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An inverse solitary wave problem related to microstructured materials

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Abstract

An inverse problem for determining coefficients of a one-dimensional wave equation of nonlinear microstructured material is considered. The solution of the problem is based on measurements gathered from two independent solitary waves. Uniqueness for the inverse problem is proved and a stability estimate is derived.

1. Introduction and description of model

According to Mindlin [1], a material is interpreted as an elastic continuum including microstructure that could be 'a molecule of a polymer, a crystallite of a polycrystal or a grain of a granular material'. This microstructure is modelled by microelements within the macrostructure. Microstructured materials are gaining a growing importance in contemporary high technology [2, 3] because combining the mechanical properties of different constituencies as in functionally graded materials or composites yields better (optimal) properties of solids. In addition, it opens new possibilities for designing materials with certain predetermined properties. Besides materials science, such modelling can also be used in geophysics for describing waves in the earth's crust [4].

The utilization of microstructured materials requires proper methods of nondestructive testing (NDT) of these materials. Since the microstructural properties considerably affect the macro-behaviour of a material [5–8], macro-scale wave processes could be informative in NDT. For instance, the NDT problem in a linear setting is studied using harmonic waves and Gaussian wave packets [5].

The microstructure introduces dispersive effects in wave propagation. In the case where the dispersion is balanced by nonlinearity, solitary waves may emerge. This has been proved both theoretically [6–9] and empirically [4, 10, 11]. Experimental studies show the possibility of exciting and measuring solitary waves in the laboratory. Solitary waves have been shown to exist in plexiglas [10] and also in tungsten-epoxy composites with reference samples made of pure aluminium [11]. In both the mentioned papers the experimentally generated solitary waves are asymmetric; in [11] the asymmetry is clearly pointed out and discussed.

The physical and geometrical properties of solitary waves, e.g. velocity, amplitude, asymmetry, contain information about the physical properties of the material. This encourages the usage of such waves in NDT.

In this paper we pose and study an inverse problem to identify the coefficients of a wave equation of nonlinear microstructured material. These coefficients are related to material properties. To recover the coefficients we use information gathered from two independent solitary waves. We prove the uniqueness and stability for the posed inverse problem. To the authors' knowledge, such an inverse problem to determine material properties using measurements of asymmetric solitary waves is posed and studied for the first time. However, it is based on the evidence of such waves in earlier studies [6, 11–13].

Let us describe the underlying mathematical model of the problem. Here we follow Mindlin's ideas [1] and use the hierarchical approach by Engelbrecht and Pastrone [14]. The basic one-dimensional model for longitudinal waves in materials possessing microstructure, is

$$\rho u_{tt} = \sigma_x, \qquad I\psi_{tt} = \eta_x - \tau, \tag{1}$$

where u is the macrodisplacement, ψ is the microdeformation, ρ and I are the macrodensity and microinertia, respectively, σ is the macrostress, η is the microstress and τ is the interactive force. We suppose that the material is physically nonlinear, although we have kept linear deformation and material coordinates x. We neglect possible dissipation and consider only dispersive and nonlinear effects.

To derive a system for u and ψ we assume that the free energy function W has the form $W = W_2 + W_3$, where W_2 is the simplest quadratic function

$$W_2 = \tfrac{1}{2}au_x^2 + \tfrac{1}{2}B\psi^2 + \tfrac{1}{2}C\psi_x^2 + D\psi u_x$$

and W_3 includes nonlinearities on both the macro- and microlevel

$$W_3 = \tfrac{1}{6}Nu_x^3 + \tfrac{1}{6}M\psi_x^3.$$

Here a, B, C, D, N and M are constants. The non-quadratic potential W_3 is the first approximation towards nonlinear theory. This is needed for describing the possible balance of nonlinear and dispersive effects that has been demonstrated in measurements [11]. The quantitative estimations for the parameters of nonlinear terms in W_3 are, for example, presented in [15] for polycrystalline materials.

Using the introduced representation of W and formulae

$$\sigma = \frac{\partial W}{\partial u_x}, \qquad \eta = \frac{\partial W}{\partial \psi_x}, \qquad \tau = \frac{\partial W}{\partial \psi},$$

we can rewrite system (1) in terms of u and ψ (cf [9, 14, 16]):

$$\rho u_{tt} = au_{xx} + Nu_x u_{xx} + D\psi_x, \tag{2}$$

$$I\psi_{tt} = C\psi_{xx} + M\psi_x\psi_{xx} - Du_x - B\psi. \tag{3}$$

Let us rewrite this system in dimensionless variables $X = \frac{x}{L}$, $T = \frac{tc_0}{L}$, $U = \frac{u}{U_0}$, where U_0 and L are certain constants and $c_0^2 = \frac{a}{\rho}$. Introducing the geometric parameters $\delta = \frac{l^2}{L^2}$, $\epsilon = \frac{U_o}{L}$, where l is the scale of the microstructure l, this system reads

$$U_{TT} = U_{XX} + \frac{N\epsilon}{\rho c_0^2}U_X U_{XX} + \frac{D}{\rho c_0^2 \epsilon}\psi_X, \tag{4}$$

$$\delta a I^* \psi_{TT} = \delta C^* \psi_{XX} + \delta^{3/2} M^* \psi_X \psi_{XX} - D\epsilon U_X - B\psi, \tag{5}$$

where $I = I^*\rho l^2$, $C = C^* l^2$ and $M = M^* l^3$.

In order to eliminate the microdeformation ψ from (4), (5) we make use of the slaving principle (cf [6, 14, 16]). This results in the following hierarchical governing equation for U:

$$U_{TT} = bU_{XX} + \frac{\mu}{2}\left(U_X^2\right)_X + \delta\left(\beta U_{TT} - \gamma U_{XX}\right)_{XX} - \delta^{3/2}\frac{\lambda}{2}\left(U_{XX}^2\right)_{XX}, \tag{6}$$

where

$$b = 1 - \frac{D^2}{aB}, \qquad \mu = \frac{N\epsilon}{a}, \qquad \beta = \frac{D^2 I^*}{B^2}, \qquad \gamma = \frac{D^2 C^*}{aB^2}, \qquad \lambda = \frac{D^3 M^* \epsilon}{aB^3}. \tag{7}$$

The inequalities

$$0 < b < 1, \qquad \delta, \beta, \gamma > 0 \tag{8}$$

are valid for the coefficients b, δ, β and γ. Equation (6) involves hierarchically two wave operators $U_{TT} - bU_{XX} - \frac{\mu}{2}\left(U_X^2\right)_X$ and $\delta\left(\beta U_{TT} - \gamma U_{XX} - \delta^{1/2}\frac{\lambda}{2}U_{XX}^2\right)_{XX}$ characteristic to the macro- and microstructure, respectively. The influence of the macro- and microstructure on the wave propagation depends on the size of the scale parameter δ [5, 14].

For future analysis, we rewrite (6) by means of lower-case letters:

$$u_{tt} = bu_{xx} + \frac{\mu}{2}\left(u_x^2\right)_x + \delta\left(\beta u_{tt} - \gamma u_{xx}\right)_{xx} - \delta^{3/2}\frac{\lambda}{2}\left(u_{xx}^2\right)_{xx}. \tag{9}$$

The related equation for the deformation $v = u_x$ reads

$$v_{tt} = bv_{xx} + \frac{\mu}{2}(v^2)_{xx} + \delta\left(\beta v_{tt} - \gamma v_{xx}\right)_{xx} - \delta^{3/2}\frac{\lambda}{2}\left(v_x^2\right)_{xxx}. \tag{10}$$

Our aim is to identify the five coefficients b, μ, β, γ and λ in equation (10) that are related to material properties. To this end we use measurements gathered from solitary waves. The quantity δ is assumed to be known. Note that from the mathematical point of view we could set $\delta = 1$ redefining in a suitable way β, γ and λ in (10). But from the physical point of view, it makes sense to preserve δ in our computations to show the scale-dependence.

2. Existence of solitary waves: formulation of the inverse problem

Travelling wave solutions of (10) have the form $v(x,t) = w(x - ct)$ where c is the velocity of the wave and $w = w(\xi)$ satisfies the equation

$$(c^2 - b)w'' - \frac{\mu}{2}(w^2)'' - \delta(\beta c^2 - \gamma)w^{IV} + \delta^{3/2}\frac{\lambda}{2}[(w')^2]''' = 0. \tag{11}$$

We interpret equation (11) in the classical sense requiring the solution to be four times continuously differentiable. Moreover, our interest is focused on solitary wave solutions vanishing at infinity, Therefore, define the following space for the solution of (12):

$$\mathcal{W} = \{w : w \not\equiv 0;\ w^{(i)} - \text{continuous in } \mathbb{R}, i = 0, \ldots, 4;\ \lim_{|\xi|\to\infty} w^{(i)}(\xi) = 0, i = 0, 1, 2\}.$$

The existence and some properties of the solitary wave solutions of (11) were proved in [9] and also, in the case $\lambda = 0$, in [7]. We summarize and complement these results in the form of lemma 1 and theorem 1 below.

