Extreme elevations and slopes of interacting Kadomtsev-Petviashvili solitons in shallow water

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Abstract. We analyse certain geometrical features of interaction of long-crested waves in the framework of two-soliton solution of the Kadomtsev-Petviashvili equation. Such interactions may be responsible for the existence of high-amplitude wave humps. Shown is that the area of extreme elevations is very narrow whereas the extreme slope of the front of the resulting structure may be eight times as high as the maximum slope of the interacting solitons. Analytical expressions for extreme slopes of interacting solitons of arbitrary amplitude are derived. Interactions of solitons of greatly different heights do not cause extreme elevations but may result in extensive bending of the crests of the counterparts.

Introduction

Many authors have suggested that an appropriate nonlinear mechanism could be responsible for extreme waves [1]. We concentrate on a specific source for considerable changes in the wave amplitudes, namely, nonlinear superposition of solitary waves in shallow water. A suitable mathematical model for the description of the interaction of soliton-like shallow water waves travelling under slightly different directions is the Kadomtsev-Petviashvili (KP) equation [2]. It is actually weakly dependent on the transverse coordinate and has been frequently stated to apply to one and one-half dimensions. This equation admits explicit formulae for multi-soliton solutions and offers a possibility to study interaction and resonance of several solitons. A well-known feature of such interactions is that they may lead to spatially localised extreme surface elevations [3–5].

Although known for a long time, this mechanism has been only recently proposed for a possible mechanism of long-living rogue wave formation in shallow water [1,6]. The reason is that it is suitable (i) provided long-crested shallow water waves can be associated with solitons and (ii) provided the KP equation is a valid model for such waves. These conditions may be uncommon for storm waves; however, they may be satisfied when two or more systems of swell approach a certain shelf sea area from different directions. Since a moving pressure disturbance can generate solitary waves also in open sea areas [7,8], unexpected wave humps may occur in areas hosting intense fast ferry traffic [9,10] owing to interaction of wake wash from different ships.

Geometry of interacting waves

The spatial extension of the high hump in the framework of soliton interactions is frequently associated with the area where the interacting waves have a common crest [6]. For equal amplitude incoming solitons the area where elevation exceeds the sum of amplitudes of the counterparts may considerably exceed the estimates based on the geometry of the wave crests [11]. The limits of the amplitude, the spatial occupancy of the high elevation and the slope of the front of the interaction pattern were analysed in some detail for interactions of solitons of equal amplitude [6,7] that are equivalent to the Mach reflection [12,13] of a single soliton and have specific symmetry properties. A pronounced feature of freak waves is that they are particularly steep. Nonlinear interaction in the framework of the KP equation substantially modifies the profile of the two-soliton solution [11]. The slope of the high wave hump may be 8 times as large as the slope of the incoming waves. Thus in this case the nonlinear interaction leads to an extraordinarily high and narrow structure. Such wave hump might easily break before it reaches its theoretically maximum height. The possibility of breaking of the high and nonlinear wave hump makes a hit by a near-resonant structure exceptionally dangerous.

However, nonlinear interactions of solitons of unequal amplitudes are important in many applications [4,14,15]. A part of the analysis of the geometry of high elevations is extended to the case of interacting solitons with unequal amplitudes in [16]. The location and the height of the global maximum of the two-soliton solution as well as its symmetry properties are established in [16] for the case when the maximum amplitude exceeds the sum of amplitudes of the interacting solitons. The relative increase of the amplitude (compared to the sum of amplitudes of the incoming solitons) is largest when the counterparts are equal whereas elevations greatly exceeding the sum of amplitudes of the counterparts only occur when the amplitudes of the intersecting solitons are comparable.

In this paper, we extend a part of the analysis of extremely large slopes of the nonlinear interaction pattern to incoming solitons with unequal amplitudes. The maximum slope of the two-soliton solution in the principal propagation direction is established for the case when its amplitude exceeds the sum of amplitudes of the incoming solitons. The maximum increase of the slope of the interaction pattern occurs in the case of equal amplitude solitons and decreases if the amplitudes of the counterparts become different.

