the wave propagating to the right. In this case we have

$$k = k(\omega) = \omega[((Q^2 + 4\delta\omega^2)^{1/2} - Q)(2\delta\gamma\omega^2)^{-1}]^{1/2}, \quad Q = 1 - b - \delta\beta\omega^2. \tag{27}$$

The sought solution corresponding to these conditions is

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) \exp[\mathrm{i}(k(\omega)x - \omega t)]\, \mathrm{d}\omega, \tag{28}$$

where

$$G(\omega) = \int_{-\infty}^{\infty} g(t) \exp(\mathrm{i}\omega t)\, \mathrm{d}t. \tag{29}$$

4.2. Solution for a localized boundary condition

We specify now $g(t)$ assuming

$$g(t) = A \exp\left(-\frac{t^2}{4\mu^2}\right) \exp(-\mathrm{i}\eta t) \tag{30}$$

for $-\infty < t < \infty$. Here $A > 0$, $\mu > 0$ are given and $\eta \in \mathbf{R}$ is the fixed frequency. To evaluate the integrals (28) and (29) we are going to use the ideas described in ([22], Section 12.5). Substituting (30) into (29), we obtain

$$G(\omega) = 2A\mu\sqrt{\pi}\,\exp[-\mu^2(\omega-\eta)^2]. \tag{31}$$

Combining (28) and (31), the sought solution is

$$u(x,t) = \frac{A\mu}{\sqrt{\pi}}\int_{-\infty}^{\infty}\exp[-\mu^2(\omega-\eta)^2]\,\exp[\mathrm{i}(k(\omega)x-\omega t)]\,\mathrm{d}\omega. \tag{32}$$

As far as $k(\omega)$ expressed by (27) is a complicated function, the integral in (32) can be evaluated only as an approximation. Observing that integrand in (32) is rapidly decreasing with increasing $|\omega-\eta|$ we derive $k(\omega)$ into the Taylor series around $\omega = \eta$. Keeping three first terms, we have

$$k(\omega) \approx k(\eta) + k'(\eta)(\omega-\eta) + \frac{1}{2}k''(\eta)(\omega-\eta)^2, \tag{33}$$

where prime denotes differentiation. From the definition of the phase velocity c_{ph} we determine

$$k(\eta) = \frac{\eta}{c_{\mathrm{ph}}}. \tag{34}$$

Next, from the definition of the group velocity $c_{\mathrm{g}} = \omega'(k)|_{k=k(\eta)} = 1/k'(\eta)$ we obtain

$$k'(\eta) = \frac{1}{c_{\mathrm{g}}}. \tag{35}$$

Denoting $\mathrm{d} = \frac{1}{2}k''(\eta)$ and using (34), (35), expression (32) can be written

$$k(\omega) \approx \frac{\eta}{c_{\mathrm{ph}}} + \frac{\omega-\eta}{c_{\mathrm{g}}} + d(\omega-\eta)^2. \tag{36}$$

Substituting (36) into (32), we get after integration the approximation $\tilde{u}$ for u of the form:

$$\tilde{u}(x,t) = A\mu(\mu^2 - \mathrm{i}xd)^{-1/2}\exp(-f(x,t))\exp\left[\mathrm{i}\eta\left(\frac{x}{c_{\mathrm{ph}}} - t\right)\right], \tag{37}$$

where

$$f(x,t) = \frac{1}{4}\left(\frac{x}{c_{\mathrm{g}}-t}\right)^2(\mu^2 - \mathrm{i}xd)^{-1}.$$

For practical purposes we need the real part of the solution. After tedious operations the final result reads

$$\operatorname{Re}\tilde{u}(x,t) = A_1(x)\exp[-\mu^2 f_1(x,t)]\cos\left[\eta\left(\frac{x}{c_{\mathrm{ph}}} - t\right) + \Phi(x) - x\,\mathrm{d}f_1(x,t)\right], \tag{38}$$

$$A_1(x) = A\mu(\mu^4 + x^2d^2)^{-1/4}, \tag{39}$$

$$f_1(x,t) = \frac{1}{4}\left(\frac{x}{c_{\mathrm{g}}} - t\right)^2(\mu^4 + x^2d^2)^{-1}, \tag{40}$$

$$\Phi(x) = \frac{\arctan\frac{1}{2}xd}{\mu^2}. \tag{41}$$

Clearly the amplitude is decreasing with increasing x (expression (39)) and the dispersion of the normal distribution in (30) is increasing

$$\mu_1(x) = \mu \left(\frac{1 + x^2 d^2}{\mu^4} \right)^{\frac{1}{2}}. \tag{42}$$

Both processes depend on the value of d. The harmonic part in (38) is influenced by the phase shift $\Phi(x)$ and a frequency changing term $x\,\mathrm{d} f_1(x,t)$. The latter term shows increasing the frequency if t departs from $t = x/c_g$. This actually shows that the accuracy of the approximation (33) is sufficient close to $t = x/c_g$ where the absolute value of $\operatorname{Re} \tilde{u}(x,t)$ is large enough.