Lemma 1. *If (11) has a solution in* $\mathcal{W}$ *then* $\mu \neq 0$, $\beta c^2 - \gamma \neq 0$, $c^2 - b \neq 0$ *and* $\frac{c^2-b}{\beta c^2-\gamma} > 0$. *Moreover, equation (11) in* $\mathcal{W}$ *is equivalent to the following equation of the first order*

$$(w')^2 - \alpha(w')^3 = \kappa^2 w^2\left(1 - \frac{w}{A}\right), \tag{12}$$

where

$$\kappa = \sqrt{\frac{c^2 - b}{\delta(\beta c^2 - \gamma)}}, \qquad A = \frac{3(c^2 - b)}{\mu}, \qquad \alpha = \frac{2\delta^{1/2}\lambda}{3(\beta c^2 - \gamma)}. \tag{13}$$

Proof. Lemma 1 was completely proved in section 2 of [9]. However, for the reader's convenience we repeat the proof of equivalence of (11) and (12) here. Integrating twice (11) we have

$$(c^2 - b)w - \frac{\mu}{2}w^2 - \delta(\beta c^2 - \gamma)w'' + \delta^{3/2}\frac{\lambda}{2}[(w')^2]' = C_1\xi + C_2, \tag{14}$$

where C_1 and C_2 are arbitrary constants. In view of the asymptotical conditions for w in $\mathcal{W}$ we have $C_1 = C_2 = 0$. Therefore, (11) is equivalent to the equation of the second order

$$(c^2 - b)w - \frac{\mu}{2}w^2 - \delta(\beta c^2 - \gamma)w'' + \delta^{3/2}\lambda w'w'' = 0. \tag{15}$$

Multiplying this equation by w' and integrating again we obtain

$$\frac{c^2 - b}{2}w^2 - \frac{\mu}{6}w^3 - \frac{\delta(\beta c^2 - \gamma)}{2}(w')^2 + \frac{\delta^{3/2}\lambda}{3}(w')^3 + C = 0 \tag{16}$$

with an arbitrary constant C. Again, due to the asymptotical conditions for w in $\mathcal{W}$ we have $C = 0$. Thus, (16) is equivalent to (12) with (13). □

Theorem 1. *Let $\mu \neq 0$, $\beta c^2 - \gamma \neq 0$ and $c^2 - b \neq 0$. Equation (11) has a solution in $\mathcal{W}$ if and only if*

$$\left(\frac{\beta c^2 - \gamma}{c^2 - b}\right)^3 > \frac{4\lambda^2}{\mu^2} \qquad (\textit{equivalently, } |A\alpha\kappa| < 1)\,. \tag{17}$$

In case (17) holds, the set of all solutions in $\mathcal{W}$ has the form $\{w_C(\xi) = w_0(\xi + C) : C \in \mathbb{R}\}$, where $w = w_0 \in \mathcal{W}$ is an infinitely differentiable function in $\mathbb{R}$, which has the following properties:

- *(a)* $\ln|w(\xi)| \sim -\kappa|\xi|$ *as* $|\xi| \to \infty$;
- *(b)* $A^{-1}w(\xi) \in (0, 1)$ *if* $\xi \neq 0$ *and* $w(0) = A$, *i.e.,* A *is the amplitude;*
- *(c)* $Aw'(\xi) > 0$ *if* $\xi < 0$, $Aw'(\xi) < 0$ *if* $\xi > 0$ *and* $w'(0) = 0$; w' *has exactly two relative extrema occurring at* $\xi = \xi^- < 0$ *and* $\xi = \xi^+ > 0$ *such that* $w(\xi^-) = w(\xi^+) = \frac{2A}{3}$;
- *(d)* $|w'(\xi)| < \frac{2|A|\kappa}{3} < \frac{2}{3|\alpha|}$ *for* $\xi \in \mathbb{R}$;
- *(e)* $|w(\xi)| > |w(-\xi)|$ *for any* $\xi > 0$ *in case* $\mu\lambda > 0 (A\alpha > 0)$, $|w(\xi)| < |w(-\xi)|$ *for any* $\xi > 0$ *in case* $\mu\lambda < 0 (A\alpha < 0)$;
- *(f) if* $\lambda = 0 (\alpha = 0)$ *then* $w(\xi) = A\cosh^{-2}\left(\frac{\kappa\xi}{2}\right)$.

Proof. The existence and uniqueness as well as properties (a)–(c), (f) were proved in section 3 of [9] in the canonical case when $\alpha = A = 1$. These assertions in the non-canonical case follow due to the simple one-to-one relations

$$y = \frac{1}{A}w, \qquad \zeta = \kappa\xi \tag{18}$$

between canonical variables ζ, $y(\zeta)$ and the non-canonical ones ξ, $w(\xi)$ (see (32), (33) of [9]). The assertion (e) follows from lemma 2 in [9] and (18). Finally, let us prove (d). In section 3 of [9] the relation $|y'(\zeta)| < \frac{2}{3|\Theta|} = \frac{2}{3|A\alpha\kappa|}$ was shown. This by (18) implies $|w'(\xi)| < \frac{2}{3|\alpha|}$. Thus, the right inequality in (d) is shown. Moreover, by this relation

$$|(w')^2 - \alpha(w')^3| > (w')^2 - |\alpha|\frac{2}{3|\alpha|}(w')^2 = \frac{1}{3}(w')^2. \tag{19}$$

Further, by (12) and $0 < \frac{w}{A} \leqslant 1$ following from (a) we have

$$|(w')^2 - \alpha(w')^3| = \kappa^2 w^2\left(1 - \frac{w}{A}\right) \leqslant \frac{4\kappa^2 A^2}{27}. \tag{20}$$

Relations (19) and (20) imply the left inequality in (d). The theorem is proved. □

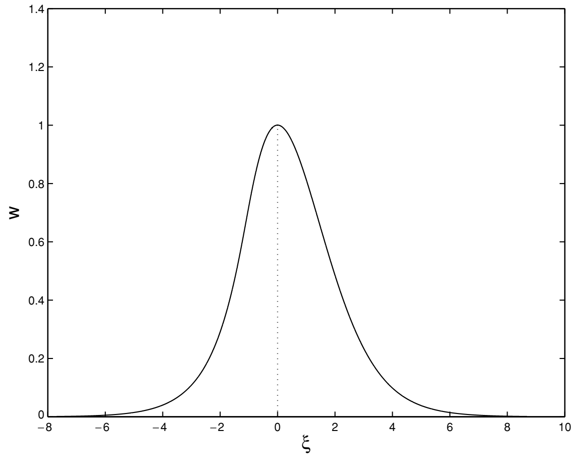

Figure 1. Solitary wave in the case $A = \kappa = 1, \alpha = 0.9$.

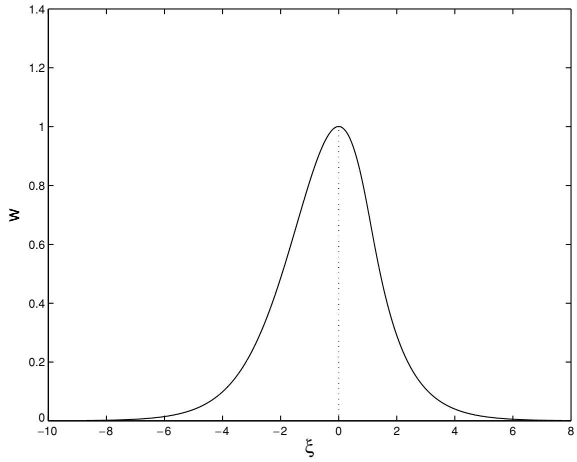

Figure 2. Solitary wave in the case $A = \kappa = 1, \alpha = -0.9$.

In the case where the nonlinearity in the microscale is included, i.e. $\lambda \neq 0$, the solitary wave is asymmetric with the asymmetry depending on sign $\mu\lambda$ = sign $A\alpha$ (statement (e)). This is illustrated by two examples below (figures 1 and 2).

We remark that a single solitary wave does not contain enough information to recover all the five unknowns $b, \mu, \beta, \gamma, \lambda$. Indeed, equation (12) depends upon the three parameters A, κ, α. Thus, measuring the whole wave $w(\xi)$ we can recover maximally A, κ and α. But system (13) has infinitely many solutions $b, \mu, \beta, \gamma, \lambda$ for given A, κ, α and c^2. Summing up, we have to measure at least two waves with different c^2s.

Let us take two waves $w[c_1]$ and $w[c_2]$ with the velocities c_1 and c_2 satisfying $c_1^2 \neq c_2^2$ and the amplitudes A_1 and A_2, respectively. From (13) we have the system $3b + A_j\mu = 3c_j^2$, $j = 1, 2$ for unknowns b and μ. The assumption $c_1^2 \neq c_2^2$ implies $A_1 \neq A_2$; hence this system is regular. This means that the coefficients b and μ are uniquely recovered by amplitudes of two waves. However, A cannot be used to determine other unknowns β, γ, λ. To this end, we have to gather some additional information from solitary waves.