Counterparts of the two-soliton solution

The standard KP equation in normalised variables (η, x, y, t) reads

$$(\eta_t + 6\eta\eta_x + \eta_{xxx})_x + 3\eta_{yy} = 0, \tag{1}$$

where $\eta = (x, y, t)$ describes a certain disturbance, e.g., the elevation of the water surface. A two-soliton solution of Eq. (1) is $\eta = 2\partial^2 \ln\left(1 + e^{\varphi_1} + e^{\varphi_2} + A_{12}e^{\varphi_1+\varphi_2}\right)/\partial x^2$, where $\varphi_i = k_i x + l_i y + \omega_i t$ are phase variables, $\boldsymbol{\kappa}_i = (k_i, l_i)$, $i = 1,2$ are the wave vectors, the frequencies ω_i satisfy the dispersion relation $P(k_i, l_i, \omega_i) = k_i\omega_i + k_i^4 + 3l_i^2 = 0$ of the linearised KP equation, $A_{12} = -P(2k_-, 2l_-, \omega_1 - \omega_2)P^{-1}(2k_+, 2l_+, \omega_1 + \omega_2)$ is the phase shift parameter, $k_\pm = \frac{1}{2}(k_1 \pm k_2)$ and $l_\pm = \frac{1}{2}(l_1 \pm l_2)$ [15,17,18]. In the following we take $t = 0$ without loss of generality. Doing so is equivalent to introducing of a proper coordinate frame moving in a certain direction. This solution can be decomposed into a sum $\eta = s_1 + s_2 + s_{12}$ of two incoming solitons s_1, s_2 and an interaction soliton s_{12} [18]:

$$s_{1,2} = A_{12}^{1/2} k_{1,2}^2 \Theta^{-2} \cosh\left(\varphi_{2,1} + \ln A_{12}^{1/2}\right),\ s_{12} = \tfrac{1}{2} B\Theta^{-2},\quad B = 4A_{12}k_+^2 + 4k_-^2,$$
$$\Theta = \cosh[(\varphi_1 - \varphi_2)/2] + A_{12}^{1/2}\cosh[(\varphi_1 + \varphi_2 + \ln A_{12})/2], \qquad \mathbf{(2)}$$

The solution $\eta(x, y)$ is symmetric with respect to rotations by 180º around the point $x_0 = l_-(k_1 l_2 - k_2 l_1)^{-1} \ln A_{12}$, $y_0 = -k_- l_-^{-1} x_0$ corresponding to $\varphi_1 = \varphi_2 = -\ln A_{12}^{1/2}$ and called the interaction centre. The maximum heights (amplitudes) $a_{1,2} = \frac{1}{2} k_{1,2}^2$ of the counterparts $s_{1,2}$ occur infinitely far from the interaction centre. The phase shift $\Delta_{12} = -\ln A_{12}$ of the counterparts may be either positive or negative. In what follows we consider the negative phase shift case $\Delta_{12} < 0$, $A_{12} > 1$ when $\max\eta \geq a_1 + a_2$.
In the case of equal amplitude solitons $k_1 = k_2$ with $l_1 = -l_2 = l$ the interaction soliton s_{12} has two axes of symmetry: the *x*-axis and the line $k_+ x = -\ln A_{12}^{1/2}$. The incoming solitons s_1, s_2 are the mirror images of each other with respect to these axes. The solution $\eta(x, y)$ is symmetric with respect to these axes. This symmetry is lost in interactions of solitons of unequal amplitudes; however, the interaction soliton is symmetric with respect to both the coordinate axes in the (φ_-, φ_+)-plane, where

$$\varphi_- = \frac{\varphi_1 - \varphi_2}{2},\ \varphi_+ = \frac{\varphi_1 + \varphi_2}{2} + \ln A_{12}^{1/2} \qquad \mathbf{(3)}$$

and expressions for s_1, s_2 and Θ have the particularly simple form [16]:

$$s_1 = A_{12}^{1/2} k_1^2 \Theta^{-2} \cosh(\varphi_+ - \varphi_-),\ s_2 = A_{12}^{1/2} k_2^2 \Theta^{-2} \cosh(\varphi_+ + \varphi_-),$$
$$\Theta = \cosh\varphi_- + A_{12}^{1/2}\cosh\varphi_+ \qquad \mathbf{(4)}$$