4.3. Posing and solving the inverse problem

Let us pose the following *inverse problem*: given the phase and group velocities c_{ph}, c_g and the number d, determine the parameters b, β, γ. Here d can be obtained either from (39) using measured amplitude $A_1(x)$ at some $x > 0$ or from (41) using measured phase shift at some $x > 0$.

Remind that (see above):

$$k(\eta) = \frac{\eta}{c_{ph}}, \qquad k'(\eta) = \frac{1}{c_g}, \qquad k''(\eta) = 2d. \tag{43}$$

Further, instead of $k(\omega)$ its inverse function $\omega(k)$ will be used because it is easier to determine the derivatives. In this case

$$\omega'(k) = \frac{1}{k'(\omega)}, \qquad \omega''(k) = \frac{-k''(\omega)}{[k'(\omega)]^3}. \tag{44}$$

Suppose m is the wave number corresponding to the frequency η. Then

$$m = \frac{\eta}{c_{ph}}, \tag{45}$$

and

$$\omega(m) = \eta, \qquad \omega'(m) = c_g, \qquad \omega''(m) = -2dc_g^3. \tag{46}$$

Now we use (11) to calculate the derivatives and deduce from (45) and (46) the following system of equations for b, β, γ:

$$m c_{ph}(m) = \eta, \tag{47}$$

$$c_{ph}(m) + \frac{\delta m^2}{c_{ph}(m)} f_2(m) = c_g, \tag{48}$$

$$f_2(m) \left[\frac{\delta m}{c_{ph}(m)^2} \left(c_{ph}(m) - \frac{\delta m^2}{c_{ph}(m)} f_2(m) \right) + \frac{2\delta m}{c_{ph}(m)} \frac{1 - \delta\beta m^2}{\delta\beta m^2 + 1} \right] = -2dc_g^3, \tag{49}$$

where m is given by (45):

$$f_2(m) = \frac{\gamma - \beta(1-b)}{(\delta\beta m^2 + 1)^2} \quad \text{and} \quad c_{ph}(m) = \left(\frac{\delta\gamma m^2 + 1 - b}{\delta\beta m^2 + 1} \right)^{1/2}. \tag{50}$$

We can simplify the system (47)–(49) substituting c_g for $c_{ph}(m) + \frac{\delta m^2}{c_{ph}(m)} f_2(m)$ in Eq. (49) and η/m for $c_{ph}(m)$ in (48) and (49). After these simplifications it is not difficult to solve the system. This leads us to the following result:

(i) in case $c_g \neq c_{ph}$ all parameters b, β, γ are uniquely determined by the formulas

$$\beta = \frac{1}{\delta m^2}(4F(m) - 1), \tag{51}$$

$$\gamma = \frac{c_{ph}}{\delta m^2}(4c_g F(m) - c_{ph}), \tag{52}$$

$$b = 1 + c_{ph}[4(c_g - c_{ph})F(m) - c_{ph}] \tag{53}$$

with

$$F(m) = \left[\frac{c_g}{c_{ph}} - \frac{2dmc_g^3}{c_g - c_{ph}}\right]^{-1}; \tag{54}$$

remark that in order to get positive β, the condition $0 < F^{-1}(m) < 4$ must be satisfied;

(ii) in case $c_g = c_{ph}$ necessarily $d = 0$; then only b and the ratio γ/β are uniquely determined by the formulas

$$\frac{\gamma}{\beta} = c_{ph}^2, \quad b = 1 - c_{ph}^2. \tag{55}$$

The case (ii), although being mathematically correct, is not possible physically in a dispersive medium.

5. Discussion

Our attention above was focused on proving the possibility to solve the inverse problem of establishing the material parameters of microstructured solids from measuring the wave velocities. As shown by experimental studies [1], these measurements are rather easily realized and their accuracy is high enough to make the conclusions. However, the existence and uniqueness of the solution of an inverse problem and the accuracy of the realization should also be established correctly.

There are two main points to be discussed. First, the place of the mathematical model used with respect to other models and second, what actually is possible to determine.

The mathematical model used (Eq. (8)) is based on the Mindlin theory [15] that takes the deformation of microstructure into account. If microstructure (a unit cell of [15]) is rigid, then Mindlin's equations reduce to those of a Cosserat material. In our case, the model derived form Eqs. (3) and (4) leads to the dispersion curves like in [15], where optical and acoustic branches are separated. The approximated governing Eq. (8) corresponds to the acoustic branch. The comparative analysis of the exact and approximate dispersion relations has shown a good matching [17]. Clearly, the *convexity* of *dispersion* curve is a characteristic feature like in [15,20] and here.