Let us fix two numbers w_{11}, w_{12} which lie between 0 and A_1 and a number w_2 which lies between 0 and A_2. Concerning the first wave we register the time when it attains the level w_{11}, the extremum $w = A_1$, and the time when it drops below the level w_{12}. Using the velocity c_1 we can then compute the relative coordinates $\xi = \xi_{11} > 0$ and $\xi = \xi_{12} < 0$ such that $w[c_1](\xi_{1l}) = w_{1l}, l = 1, 2$. Similarly, for the second wave $w[c_2]$ we register the time when it either attains the level w_2 (case (a)) or drops below it (case (b)). Then, using the arrival time of the extremum $w = A_2$ and the velocity c_2 we can compute $\xi_2 \neq 0$ such that $w[c_2](\xi_2) = w_2$. The coordinate ξ_2 is positive in case (a) and negative in case (b).

We pose the following inverse problem: given b, μ, the points $(\xi_{1l}, w_{1l}), l = 1, 2$ with $\xi_{11} > 0, \xi_{12} < 0$ on the graph of the first wave $w[c_1]$ and the point (ξ_2, w_2) with $\xi_2 \neq 0$ on the graph of the second wave $w[c_2]$, determine the triplet $S = (\beta, \gamma, \lambda)$.

We close this section introducing some notation that we will use throughout the next sections. We will work both with the cases of positive and negative amplitudes. In the former case $w \in (0, A]$ but in the latter case $w \in [A, 0)$. To unify the notation we give to intervals of real numbers the following generalized meaning:

$$(d, e) = \{x : d < x < e\} \quad \text{in case} \quad d < e; \qquad (d, e) = \{x : e < x < d\} \quad \text{in case} \quad d > e.$$

As usual, $[d, e) = (d, e) \cup \{d\}$, $(d, e] = (d, e) \cup \{e\}$ and $[d, e] = (d, e) \cup \{d; e\}$. In this notation we may write $w \in (0, A]$ in both the cases of positive and negative A.

Furthermore, it turns out that it is more convenient to work with the inverse of the wave function $w(\xi)$ than with $w(\xi)$ itself in the analysis of the inverse problem. Due to assertions of theorem 1 the inverse of function $w(\xi)$ has the two single-valued branches $\xi^+(w)$ and $\xi^-(w)$ which are defined for $w \in (0, A]$ and satisfy the conditions $\xi^\pm(A) = 0$ and $\xi^+(w) > 0, \xi^-(w) < 0$ for $w \in (0, A)$. Moreover, there hold the relations $A\xi^{+'}(w) < 0, A\xi^{-'}(w) > 0$ for $w \in (0, A)$ and $\xi^+(w), \xi^-(w)$ have inflection points at $w = 2A/3$. We also emphasize that $\operatorname{sign} w = \operatorname{sign} A$ and $1 - \frac{w}{A} \in [0, 1)$.

3. Series expansion of the solution of the direct problem

In this section we expand the inverse of $w(\xi)$ into series. This form will be applied in the analysis of the inverse problem. By (12) the derivative of $\xi(w) = \xi^\pm(w)$ is a solution to the following equation with fixed $w \in (0, A)$:

$$\xi'(w) - \alpha = \kappa^2 w^2 \left(1 - \frac{w}{A}\right) [\xi'(w)]^3. \tag{21}$$

Introducing new variables y and $\zeta = \zeta(y)$ by

$$y = \alpha\kappa w\sqrt{1 - \frac{w}{A}}, \qquad \xi'(w) = \frac{1}{\kappa w\sqrt{1 - \frac{w}{A}}}\zeta\left(\alpha\kappa w\sqrt{1 - \frac{w}{A}}\right) \tag{22}$$

equation (21) for $\xi'(w)$ is equivalent to the following cubic equation for $\zeta(y)$:

$$[\zeta(y)]^3 - \zeta(y) + y = 0. \tag{23}$$

The discriminant of this equation $D = 4 - 27y^2$ differs from zero for $|y| < \frac{2}{3\sqrt{3}}$. This by a theorem for algebraic equations with meromorphic coefficients [17, chapter 6, theorem 14.2] implies that equation (23) has three solutions $\zeta(y)$ which are holomorphic and differ from each other for $|y| < \frac{2}{3\sqrt{3}}$. Consequently, by the Taylor theorem any of these solutions is expandable into the series $\zeta(y) = \sum_{i=0}^{\infty} a_i y^i$ which is uniformly convergent in every compact subset of $\left(-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right)$. Plugging this series into (23) and equalizing the coefficients of powers of y, we deduce the following recursive formulae for a_i:

$$a_0^3 - a_0 = 0, \qquad a_1 = \left(1 - 3a_0^2\right)^{-1}, \qquad a_i = \left(1 - 3a_0^2\right)^{-1} \sum_{\substack{0\leqslant i_1,i_2,i_3<i \\ i_1+i_2+i_3=i}} a_{i_1}a_{i_2}a_{i_3}, \quad i \geqslant 2. \tag{24}$$

We are interested in sequences starting with $a_0 = \pm 1$. Then first four values of a_i are: $a_0 = \pm 1, a_1 = -\frac{1}{2}, a_2 = \mp\frac{3}{8}, a_3 = -\frac{11}{32}$. The remaining case $a_0 = 0$ leads to the third solution of (21) which is not related to the solitary wave. Substituting the series for ζ in (22) we obtain

$$\xi'(w) = \frac{a_0}{\kappa}\left[w\sqrt{1-\frac{w}{A}}\right]^{-1} + \alpha \sum_{i=0}^{\infty} a_{i+1}(\alpha\kappa)^i \left[w\sqrt{1-\frac{w}{A}}\right]^i. \tag{25}$$

Since the convergence radius of $\sum_{i=0}^{\infty} a_i y^i$ is not less than $\frac{2}{3\sqrt{3}}$ and the inequality $|A\alpha\kappa| < 1$ holds (see theorem 1), the series in (25) is uniformly convergent for $w \in [0, A]$. To derive a formula for ξ we integrate (25) from A to w and use the condition $\xi(A) = 0$ (see above). Defining the sequence of w (and A)-dependent functions $I_i(w) = I_i[A](w)$ by

$$\begin{aligned} I_i(w) &= \int_A^w \left[\tau\sqrt{1-\frac{\tau}{A}}\right]^{i-1} \mathrm{d}\tau \\ &= \begin{cases} \ln\left[\sqrt{\dfrac{A}{w}}\left(1+\sqrt{1-\dfrac{w}{A}}\right)\right] & \text{if} \quad i = 0 \\ 2A^i \displaystyle\sum_{j=0}^{i-1} \binom{i-1}{j} (-1)^{j+1} \frac{\left(1-\frac{w}{A}\right)^{\frac{i+1}{2}+j}}{i+2j+1} & \text{if} \quad i \geqslant 1 \end{cases} \end{aligned} \tag{26}$$

we obtain

$$\xi(w) = \frac{a_0}{\kappa} I_0(w) + \alpha \sum_{i=1}^{\infty} a_i (\alpha\kappa)^{i-1} I_i(w). \tag{27}$$

Using the well-known theorem on passage to the limit under the integral it is not difficult to see that the series in (27) converges uniformly for any $w \in [0, A]$.

Noting that the first addend in (27) dominates as $w \to 0$ and observing its sign we see that (27) with $a_0 = \pm 1$ expresses the branch $\xi(w) = \xi^{\mp}(w)$. Obviously, the first addend in (27) $\xi_0(w) := \frac{a_0}{\kappa} I_0(w) = \frac{\pm I_0(w)}{\kappa}$ represents the inverse of the symmetric bell-shaped solution which occurs in case $\lambda = \alpha = 0$ (cf (f) of theorem 1). The second addend in (27) $a_1 I_1(w)\alpha = \frac{1}{2}(A - w)\alpha$ is the linear term w.r.t. α. Thus, $\xi_1(w) := \frac{a_0}{\kappa} I_0(w) + a_1 I_1(w)\alpha$ is the first order truncation of $\xi(w)$. The inverses $w_0(\xi)$ and $w_1(\xi)$ of the truncations $\xi_0(w)$ and $\xi_1(w)$, respectively, are illustrated in figures 3 and 4.

Using the series expansion we can immediately deduce some useful properties of $\xi^{\pm}(w)$. By virtue of the asymptotic relations $I_0(w) \sim -\frac{2}{\sqrt{|A|}}\sqrt{|w - A|}$, $I_i(w) = o(I_{i-1}(w))$ as $w \to A$, following from the definition of I_i, and the uniform convergence of the series in (27), we obtain the asymptotic relation

$$\xi(w) \sim \xi_1(w) \sim -\frac{\sqrt{|w-A|}}{|A|}\left[\frac{2a_0}{\kappa} + \frac{A\alpha}{2}\sqrt{|w-A|}\right] \qquad \text{as} \quad w \to A. \tag{28}$$

Moreover, there holds the following lemma, which provides certain relations between $\xi^{\pm}(w)$ and $I_0(w)$.

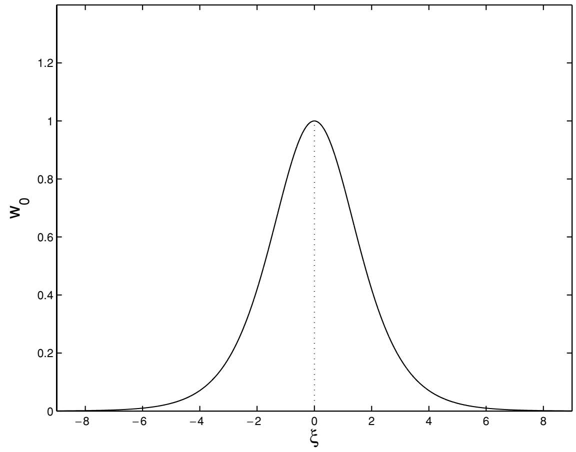

Figure 3. Function $w_0(\xi)$ in the case $A = \kappa = 1$.