Equations (3) define a regular linear affine transformation (unless the wave vectors $\boldsymbol{\kappa}_1$, $\boldsymbol{\kappa}_2$ are collinear) that maps lines of the (x, y)-plane to lines of the (φ_+, φ_-)-plane. In particular, the lines $k_+ x + l_+ y + \ln A_{12} = 0$ and $k_- x + l_- y = 0$ correspond to the φ_+- and the φ_--axes, respectively. These lines are rectangular on the (x, y)-plane and serve as the pair of axes of symmetry of the interaction soliton only provided $|\boldsymbol{\kappa}_1| = |\boldsymbol{\kappa}_2|$ [16].

Crests and lines of steepest descent of the two-soliton solution

The two-soliton solution is stationary in the coordinate system moving in the direction bisecting the angle between $\boldsymbol{\kappa}_1$ and $\boldsymbol{\kappa}_2$, that is, in the φ_+-direction on the (φ_+, φ_-)-plane. Usually, wave crests are defined as sets of points corresponding to the maximum of the wave profile in the direction of its propagation. Since the counterparts of the two-soliton solution of the KP equation propagate at only slightly different directions, it is natural to consider the crest(s) of the whole structure in the principal propagation direction [11,16]. For two-dimensional structures this definition is sometimes ambiguous, because its counterparts may propagate in different directions. If this direction is not known (e.g. there exist only a snapshot of the water surface), crests of a smooth surface $\eta(x, y)$ could be defined as lines of curvature corresponding to the minimum normal curvature of the surface and going through a maximum (minimum) of the surface [19].

A complementary problem to determining of wave crests is to estimate of the slope of the (water) surface. The lines of minimum normal curvature of a single soliton are always parallel with its crest. This feature suggests that the steepest descent of an interaction pattern may be (at least, roughly) perpendicular to the direction of wave crests. Therefore, the problem of the maximum slope has something in common with the problem of finding the lines of curvature corresponding to the maximum normal curvature. In the process of soliton interaction, the formal crests of the incoming solitons form quite a complex pattern [11,12]; however, to a large extent, the crests of the whole pattern are nearly perpendicular to the principal direction of propagation. Therefore, the steepest descent apparently exists roughly along the φ_+-direction. This property has been used heuristically by [11]. The largest slope found based on this assumption for the interaction pattern of equal amplitude solitons is eight times as large as the slope of single solitons.

We use the same heuristic argument, and look for the maximum slope of the solution containing unequal amplitude solitons (interpreted as, e.g., the water surface) along the φ_+-axis. The slopes of the counterparts in the φ_+-direction are

$$\frac{\partial s_{12}}{\partial \varphi_+} = -A_{12}^{1/2} B \Theta^{-3} \sinh \varphi_+ ,$$

$$\frac{\partial s_{1,2}}{\partial \varphi_+} = A_{12}^{1/2} k_{1,2}^2 \Theta^{-3} \left[\Theta \sinh(\varphi_+ \mp \varphi_-) - 2\cosh(\varphi_+ \mp \varphi_-) A_{12}^{1/2} \sinh \varphi_+\right], \tag{5}$$

where the upper sign corresponds to s_1 and the lower sign to s_2. The slope of the surface is

$$S = \frac{A_{12}^{1/2}}{\Theta^3} \left\{ \begin{matrix} -B \sinh \varphi_+ + k_1^2 \left[\Theta \sinh(\varphi_+ - \varphi_-) - 2A_{12}^{1/2} \cosh(\varphi_+ - \varphi_-) \sinh \varphi_+\right] + \\ + k_2^2 \left[\Theta \sinh(\varphi_+ + \varphi_-) - 2A_{12}^{1/2} \cosh(\varphi_+ + \varphi_-) \sinh \varphi_+\right] \end{matrix} \right\}. \tag{6}$$