Leaving aside the lattice theory, a special case of periodic continua is of importance. This is the case of a periodic microstructure, like a periodic composite. The pioneering studies [20] have demonstrated how an "effective stiffness theory" can be elaborated assuming that the displacements in each layer of a laminated body are described by Legendre polynomials. Again, the comparison with exact dispersion curves has shown the good accuracy. The more recent study [19] has explicitly shown the role of microinertia in governing equations and dispersion relations due to the effective body force term in macro equations of motion. The mathematical model used above includes also the effect of microinertia but characteristically to the Mindlin's theory [15], the elastic properties of the microstructure are included as well. From another viewpoint, the attention could involve only the elastic properties of periodic structure [18]. In all these cases the curvature of the dispersion curves gives the evidence of the internal structure (either periodic or non-periodic). The numerical methods allow to inscribe the material properties to every computational cell (every layer) and are widely used. The numerical simulation of waves in periodic structures (composites) has clearly demonstrated the influence of layering [18,19,23,24]. In nonperiodic microstructured materials the velocity

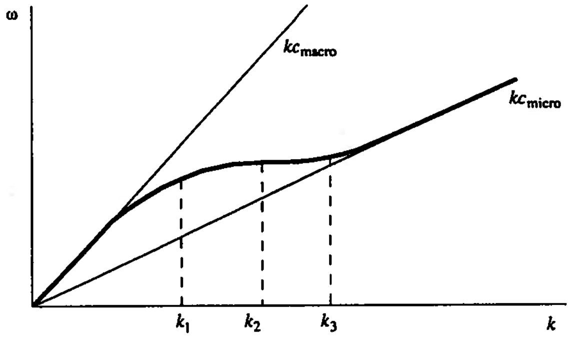

Fig. 1. Qualitative dispersion curve for Eq. (8).

dependencies on internal structure are also clearly seen, as shown in [13,17,24,25]. In [25], for example, the waves in compound material are studied for a 2D case with randomly embedded ceramic particles in a metal matrix with prescribed volume fraction (see also [17]).

The main aim of this paper is to understand what kind of information can be extracted from the velocity measurements in microstructured materials. The inverse problems analysed above give an answer to the following question: given the mathematical model (Eq. (8)), what should be with its unknown coefficients measured in order to establish the values of these coefficients and then, back to proper material parameters in the free energy function (6). As follows form the analysis, three physical parameters β, γ, b of Eq. (8) are to be determined. The fourth parameter in Eq. (8) is the geometrical parameter δ and this is assumed to be known. If this is not possible then we have to deal with $\delta\beta$ and $\delta\gamma$ as targets.

In case of harmonic waves (Section 3), the solution of the inverse problem needs knowing of three different phase velocities. Indeed, the dispersion relation $\omega = kc_{\mathrm{ph}}$ with c_{ph} determined by (11) demonstrates the changes due to smaller and larger influence of the microstructure (see Figs. 1 and 2). The behaviour of c_{ph} (Fig. 2) gives also a hint, how to choose k_i's for more effective measurements – one $c_{\mathrm{ph}}(k_i)$ should be measured between c_{macro} and c_{macro}, other two closer to c_{macro} and c_{macro}, respectively. Here c_{macro} and c_{macro} denote the velocities in pure macro- and microstructure, respectively.

The localized harmonic excitation (more realistic for experimental studies) leads to much more complicated solution (38) involving amplitude changes, localization changes, phase shift and also frequency–shift. However, in this case also phase and group velocities are informative, but in addition, the rate of changing of the group velocity (denoted by d) is used. The latter can be determined either from the amplitude change or the phase–shift. Again from three measured quantities three parameters can be evaluated.

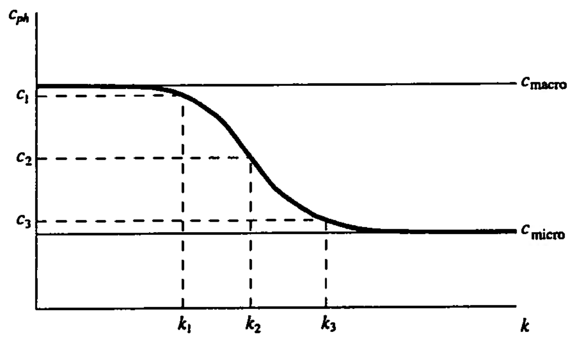

Fig. 2. Qualitative behaviour of phase velocities.