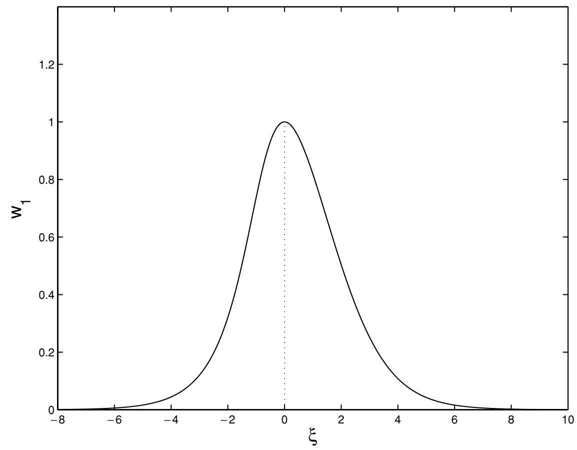

Figure 4. Function $w_1(\xi)$ in the case $A = \kappa = 1, \alpha = 0.9$.

Lemma 2. *For any* $w, \overline{w}, \widehat{w} \in (0, A)$ *the following estimates are valid:*

$$\frac{1}{\sqrt{5}}\left|\frac{\xi^{\pm'}(w)}{I_0'(w)}\right| < \left|\frac{\xi^{\pm}(\overline{w})}{I_0(\overline{w})}\right| < \sqrt{5}\left|\frac{\xi^{\pm'}(\widehat{w})}{I_0'(\widehat{w})}\right|. \tag{29}$$

Proof. Dividing (21) by $\kappa^2\xi'(w)$ and observing the definition of I_0, we have the relation $\left(\frac{\xi'(w)}{I_0'(w)}\right)^2 = \frac{1}{\kappa^2}\left(1 - \frac{\alpha}{\xi'(w)}\right)$. Here $|\xi'(w)^{-1}| < 2/(3|\alpha|)$ due to assertion (d) of theorem 1. Thus, we get the inequalities

$$\frac{1}{3\kappa^2} < \left(\frac{\xi'(w)}{I_0'(w)}\right)^2 < \frac{5}{3\kappa^2} \tag{30}$$

which hold for any $w \in (0, A)$. By the Cauchy mean value theorem and the relations $\xi(A) = I_0(A) = 0$, for any $w \in (0, A)$, there exists $v \in (w, A)$ such that $\frac{\xi'(v)}{I_0'(v)} = \frac{\xi(w)}{I_0(w)}$. Using (30) with w replaced by v, we obtain

$$\frac{1}{3\kappa^2} < \left(\frac{\xi(w)}{I_0(w)}\right)^2 < \frac{5}{3\kappa^2}, \tag{31}$$

which holds for any $w \in (0, A)$. Combining (30) with (31) we prove (29). □

4. Basic estimates

Let us rewrite equation (21) for the derivative of $\xi = \xi^{\pm}$ in the form

$$3\delta(\beta c^2 - \gamma)\xi'(w) - 2\delta^{3/2}\lambda = \{3(c^2 - b)w^2 - \mu w^3\}[\xi'(w)]^3. \tag{32}$$

To emphasize the dependence of $\xi^{\pm}(w)$ on the triplet $S = (\beta, \gamma, \lambda)$ and the velocity c we write either $\xi^{\pm}(w) = \xi^{\pm}[S](w)$ or $\xi^{\pm}(w) = \xi^{\pm}[S, c](w)$ depending on the necessity.

According to the definition of $\xi^{\pm}[S, c](w)$ we can rewrite the inverse problem in the form of the system of nonlinear equations

$$\xi^{+}[S, c_1](w_{11}) = \xi_{11}, \qquad \xi^{-}[S, c_1](w_{12}) = \xi_{12}, \qquad \xi[S, c_2](w_2) = \xi_2 \tag{33}$$

for the triplet $S = (\beta, \gamma, \lambda)$ with the data $d = (w_{11}, w_{12}, w_2, \xi_{11}, \xi_{12}, \xi_2)$. Here $\xi[S, c_2](w_2) = \xi^{+}[S, c_2](w_2)$ in case $\xi_2 > 0$ and $\xi[S, c_2](w_2) = \xi^{-}[S, c_2](w_2)$ in case $\xi_2 < 0$.

To study the inverse problem, we have to estimate the triplet S in terms of w_{11}, w_{12}, w_2 and $\xi^{+}[S, c_1](w_{11}), \xi^{-}[S, c_1](w_{12}), \xi[S, c_2](w_2)$. To explain the ideas how this can be done, let us for a moment consider a slightly different problem. Namely, let us try to identify the triplet S from the data consisting of the quantities w_{11}, w_{12}, w_2 and the derivatives $\xi^{+}[S, c_1]'(w_{11}), \xi^{-}[S, c_1]'(w_{12}), \xi[S, c_2]'(w_2)$. It turns out that such a problem is easier to solve. Indeed, from (32) we immediately obtain the following 3×3 system for the vector $(3\delta\beta, 3\delta\gamma, 2\delta^{3/2}\lambda)$:

$$\begin{pmatrix} c_1^2\xi^{+}[S, c_1]'(w_{11}) & \xi^{+}[S, c_1]'(w_{11}) & -1 \\ c_1^2\xi^{-}[S, c_1]'(w_{12}) & \xi^{-}[S, c_1]'(w_{12}) & -1 \\ c_2^2\xi[S, c_2]'(w_2) & \xi[S, c_2]'(w_2) & -1 \end{pmatrix} \begin{pmatrix} 3\delta\beta \\ 3\delta\gamma \\ 2\delta^{\frac{3}{2}}\lambda \end{pmatrix} = \begin{pmatrix} q(c_1, w_{11})[\xi^{+}[S, c_1]'(w_{11})]^3 \\ q(c_1, w_{12})[\xi^{-}[S, c_1]'(w_{12})]^3 \\ q(c_2, w_2)[\xi[S, c_2]'(w_2)]^3 \end{pmatrix}$$

where $q(c, w) = 3(c^2 - b)w^2 - \mu w^3$. The determinant of this system equals $\left(c_1^2 - c_2^2\right)\xi[S, c_2]'(w_2)\{\xi^{+}[S, c_1]'(w_{11}) - \xi^{-}[S, c_1]'(w_{12})\}$. It differs from zero in case $c_1^2 \neq c_2^2$ due to the relations $\xi[S, c_2]'(w_2), \xi^{+}[S, c_1]'(w_{11}), \xi^{-}[S, c_1]'(w_{12}) \neq 0$ and $\operatorname{sign}\xi^{+}[S, c_1]'(w_{11}) = -\operatorname{sign}\xi^{-}[S, c_1]'(w_{12})$ following from the properties of the functions $\xi^{\pm}(w)$ listed at the end of section 2. Therefore, the considered system is uniquely solvable and the solution can be explicitly expressed by the Cramer formulae. Moreover, an estimate of $S = (\beta, \gamma, \lambda)$ in terms of $w_{11}, w_{12}, w_2, \xi^{+}[S, c_1]'(w_{11}), \xi^{-}[S, c_1]'(w_{12}), \xi[S, c_2]'(w_2)$ can be derived as well.

Let us return to the posed inverse problem. To derive an estimate of $S = (\beta, \gamma, \lambda)$ in terms of w_{11}, w_{12}, w_2 and $\xi^{+}[S, c_1](w_{11}), \xi^{-}[S, c_1](w_{12}), \xi[S, c_2](w_2)$, we construct again by means of (32) a system containing the derivatives of $\xi^{+}[S, c_1](w), \xi^{-}[S, c_1](w), \xi[S, c_2](w)$ and go over from the derivatives to the original functions in this system making use of a mean value theorem. This technique is demonstrated in the following proposition.