Since the global maximum of the interaction pattern for the negative phase shift case is at the interaction centre [12] and since the whole structure is most shrinked in the vicinity of this centre, the largest slope apparently exists near this point. For that reason, we only consider the slope along the φ_+-axis ($\varphi_- = 0$). Expression (6) can be simplified as follows:

$$\tilde{S} = \frac{A_{12}^{1/2}}{\Theta_0^3}\left[-B + \left(k_1^2 + k_2^2\right)\left(1 - A_{12}^{1/2}\cosh\varphi_+\right)\right]\sinh\varphi_+ , \quad \Theta_0 = 1 + A_{12}^{1/2}\cosh\varphi_+ . \qquad \textbf{(7)}$$

The slope obviously is zero at $\varphi_+ = 0$. For any other φ_+ where $\tilde{S} = 0$ we have

$$A_{12}^{1/2}\cosh\varphi_+ = 1 - \frac{B}{k_1^2 + k_2^2} . \qquad \textbf{(8)}$$

For the negative phase shift case $A_{12} > 1$ this condition cannot be satisfied. However, for the positive phase shift Eq. (9) may have real solutions. If this happens, there exists exactly one additional point of zero slope at each side of the axis of the interaction pattern. For example, for equal amplitude solitons $k_1 = k_2$, condition (8) can be reduced to $A_{12}^{1/2}\cosh\varphi_+ = 1 - 2A_{12}$ and may have real solutions if $A_{12}^{1/2} < 1 - 2A_{12}$, or $A_{12} < 1/4$ [11,12].

The location of the maximum slope in the φ_+-direction can be found from the condition $\partial S/\partial\varphi_+ = 0$. From expressions (5) we have that

$$\frac{\partial^2 s_{12}}{\partial\varphi_+^2} = \frac{A_{12}^{1/2}B}{\Theta^4}\left(3A_{12}^{1/2}\sinh^2\varphi_+ - \Theta\cosh\varphi_+\right),$$
$$\frac{\partial^2 s_{1,2}}{\partial\varphi_+^2} = \frac{A_{12}^{1/2}k_{1,2}^2}{\Theta_4}\begin{bmatrix} -4\Theta A_{12}^{1/2}\sinh(\varphi_+ \mp \varphi_-)\sinh\varphi_+ - 2A_{12}^{1/2}\Theta\cosh(\varphi_+ \mp \varphi_-)\cosh\varphi_+ \\ +6A_{12}^{1/2}A_{12}^{1/2}\cosh(\varphi_+ \mp \varphi_-)\sinh^2\varphi_+ + \Theta^2\cosh(\varphi_+ \mp \varphi_-) \end{bmatrix} . \qquad \textbf{(9)}$$

At the φ_+-axis, $\Theta = \Theta_0 = 1 + A_{12}^{1/2}\cosh\varphi_+$ and the condition $\partial S/\partial\varphi_+ = 0$ at this axis can be written as

$$\begin{aligned} &A_{12}\left(k_1^2 + k_2^2\right)\cosh^3\varphi_+ + 2A_{12}^{1/2}\left[B - 2\left(k_1^2 + k_2^2\right)\right]\cosh^2\varphi_+ + \\ &+\left[\left(1 - 2A_{12}\right)\left(k_1^2 + k_2^2\right) - B\right]\cosh\varphi_+ + A_{12}^{1/2}\left[4\left(k_1^2 + k_2^2\right) - 3B\right] = 0 \end{aligned} . \qquad \textbf{(10)}$$

This is a cubic equation with respect to $\cosh\varphi_+$ and serves as a generalisation of Eq. (29) in [11]. Certain differences between these equations are caused by different coordinate systems. The sum of all the coefficients of Eq. (10) is $\left(1 - 3A_{12}^{1/2} - A_{12}\right)\left(k_1^2 + k_2^2\right) - A_{12}^{1/2}B - B$. It is negative for the negative phase shift case $A_{12} > 1$, and may become positive only for very small A_{12}. Therefore, there exists always at least one solution $\cosh\varphi_+ \geq 1$ corresponding to the maximum slope provided $A_{12} > 1$. Physically, the existence of such a solution is obvious, because the slope of the surface is zero at the interaction centre, becomes negative in the positive direction of the *x*-axis, and approaches zero at infinity.