Returning to the dimensional coefficients, it is clear that our analysis should involve knowledge from the fundamental elastic properties as a prerequisite. Indeed, remind that

$$b = \frac{A^2}{\alpha B^2}, \qquad \beta = \delta \frac{A^2}{B^2} I^*, \qquad \gamma = \delta \frac{A^2}{b^2} \frac{C^*}{\alpha}. \tag{56}$$

If we known α, C (i.e. C^*) and δ, then the coefficients A, B, I can be determined. The present study has proved the possibility of solving the inverse problems stated in Sections 3 and 4. For practical purposes in NDT, the accuracy of such an approach should be analyzed in detail starting from the analysis of the accuracy of measurements. This work is in progress.

Finally, the present study is based on the 1D geometry. This is a first step in generalizing the governing equations for more complicated cases like in [15] bearing in mind nonperiodic microstructures, typical for functionally graded materials (cf. numerical simulation in [25]).

Acknowledgement

Authors thank Prof. A. Salupere for a very useful discussion.

References

[1] V. Hauk, Structural and Residual Stress Analysis by Nondestructive Methods, Elsevier, Amsterdam, 1997.
[2] A.C. Eringen, Microcontinuum Field Theories. Foundations and Solids, Springer, New York, 1999.
[3] G.A. Maugin, Material Inhomogeneities in Elasticity, Chapman & Hall, London, 1993.
[4] P. Suquet (Ed.), Continuum Micromechanics, Springer, Vienna, 1997.
[5] G.G. Sih (Ed.), Mesomechanics, vol. I, Tsinghua University, Beijing, 2000.
[6] S. Suresh, A. Mortensen, Fundamentals of Functionally Graded Materials, IOM Communications, Ltd., London, 1998.
[7] R. Phillips, Crystals, Defects and Microstructures. Modeling Across Scales, Cambridge University Press, Cambridge, 2001.
[8] J. Engelbrecht, A. Ravasoo, From continuum mechanics to applications in the nondestructive testing, Bull. Techn. Univ. Istanbul 47 (1994) 83–103 (Special issue Suhubi and Continuum Mechanics, E. Ihan (Ed.)).
[9] J.W. Nunziato, E.K. Welsch, K.W. Schuler, L.M. Barker, Wave Propagation in Nonlinear Viscoelastic Solids Handbuch der Physik B, VI/4, Springer, Berlin, 1970, pp. 1–108.
[10] A. Ravasoo, J. Janno, Nondestructive characterization of materials with variable properties, Acta Mechanica 151 (2001) 217–233.
[11] F. Ziegler, Wave propagation in periodic and disordered layered composite elastic materials, Int. J. Solids Struct. 13 (1977) 293–305.
[12] J. Engelbrecht, F. Pastrone, Waves in microstructured solids with strong nonlinearities in microscale, Proc. Estonian Acad. Sci. Phys. Math. 52 (2003) 12–20.
[13] J. Engelbrecht, F. Pastrone, A. Berezovski, Wave propagation in microstructured solids, Proceedings of the WCU 2003, Paris, CD-ROM, 2003, pp. 495–497.
[14] G.A. Maugin, Nonlinear Waves in Elastic Crystals, Oxford University Press, Oxford, 1999.
[15] R.D. Mindlin, Micro-structure in linear elasticity, Arch. Rat. Mech. Anal. 16 (1964) 51–78.
[16] G.B. Whitham, Linear and Nonlinear Waves, Wiley, New York, 1974.
[17] J. Engelbrecht, A. Berezovski, F. Pastrone, M. Braun, Waves in microstructured materials and dispersion, Phil. Mag. in press.
[18] F. Santosa, W.W. Symes, A dispersive effective medium for wave propagation in periodic composities, SIAM J. Appl. Math. 51 (1991) 984–1005.
[19] Z.-P. Wang, C.T. Sun, Modelling micro-inertia in heterogeneous materials under dynamic loading, Wave Motion 36 (2002) 473–485.
[20] C.T. Sun, J.D. Achenbach, G. Herrmann, Continuum theory for a laminated medium, J. Appl. Mech. 35 (1968) 467–475.
[21] A.M. Samsonov, Strain Solitons in Solids and How to Construct Them, Chapman & Hall, London, 2001.
[22] W.C. Elmore, M.A. Heald, Physics of Waves, Dover Publications, Inc., New York, 1969.
[23] S.L. Weekes, Numerical computation of wave propagation in dynamic materials, Appl. Num. Math. 37 (2001) 417–440.
[24] T. Fogarty, R.J. Leveque, High-resolution on finite volume methods for acoustics in perodic or random media, J. Acoust. Soc. Am. 106 (1999) 17–28.
[25] A. Berezovski, J. Engelbrecht, G.A. Maugin, Numerical simulation of two-dimensional wave propagation in functionally graded materials, Eur. J. Mech. A Solids 22 (2003) 257–265.