Proposition 1. *Let $c_1^2 \neq c_2^2$. Then for any $w_{11}, w_{12} \in (0, A_1) = \left(0, 3\left(c_1^2 - b\right)/\mu\right)$ and $w_2 \in (0, A_2) = \left(0, 3\left(c_2^2 - b\right)/\mu\right)$ the following estimate holds:*

$$\max\{\delta|\beta|; \delta|\gamma|; \delta^{2/3}|\lambda|\} \leqslant \frac{|\mu| \max\left\{1; c_1^2; c_2^2\right\} \max\left\{A_1^2; A_2^2\right\}}{3\left|c_1^2 - c_2^2\right|} K(d), \tag{34}$$

where $d = (w_{11}, w_{12}, w_2, \xi_{11}, \xi_{12}, \xi_2)$ *with* $\xi_{11} = \xi^+[S, c_1](w_{11}), \xi_{12} = \xi^-[S, c_1](w_{12})$, $\xi_2 = \xi^\pm[S, c_2](w_2)$ *and* K *is the following function of* d *depending also on* A_1, A_2*:*

$$K(d) = \left[\frac{|w_{11} - A_1|}{|\xi_{11}|} + \frac{|w_{12} - A_1|}{|\xi_{12}|} + \frac{|w_2 - A_2|}{|\xi_2|}\right] \times \left[\frac{|\xi_{11}|^3}{|w_{11} - A_1|^2} + \frac{|\xi_{12}|^3}{|w_{12} - A_1|^2} + \frac{|\xi_2|^3}{|w_2 - A_2|^2}\right]. \tag{35}$$

Proof. Since $\xi^\pm[S, c_j](A_j) = 0$, $j = 1, 2$, by the Lagrange mean value theorem there exist $v_{1l} \in (w_{1l}, A_1)$, $l = 1, 2$ and $v_2^\pm \in (w_2, A_2)$ such that

$$\xi^+[S, c_1]'(v_{11}) = \frac{\xi_{11}}{w_{11} - A_1}, \qquad \xi^-[S, c_1]'(v_{12}) = \frac{\xi_{12}}{w_{12} - A_1}, \qquad \xi^\pm[S, c_2]'\left(v_2^\pm\right) = \frac{\xi_2}{w_2 - A_2}. \tag{36}$$

Let us take equation (32) for $\xi'(w) = \xi^+[S, c_1]'(w)$, $\xi'(w) = \xi^-[S, c_1]'(w)$ and $\xi'(w) = \xi^\pm[S, c_1]'(w)$ with $w = v_{11}$, $w = v_{12}$ and $w = v_2^\pm$, respectively. We obtain a 3×3 linear system $\mathcal{A}S_* = Y$ for the vector $S_* = \left(\frac{3\delta}{\mu}\beta, \frac{3\delta}{\mu}\gamma, \frac{2\delta^{3/2}}{\mu}\lambda\right)^T$, where T denotes the transposition. In view of (36) the matrix and free term of this system read as

$$\mathcal{A} = \begin{pmatrix} c_1^2\xi_{1l}(w_{1l} - A_1)^{-1} & -\xi_{1l}(w_{1l} - A_1)^{-1} & -1 & (l = 1, 2) \\ c_2^2\xi_2(w_2 - A_2)^{-1} & -\xi_2(w_2 - A_2)^{-1} & -1 & \end{pmatrix} \tag{37}$$

and

$$Y = \left(v_{1l}^2(A_1 - v_{1l})\left[\frac{\xi_{1l}}{w_{1l} - A_1}\right]^3 (l = 1, 2), \left(v_2^\pm\right)^2\left(A_2 - v_2^\pm\right)\left[\frac{\xi_2}{w_2 - A_2}\right]^3\right)^T, \tag{38}$$

respectively. Note that $\det\mathcal{A} = \frac{(c_2^2 - c_1^2)\xi_2}{w_2 - A_2}\left[\frac{\xi_{11}}{w_{11} - A_1} - \frac{\xi_{12}}{w_{12} - A_1}\right]$. Here $\xi_{11} > 0$ and $\xi_{12} < 0$ by the definition of ξ_{11}, ξ_{12}. Moreover, $\operatorname{sign}(w_{11} - A_1) = \operatorname{sign}(w_{12} - A_1) = -\operatorname{sign} A_1$. Due to these relations we obtain $|\det\mathcal{A}| = \frac{|c_2^2 - c_1^2||\xi_2|}{|w_2 - A_2|}\left[\frac{|\xi_{11}|}{|w_{11} - A_1|} + \frac{|\xi_{12}|}{|w_{12} - A_1|}\right]$. By this formula from (37) we derive the following estimate for the components of $\mathcal{A}^{-1}$:

$$|\widehat{a}_{ij}| \leqslant \frac{\max\left\{1; c_1^2; c_2^2\right\}}{\left|c_1^2 - c_2^2\right|}\left[\frac{|w_{11} - A_1|}{|\xi_{11}|} + \frac{|w_{12} - A_1|}{|\xi_{12}|} + \frac{|w_2 - A_2|}{|\xi_2|}\right], \qquad \mathcal{A}^{-1} = (\widehat{a}_{ij}). \tag{39}$$

Further, some terms in (38) can be estimated as follows:

$$|v_{1l}| \leqslant |A_1|, \qquad |A_1 - v_{1l}| \leqslant |A_1 - w_{1l}|, \qquad \left|v_2^\pm\right| \leqslant |A_2|, \qquad \left|A_2 - v_2^\pm\right| \leqslant |A_2 - w_2|. \tag{40}$$

Finally, using (38)–(40) in the equation $S_* = \mathcal{A}^{-1}Y$ we prove (34) with (35). □

In order to study the stability of the inverse problem, we have to estimate the difference of S in terms of differences of w_{11}, w_{12}, w_2 and $\xi^+[S, c_1](w_{11}), \xi^-[S, c_1](w_{12}), \xi[S, c_2](w_2)$. To derive such an estimate, we can follow the same ideas as above. But, first of all, we need an auxiliary lemma whose proof is deferred to the appendix.

Lemma 3. *Let we be given two triplets* $S^i = (\beta^i, \gamma^i, \lambda^i)$, $i = 1, 2$. *Then for any* $w^i \in (0, A) = (0, 3(c^2 - b)/\mu)$, $i = 1, 2$ *there exist* $u_i^\pm = u_i^\pm(w^1, w^2, A) \in (w^i, A)$, $i = 1, 2$ *such that the estimates*

$$\left|\xi^\pm[S^1]'\left(u_1^\pm\right) - \xi^\pm[S^2]'\left(u_2^\pm\right)\right| \leqslant \frac{C_1}{\kappa}\left[\left|\xi^\pm[S^1](w^1) - \xi^\pm[S^2](w^2)\right| + \frac{M}{|w^2|\kappa^{1/2}}|w^1 - w^2|\right] \tag{41}$$

and

$$\left|\left(u_1^{\pm}\right)^2\left(A-u_1^{\pm}\right)\left(\xi^{\pm}[S^1]'\left(u_1^{\pm}\right)\right)^3-\left(u_2^{\pm}\right)^2\left(A-u_2^{\pm}\right)\left(\xi^{\pm}[S^2]'\left(u_2^{\pm}\right)\right)^3\right|$$
$$\leqslant \frac{C_2M^2}{\kappa}\left[|\xi^{\pm}[S^1](w^1)-\xi^{\pm}[S^2](w^2)|+\frac{M}{\Theta^3\kappa^{1/2}}|w^1-w^2|\right] \tag{42}$$

hold. Here $M=\max\limits_{i=1,2}\frac{|\xi^{\pm}[S^i](w^i)|}{|I_0(w^i)|}$, $\Theta=\min\limits_{i=1,2}|w^i|$, $\kappa=\min\limits_{i=1,2}|w^i-A|$ *and* C_1, C_2 *are some constants depending on* $|A|$.

Now we are in the situation to prove an estimate for the difference of S.

Proposition 2. *Let* $c_1^2\neq c_2^2$ *and we be given two triplets* $S^i=(\beta^i,\gamma^i,\lambda^i)$, $i=1,2$. *Then for any* $w_{11}^i, w_{12}^i\in(0,A_1)=\left(0,3\left(c_1^2-b\right)/\mu\right)$, $w_2^i\in(0,A_2)=\left(0,3\left(c_2^2-b\right)/\mu\right)$, $i=1,2$ *the following estimate holds:*

$$\max\{\delta|\beta^1-\beta^2|;\,\delta|\gamma^1-\gamma^2|;\,\delta^{3/2}|\lambda^1-\lambda^2|\}\leqslant\frac{C_3}{\left|c_1^2-c_2^2\right|^2}[N(d^1,d^2)\varepsilon_\xi+\overline{N}(d^1,d^2)\varepsilon_w] \tag{43}$$

where C_3 *is a constant depending on* μ, $\max\limits_{j=1,2}|A_j|$ *and* $\max\limits_{j=1,2}|c_j|$, *the vectors* $d^i=\left(w_{11}^i, w_{12}^i, w_2^i, \xi_{11}^i, \xi_{12}^i, \xi_2^i\right)$, $i=1,2$ *contain the components*

$$\xi_{11}^i=\xi^+[S^i,c_1]\left(w_{11}^i\right),\qquad \xi_{12}^i=\xi^-[S^i,c_1]\left(w_{12}^i\right),$$
$$\xi_2^i=\xi^{\pm}[S^i,c_2]\left(w_2^i\right),\qquad i=1,2; \tag{44}$$

$$\varepsilon_\xi=\left|\xi_{11}^1-\xi_{11}^2\right|+\left|\xi_{12}^1-\xi_{12}^2\right|+\left|\xi_2^1-\xi_2^2\right|,$$
$$\varepsilon_w=\left|w_{11}^1-w_{11}^2\right|+\left|w_{12}^1-w_{12}^2\right|+\left|w_2^1-w_2^2\right|, \tag{45}$$

$N, \overline{N}$ *are the following functions of* d^1, d^2 *depending also on* c_1, c_2, A_1, A_2:

$$N(d^1,d^2)=\frac{P[Q^2+K(d^2)]}{\nu},\qquad \overline{N}(d^1,d^2)=\frac{PQ}{\nu^{3/2}}\left(\frac{Q^2}{\sigma^3}+\frac{K(d^2)}{\sigma}\right), \tag{46}$$