In the particular case $A_{12} = 1$, $B = 2\left(k_1^2 + k_2^2\right)$ Eq. (10) has the form $\cosh^3 \varphi_+ - 3\cosh \varphi_+ - 2 = 0$ and has an obvious solution $\cosh_1 \varphi_+ = 2$, $\sinh_1 \varphi_+ = \sqrt{3}$ (cf. [11]). This case is similar to the linear superposition of waves, because neither phase shift nor changes in the resulting wave amplitude occur; however, this is possible only if one of the incoming waves is infinitesimally small [15]. The maximum slope in this case is

$$\tilde{S}_1 = \frac{1}{3\sqrt{3}}\left(k_1^2 + k_2^2\right). \tag{11}$$

For near-resonant case $A_{12} \to \infty$ Eq. (10) has an asymptotic solution $\cosh_\infty \varphi_+ = \sqrt{3/2}$, $\sinh_\infty \varphi_+ = \sqrt{1/2}$. In [11] the corresponding point at the *x*-axis is located at the distance $\sim \ln A_{12}^{1/2}$ from the origin. This location tends to infinity when $A_{12} \to \infty$ and the maximum slope calculated in [11] is, strictly speaking, correct only asymptotically. In the coordinate system used in the current paper the point of the maximum slope is located at a finite distance from the origin. The slope at this point at the resonance case $A_{12} \to \infty$ and the corresponding slope amplification factor are

$$\tilde{S}_\infty = \frac{2\left(k_1 + k_2\right)^2}{3\sqrt{3}}, \quad m_s = \frac{\tilde{S}_\infty}{\tilde{S}_0} = \frac{2\left(k_1 + k_2\right)^2}{\left(k_1^2 + k_2^2\right)}, \tag{12}$$

respectively. The latter expression is the generalisation of an analogous result for equal amplitude solitons 11. This factor is exactly twice the analogous amplitude amplification factor [12]. In the case of equal amplitude solitons $m_s = 4$, and it decreases to 1 for solitons of greatly different amplitudes.

Extent of the area of nonlinear effects of the two-soliton solution

The spatial extent of the extreme slopes apparently follows the extent of the extreme elevations and the area where extensive deformation of the crests of the incoming solitons occur and where the whole structure has a single crest. This extent for the special case of equal amplitude solitons is studied in [11]. To the first approximation, the area where the two-soliton solution exceeds the amplitude occurring in the process of linear superposition of s_1 and s_2 can be well described with the use of the (geometric) length of the idealised common part L_{12} (Fig. 1) of the crests of the incoming soliton [6,11].

Since the interaction pattern of the two-soliton solution of the KP equation only depends on the amplitudes of the incoming solitons and the angle between their crests (that define the parameter λ), the coordinate system for description of the instantaneous interaction pattern can be always chosen so that $l_1 = -l_2 = l$. The whole pattern is not necessarily steady in such coordinates.

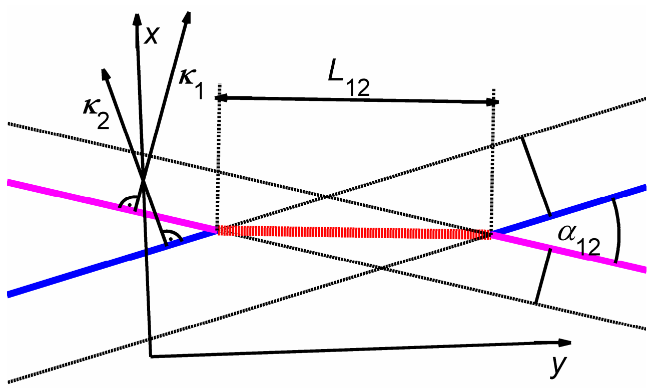

Fig. 1. Idealised patterns of crests of incoming solitons (blue and magenta lines), their position in the absence of interaction (dashed lines) and the interaction soliton (bold-dashed red line) corresponding to the negative phase shift case.