$K(d)$ *is given by* (35) *and*

$$P=\max\left\{\frac{\left|I_0[A_1]\left(w_{1l}^1\right)\right|}{\left|\xi_{1l}^1\right|}\sqrt{\left|A_1-w_{1l}^1\right|}(l=1,2);\ \frac{\left|I_0[A_2]\left(w_2^1\right)\right|}{\left|\xi_2^1\right|}\sqrt{\left|A_2-w_2^1\right|}\right\}, \tag{47}$$

$$Q=\max_{i,l=1,2}\left\{\frac{\left|\xi_{1l}^i\right|}{\left|I_0[A_1]\left(w_{1l}^i\right)\right|};\ \frac{\left|\xi_2^i\right|}{\left|I_0[A_2]\left(w_2^i\right)\right|}\right\}, \tag{48}$$

$$\sigma=\min_{i,l=1,2}\left\{\left|w_{1l}^i\right|;\left|w_2^i\right|\right\},\ \nu=\min_{i,l=1,2}\left\{\left|w_{1l}^i-A_1\right|;\left|w_2^i-A_2\right|\right\}. \tag{49}$$

Proof. Making use of functions $u_i^{\pm}$ from lemma 3 we define

$$u_{11}^i=u_i^+\left(w_{11}^1,w_{11}^2,A_1\right),\qquad u_{12}^i=u_i^-\left(w_{12}^1,w_{12}^2,A_1\right),$$
$$u_2^i=u_i^{\pm}\left(w_2^1,w_2^2,A_2\right),\qquad i=1,2, \tag{50}$$

$$\psi_{11}^i=\xi^+[S^i,c_1]'\left(u_{11}^i\right),\qquad \psi_{12}^i=\xi^-[S^i,c_1]'\left(u_{12}^i\right),\qquad \psi_2^i=\xi^{\pm}[S^i,c_2]'\left(u_2^i\right). \tag{51}$$

Equation (32) for $\xi'(w)=\xi^+[S^i,c_1]'(w)$, $\xi'(w)=\xi^-[S^i,c_1]'(w)$ and $\xi'(w)=\xi^{\pm}[S^i,c_2]'(w)$, with $w=u_{11}^i$, $w=u_{12}^i$ and $w=u_2^i$, respectively, yields 3×3 systems

$\mathcal{B}_i S_*^i = Z^i$, $i = 1, 2$ for $S_*^i = \left(\frac{3\delta}{\mu}\beta^i, \frac{3\delta}{\mu}\gamma^i, \frac{2\delta^{3/2}}{\mu}\lambda^i\right)^T$, $i = 1, 2$. Due to (51) the matrices and free terms of these systems are representable as follows:

$$\mathcal{B}_i = \begin{pmatrix} c_1^2\psi_{1l}^i & -\psi_{1l}^i & -1 & (l=1,2) \\ c_2^2\psi_2^i & -\psi_2^i & -1 & \end{pmatrix}, \qquad i = 1, 2, \tag{52}$$

$$Z^i = \left(\left(u_{1l}^i\right)^2\left(A_1 - u_{1l}^i\right)\left(\psi_{1l}^i\right)^3 (l=1,2), \left(u_2^i\right)^2\left(A_2 - u_2^i\right)\left(\psi_2^i\right)^3\right)^T, \qquad i = 1, 2. \tag{53}$$

The difference $S_*^1 - S_*^2$ can be expressed by $S_*^1 - S_*^2 = (\mathcal{B}_1)^{-1}V$, where $V = Z^1 - Z^2 + (\mathcal{B}_2 - \mathcal{B}_1)S_*^2$. To prove (43), we must estimate $(\mathcal{B}_1)^{-1}$ and V. Note that $\det\mathcal{B}_1 = \left(c_2^2 - c_1^2\right)\psi_2^1\left[\psi_{11}^1 - \psi_{12}^1\right]$. By the definition of ξ^+ and ξ^- the numbers $\psi_{11}^1 = \xi^+[S^1, c_1]'\left(u_{11}^1\right)$ and $\psi_{12}^1 = \xi^-[S^1, c_1]'\left(u_{12}^1\right)$ have different signs. Thus, we obtain $|\det\mathcal{B}_1| = \left|c_2^2 - c_1^2\right|\left|\psi_2^1\right|\left[\left|\psi_{11}^1\right| + \left|\psi_{12}^1\right|\right]$. Using this relation from (52), we derive the following estimate for the components of the inverse of $\mathcal{B}_1$:

$$|\widehat{b}_{ij}| \leqslant \frac{\max\left\{1; c_1^2; c_2^2\right\}}{\left|c_1^2 - c_2^2\right|}\left[\frac{1}{\left|\psi_{11}^1\right|} + \frac{1}{\left|\psi_{12}^1\right|} + \frac{1}{\left|\psi_2^1\right|}\right], \qquad (\mathcal{B}_1)^{-1} = (\widehat{b}_{ij}). \tag{54}$$

By virtue of lemma 2, the definitions of ψ_{11}^1, ξ_{11}^1, $I_0[A](w)$ and the inequalities $\left|w_{11}^1\right| < \left|u_{11}^1\right| < |A_1|$ we have

$$\left|\psi_{11}^1\right| > \frac{\left|I_0[A_1]'\left(u_{11}^1\right)\right|\left|\xi_{11}^1\right|}{\sqrt{5}\left|I_0[A_1]\left(w_{11}^1\right)\right|} = \frac{\left[u_{11}^1\sqrt{1 - \frac{u_{11}^1}{A_1}}\right]^{-1}|\xi_{11}|}{\sqrt{5}\left|I_0[A_1]\left(w_{11}^1\right)\right|} > \frac{\left[\sqrt{|A_1 - w_{11}|}\right]^{-1}|\xi_{11}|}{\sqrt{5}\left|I_0[A_1]\left(w_{11}^1\right)\right|} \geqslant \frac{1}{\sqrt{5}P}.$$

Similarly, $\left|\psi_{12}^1\right| > \frac{1}{\sqrt{5}P}$ and $\left|\psi_2^1\right| > \frac{1}{\sqrt{5}P}$. Using these inequalities in (54), we obtain

$$|\widehat{b}_{ij}| \leqslant \frac{3\sqrt{5}\max\left\{1; c_1^2; c_2^2\right\}}{\left|c_1^2 - c_2^2\right|}P, \qquad (\mathcal{B}_1)^{-1} = (\widehat{b}_{ij}). \tag{55}$$

Next let us estimate $V = (V_1, V_2, V_3)^T = Z^1 - Z^2 + (\mathcal{B}_2 - \mathcal{B}_1)S_*^2$. For the first component $V_1 = \left(u_{11}^1\right)^2\left(A_1 - u_{11}^1\right)\left(\psi_{11}^1\right)^3 - \left(u_{11}^2\right)^2\left(A_1 - u_{11}^2\right)\left(\psi_{11}^2\right)^3 - c_1^2\left(\psi_{11}^2 - \psi_{11}^1\right)\frac{3\delta}{\mu}\beta^2 - \left(\psi_{11}^2 - \psi_{11}^1\right)\frac{3\delta}{\mu}\gamma^2$ by means of lemma 3 and proposition 1 we obtain

$$\begin{aligned} |V_1| &\leqslant \frac{C_2M^2}{\kappa}\left[\left|\xi_{11}^1 - \xi_{11}^2\right| + \frac{M}{\Theta^3\kappa^{1/2}}\left|w_{11}^1 - w_{11}^2\right|\right] \\ &\quad + \frac{C_4K(d^2)}{\left|c_1^2 - c_2^2\right|\kappa}\left[\left|\xi_{11}^1 - \xi_{11}^2\right| + \frac{M}{\Theta\kappa^{1/2}}\left|w_{11}^1 - w_{11}^2\right|\right] \\ &\leqslant \frac{C_5}{\left|c_1^2 - c_2^2\right|}\left[\frac{Q^2 + K(d^2)}{\nu}\varepsilon_\xi + \frac{Q}{\nu^{3/2}}\left(\frac{Q^2}{\sigma^3} + \frac{K(d^2)}{\sigma}\right)\varepsilon_w\right]. \end{aligned} \tag{56}$$

Here $M = \max_{i=1,2}\frac{\left|\xi_{11}^i\right|}{\left|I_0[A_1]\left(w_{11}^i\right)\right|}$, $\Theta = \min_{i=1,2}\left|w_{11}^i\right|$, $\kappa = \min_{i=1,2}\left|w_{11}^i - A_1\right|$ and C_4, C_5 are constants depending on μ, $|A_1|$ and $\max_{i=1,2}|c_i|$. Similarly we derive

$$|V_2|, |V_3| \leqslant \frac{C_5}{\left|c_1^2 - c_2^2\right|}\left[\frac{Q^2 + K(d^2)}{\nu}\varepsilon_\xi + \frac{Q}{\nu^{3/2}}\left(\frac{Q^2}{\sigma^3} + \frac{K(d^2)}{\sigma}\right)\varepsilon_w\right]. \tag{57}$$

Finally, estimating $S_*^1 - S_*^2 = (\mathcal{B}_1)^{-1}V$ by means of (55)–(57) we prove (43). $\square$

5. Uniqueness and stability for inverse problem

As we mentioned, the inverse problem can be written in the form of the system of nonlinear equations (33) for the triplet $S = (\beta, \gamma, \lambda)$ with the data $d = (w_{11}, w_{12}, w_2, \xi_{11}, \xi_{12}, \xi_2)$. There $\xi_{11} > 0$, $\xi_{12} < 0$ and $\xi[S, c_2](w_2) = \xi^+[S, c_2](w_2)$ in case $\xi_2 > 0$ and $\xi[S, c_2](w_2) = \xi^-[S, c_2](w_2)$ in case $\xi_2 < 0$.

Important questions related to the inverse problem are the uniqueness and stability with respect to the errors in the data. We take two types of errors into account. The first type is related to the inaccuracy of fixing the levels of measurement of the waves. Levels used in the computations differ somewhat from the values w_{11}, w_{12}, w_2, where the actual measurements are performed. Let us denote these approximate levels by $\widetilde{w}_{11}, \widetilde{w}_{12}, \widetilde{w}_2$. The second type is related to the inaccuracy of the measurement of time moments during the experiment. This leads to errors in ξs. Let us denote by $\widetilde{\xi}_{11}, \widetilde{\xi}_{12}, \widetilde{\xi}_2$ the approximate values of $\xi_{11}, \xi_{12}, \xi_2$ obtained by means of measurements. Summing up, instead of (33) we solve the problem

$$\xi^+[\widetilde{S}, c_1](\widetilde{w}_{11}) = \widetilde{\xi}_{11}, \qquad \xi^-[\widetilde{S}, c_1](\widetilde{w}_{12}) = \widetilde{\xi}_{12}, \qquad \xi[\widetilde{S}, c_2](\widetilde{w}_2) = \widetilde{\xi}_2 \tag{58}$$

with the approximate data $\widetilde{d} = (\widetilde{w}_{11}, \widetilde{w}_{12}, \widetilde{w}_2, \widetilde{\xi}_{11}, \widetilde{\xi}_{12}, \widetilde{\xi}_2)$ and the solution $\widetilde{S} = (\widetilde{\beta}, \widetilde{\gamma}, \widetilde{\lambda})$. The solution is stable with respect to the data if $\widetilde{d} \to d$ implies $\widetilde{S} \to S$.

Theorem 2.

(i) The solution of the inverse problem is unique.

(ii) The solution is stable with respect to the data and satisfies the estimate

$$\max\{\delta|\beta - \widetilde{\beta}|; \delta|\gamma - \widetilde{\gamma}|; \delta^{3/2}|\lambda - \widetilde{\lambda}|\} \leqslant \frac{C_3}{|c_1^2 - c_2^2|^2}[N(d, \widetilde{d})\varepsilon_\xi + \overline{N}(d, \widetilde{d})\varepsilon_w] \tag{59}$$

where C_3 and $N, \overline{N}$ are defined in proposition 2 and

$$\begin{aligned} \varepsilon_\xi &= |\xi_{11} - \widetilde{\xi}_{11}| + |\xi_{12} - \widetilde{\xi}_{12}| + |\xi_2 - \widetilde{\xi}_2|, \\ \varepsilon_w &= |w_{11} - \widetilde{w}_{11}| + |w_{12} - \widetilde{w}_{12}| + |w_2 - \widetilde{w}_2|. \end{aligned} \tag{60}$$

Proof. Estimate (59), implying the stability, immediately follows from proposition 2. If (33) has two solutions S^1 and S^2 then estimate (43) holds with $\epsilon_w = \epsilon_\xi = 0$ on the right-hand side. Thus, $S^1 = S^2$. This proves the uniqueness, too. □

The coefficients N and $\overline{N}$ are continuous functions of $(d, \widetilde{d})$ in the domain D^2, where $D = (0, A_1)^2 \times (0, A_2) \times (0, \infty) \times (0, -\infty) \times (0, \operatorname{sign} \xi_2 \infty)$. Therefore, by (59) the solution of (33) is uniformly Lipschitz-continuous with respect to the data d in every compact subdomain of D.

Since the amplitudes do not contain any information about S, the error of the solution is expected to increase as one of the measurement levels approaches the amplitude. Let us see how this is reflected in the behaviour of the coefficients N and $\overline{N}$ in (59). For the sake of simplicity assume that $w_1 := w_{11} = w_{12}$ and consider the process $w_1 \to A_1$. Furthermore, let $\widetilde{w}_1 := \widetilde{w}_{11} = \widetilde{w}_{12}$ and $|w_1 - A_1| \sim |\widetilde{w}_1 - A_1| \sim |w_2 - A_2| \sim |\widetilde{w}_2 - A_2|$ as $w_1 \to A_1$. Due to (31) the ratios of the type $\frac{|\xi^{\pm}[S,c](w)|}{|I_0(w)|}$ in the formulae of P, Q (47), (48) are bounded from below and above by positive constants independent of w_1. Further, by the relation $I_0(w) \sim -\frac{2}{\sqrt{|A|}}\sqrt{|w - A|}$ as $w \to A$ (cf (26)) and (31), (35) the term $|K(\widetilde{d})|$ is bounded from below and above by positive constants independent of w_1. Summing up, from (46) due to (47)–(49) we obtain the inequalities

$$\frac{C_6}{\sqrt{|w_1 - A_1|}} \leqslant N(d, \widetilde{d}) \leqslant \frac{C_7}{\sqrt{|w_1 - A_1|}} \qquad \text{and} \qquad \frac{C_6}{|w_1 - A_1|} \leqslant \overline{N}(d, \widetilde{d}) \leqslant \frac{C_7}{|w_1 - A_1|}$$

for w_1 in a neighbourhood of A_1. Here C_6 and C_7 are some positive constants. Consequently, the contribution of ε_w is bigger than ε_ξ in the error estimate (59) if $w_1 \approx A_1$. This can be explained by the asymptotic behaviour of $\xi[S, c_j](w)$ as $w \to A_j$, too. Namely, by (28) we have the relations $\xi_{il}^2 \sim w_{1l} - A_1$ and $\xi_2^2 \sim w_2 - A_2$ showing that the measurements ξ_{1l} and ξ_2 approach zero faster than the levels w_{1l} and w_2 approach the amplitudes.

Further, let us study the error in the case of small levels. Let $w_1 := w_{11} = w_{12}$ and consider the process $w_1 \to 0$. Assume that $\widetilde{w}_1 := \widetilde{w}_{11} = \widetilde{w}_{12}$ and $|w_1| \sim |\widetilde{w}_1| \sim |w_2| \sim |\widetilde{w}_2|$ as $w_1 \to 0$. By (a) of theorem 1 and (35) we have $K(\widetilde{d}) \sim \text{const}|\ln|w_1||^2$ as $w_1 \to 0$. Using this relation and (31) in (46)–(49), we deduce the estimates

$$C_8|\ln|w_1||^2 \leqslant N(d, \widetilde{d}) \leqslant C_9|\ln|w_1||^2 \qquad \text{and} \qquad \frac{C_8}{|w_1|^3} \leqslant \overline{N}(d, \widetilde{d}) \leqslant \frac{C_9}{|w_1|^3}$$

for w_1 in a neighbourhood of 0 with positive constants C_8 and C_9. Both terms $N(d, \widetilde{d})$ and $\overline{N}(d, \widetilde{d})$ increase in (59). This shows the limitations of using small measurement levels in the experiment. Again, the contribution of ε_w is bigger than ε_ξ in the error estimate for small w_1. This can be explained by the behaviour of $\xi[S, c_j](w)$ as $w \to 0$ (see (a) of theorem 1).

The experiment related to the inverse problem involves measurement of the first wave at both sides of the extremum $w[c_1] = A_1$. Therefore, we have $\xi[S, c_1]$ with different superscripts $+, -$ in the first two equations of (33). We remark that the values of $\xi[S, c_1]$ with different signs enabled us to estimate $\det \mathcal{B}_1$ from below in the proof of proposition 2. We now ask: what is the situation when the first wave is measured twice from a single side of the extremum? In this case the system of equations corresponding to the inverse problem is

$$\xi[S, c_1](w_{11}) = \xi_{11}, \qquad \xi[S, c_1](w_{12}) = \xi_{12}, \qquad \xi[S, c_2](w_2) = \xi_2 \tag{61}$$

where the functions $\xi[S, c_1](w_{1l})$ occurring in the first two equations have the common superscript: either $+$ or $-$, depending on $\operatorname{sign}\xi_{11} = \operatorname{sign}\xi_{12}$, and $w_{11} \neq w_{12}$. In case $w_{11}, w_{12} \in [2A_1/3, A_1)$, using the strict monotonicity of $\xi[S, c_1]'(w)$ in the interval $(2A/3, A)$, it is again possible to estimate in a proper way $\det \mathcal{B}_1$ from below and, using a more complex technique, prove the uniqueness and stability for the system (61). However, in the general case of w_{11}, w_{12} and w_2 the uniqueness for (61) does not hold. To show this, we construct a proper counter-example.

Counter-example. Due to (a) in theorem 1 we have

$$\xi^{\pm}[S, c](w) \sim \mp\frac{1}{\kappa}\ln|w| \qquad \text{as} \quad w \to 0. \tag{62}$$

For given two triplets $S^i = (\beta^i, \gamma^i, \lambda^i)$, $i = 1, 2$ we denote

$$\kappa_j^i = \sqrt{\frac{c_j^2 - b}{\delta(\beta^i c_j^2 - \gamma^i)}}, \qquad \alpha_j^i = \frac{2\delta^{1/2}\lambda^i}{3(\beta^i c_j^2 - \gamma^i)}, \qquad A_j = \frac{3(c_j^2 - b)}{\mu}, \qquad i, j = 1, 2. \tag{63}$$

We now observe that for any $\epsilon > 0$ and $a_{01}, a_{02} \in \{-1; 1\}$ we can find S^i, $i = 1, 2$ such that

$$\beta^1 \neq \beta^2, \qquad \gamma^1 \neq \gamma^2, \qquad \lambda^1 \neq \lambda^2, \qquad |A_j\alpha_j^i\kappa_j^i| < 1, \quad i, j = 1, 2,$$
$$\operatorname{sign}\left[2a_{0j}\left(\frac{1}{\kappa_j^1} - \frac{1}{\kappa_j^2}\right) + \frac{A_j(\alpha_j^1 - \alpha_j^2)}{2}\frac{\epsilon}{2}\right] = \operatorname{sign}\left[a_{0j}\left(\frac{1}{\kappa_j^1} - \frac{1}{\kappa_j^2}\right)\right], \qquad j = 1, 2. \tag{64}$$
$$\operatorname{sign}\left[2a_{0j}\left(\frac{1}{\kappa_j^1} - \frac{1}{\kappa_j^2}\right) + \frac{A_j(\alpha_j^1 - \alpha_j^2)}{2}\epsilon\right] = -\operatorname{sign}\left[a_{0j}\left(\frac{1}{\kappa_j^1} - \frac{1}{\kappa_j^2}\right)\right], \qquad j = 1, 2.$$

Let us fix some pair $a_{01}, a_{02} \in \{-1; 1\}$. Choosing sufficiently small $\epsilon > 0$ and triplets $S^i, i = 1, 2$ satisfying (64) from relations (28) and (62), we deduce the inequalities

$$\begin{aligned} \operatorname{sign}[\xi[S^1, c_j](\omega_{1j}) - \xi[S^2, c_j](\omega_{1j})] &= -\operatorname{sign}[\xi[S^1, c_j](\omega_{2j}) - \xi[S^2, c_j](\omega_{2j})] \\ &= \operatorname{sign}[\xi[S^1, c_j](\omega_{3j}) - \xi[S^2, c_j](\omega_{3j})], \qquad j = 1, 2 \end{aligned} \tag{65}$$

where $|\omega_{1j}| = \epsilon, |\omega_{2j} - A_j| = \epsilon, |\omega_{3j} - A_j| = \epsilon/2, j = 1, 2$ and $\xi[S^i, c_j] = \xi^+[S^i, c_j]$ in case $a_{0j} = -1$ and $\xi[S^i, c_j] = \xi^-[S^i, c_j]$ in case $a_{0j} = 1$. Relations (65) imply that there exist $w_{11} \in (\omega_{11}, \omega_{12}), w_{12} \in (\omega_{12}, \omega_{13})$ and $w_2 \in (0, A_2)$ such that

$$\xi[S^1, c_1](w_{1l}) - \xi[S^2, c_1](w_{1l}) = \xi[S^1, c_2](w_2) - \xi[S^2, c_2](w_2) = 0, \qquad l = 1, 2. \tag{66}$$

Consequently, (61) has two solutions $S^1 \neq S^2$ for such $w_{11} \neq w_{12}$ and w_2.

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Appendix. Proof of lemma 3

Let us choose a pair $(\xi[S^1], \xi[S^2]) \in \{(\xi^+[S^1], \xi^+[S^2]); (\xi^-[S^1], \xi^-[S^2])\}$ and define the function $g(t) = \xi[S^1](m^1(t)) - \xi[S^2](m^2(t))$ for $t \in [0, 1]$ where

$$m^i(t) = A + t(w^i - A) - 2\operatorname{sign} A\kappa(t^{2/3} - t), \qquad t \in [0, 1], \quad i = 1, 2. \tag{A.1}$$

Functions m^i are strictly monotonic, $m^i(0) = A, m^i(1) = w^i, i = 1, 2$ and

$$|(m^i)'(t)| \geqslant \frac{\kappa}{3}, \qquad (m^2 - m^1)'(t) = w^2 - w^1, \qquad \left|\sqrt{1 - \frac{m^2(t)}{A}}(m^1)'(t)\right| \geqslant \frac{\kappa^{3/2}}{3\sqrt{|A|}}, \tag{A.2}$$

$$|m^i(t)| \leqslant A, \qquad |m^i(t)| \geqslant |w^i|, \qquad |A - m^i(t)| \geqslant \kappa t^{2/3}, \qquad i = 1, 2. \tag{A.3}$$

Since $g(0) = \xi[S^1](A) - \xi[S^2](A) = 0$, by the Lagrange mean value theorem there exists $\tau \in (0, 1)$ such that $g'(\tau) = g(1)$. Denote $u_i = u_i^{\pm} := m^i(\tau) \in (w^i, A), i = 1, 2$. Note that u_i depend on w^1, w^2 and A. By the definition of g the relation $g'(\tau) = g(1)$ has the form $(m^1)'(\tau)\xi[S^1]'(u_1) - (m^2)'(\tau)\xi[S^2]'(u_2) = \xi[S^1](w^1) - \xi[S^2](w^2)$. Thus

$$\xi[S^1]'(u_1) - \xi[S^2]'(u_2) = \frac{\xi[S^1](w^1) - \xi[S^2](w^2)}{(m^1)'(\tau)} + \frac{\xi[S^2]'(u_2)(m^2 - m^1)'(\tau)}{(m^1)'(\tau)}. \tag{A.4}$$

Using lemma 2, the formula $I_0'(w) = \left[w\sqrt{1 - \frac{w}{A}}\right]^{-1}$ and (A.2), (A.3), we obtain

$$\begin{aligned} \left|\frac{\xi[S^2]'(u_2)}{(m^1)'(\tau)}\right| &< \left|\frac{\sqrt{5}I_0'(u_2)\xi[S^2](w^2)}{(m^1)'(\tau)I_0(w^2)}\right| \\ &= \left|m^2(\tau)\sqrt{1 - \frac{m^2(\tau)}{A}}(m^1)'(\tau)\right|^{-1} \frac{\sqrt{5}|\xi[S^2](w^2)|}{|I_0(w^2)|} \leqslant \frac{3\sqrt{5|A|}M}{|w^2|\kappa^{3/2}}. \end{aligned} \tag{A.5}$$

From (A.4) by means of (A.2) and (A.5) we obtain (41).

Next let us prove (42). In view of the formula of $I_0'(w)$ above we can write

$$u_1^2(A-u_1)(\xi[S^1]'(u_1))^3 - u_2^2(A-u_2)(\xi[S^2]'(u_2))^3 = A\left[\left(\frac{\xi[S^1]'(u_1)}{I_0'(u_1)}\right)^2 + \left(\frac{\xi[S^2]'(u_2)}{I_0'(u_2)}\right)^2 + \frac{\xi[S^1]'(u_1)\xi[S^2]'(u_2)}{I_0'(u_1)I_0'(u_2)}\right](\xi[S^1]'(u_1) - \xi[S^2]'(u_2)) + |A|^{3/2}\frac{\xi[S^1]'(u_1)\xi[S^2]'(u_2)}{I_0'(u_1)I_0'(u_2)}\left[\frac{\xi[S^1]'(u_1)}{I_0'(u_1)} + \frac{\xi[S^2]'(u_2)}{I_0'(u_2)}\right] \times \frac{u_1^2(A-u_1) - u_2^2(A-u_2)}{\prod_{i=1,2} u_i\sqrt{|A-u_i|}\left[u_1\sqrt{|A-u_1|} + u_2\sqrt{|A-u_2|}\right]}. \tag{A.6}$$

By (A.3) we have $|u_i|\sqrt{|A-u_i|} = |m^i(\tau)|\sqrt{|A-m^i(\tau)|} \geqslant |w^i|\kappa^{1/2}\tau^{1/3} \geqslant \Theta\kappa^{1/2}\tau^{1/3}$. Moreover, $\left|u_1^2(A-u_1) - u_2^2(A-u_2)\right| = |[A(m^1+m^2)(\tau) - ((m^1)^2 + m^1m^2 + (m^2)^2)(\tau)] \times (w^1 - w^2)\tau| \leqslant 5|A|^2|w^1 - w^2|\tau$. Consequently,

$$\left|\frac{u_1^2(A-u_1) - u_2^2(A-u_2)}{\prod_{i=1,2} u_i\sqrt{|A-u_i|}[u_1\sqrt{|A-u_1|} + u_2\sqrt{|A-u_2|}]}\right| \leqslant \frac{5|A|^2|w^1 - w^2|}{2\Theta^3\kappa^{3/2}}. \tag{A.7}$$

Using (41), (A.7) and lemma 2 in (A.6) we deduce (42).

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