However, in this particular case the geometric length $L_{12} = l^{-1} \ln A_{12}$ [6] only depends on the l-component of the wave vectors and the phase shift parameter $\ln A_{12}$, and shows no explicit dependence on the amplitudes of the incoming solitons. Therefore, the spatial extent of appearance of nonlinear effects for interactions of solitons with drastically different amplitudes in terms of the geometric length is as large as if the amplitudes were equal (Fig. 2).

This feature of interaction of solitary waves of unequal height may be particularly important in applications where the function $\eta(x, y)$ has the meaning of surface elevation [1,6,12,20] and the extent and orientation of the near-resonant structure are equally important [15,18]. For example, in shallow sea areas near-resonant interaction of solitonic surface wave systems with radically different amplitudes apparently become evident in the form of bending of crests of the waves [15,16,20,21] rather than in the form of extreme elevations. This feature can be frequently observed in very shallow water (Fig. 3). In open sea conditions it apparently cannot be recognize in isolated form but its effect may drastically increase the probability of encountering a hit by a high wave possibly with a particularly large slope [7] and arriving from an unexpected direction.

Apart from wind-generated rogue waves, the presented mechanism may have an intriguing application in the analysis of abnormally high waves in shallow coastal areas hosting intense high speed ship traffic. The sequences of long-crested soliton-like waves are frequently excited by contemporary ships if they sail at speeds roughly equal with the maximum phase speed of gravity waves [8,10,22,23]. Groups of solitonic waves intersecting at a small angle may appear if wakes from two ships meet each other. Their interaction may be responsible for dangerous waves along shorelines.

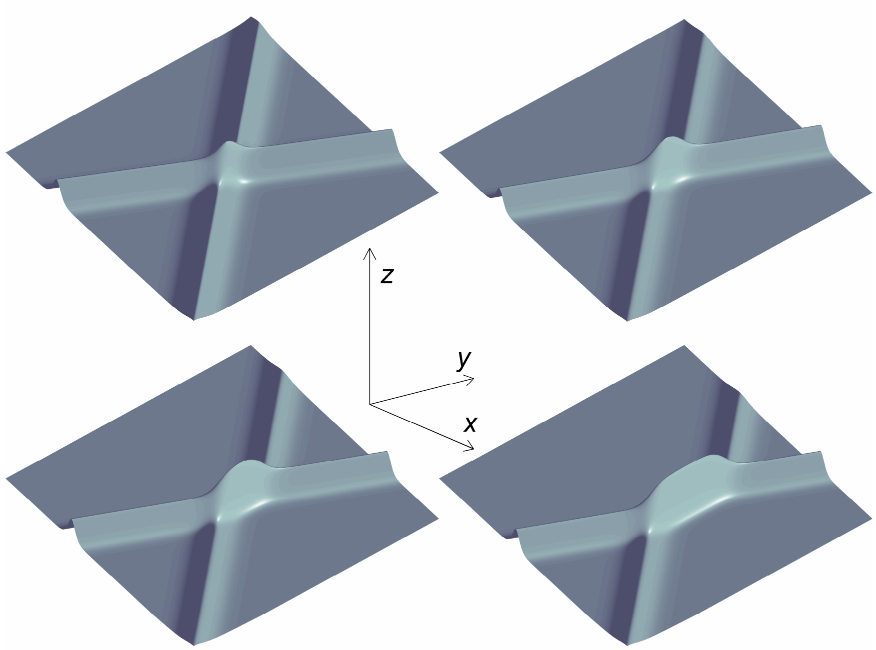

Fig. 2. Surface elevation in the vicinity of the interaction area, corresponding to incoming solitons with unequal amplitudes: $k_2 = 0.3$ $l = -l_1 = 0.2$, $k_{res} = 2/3$ and $k_1 = 0.9k_{res}$ (upper left panel), $k_1 = 0.99k_{res}$ (upper right), $k_1 = 0.999k_{res}$ (lower left), $k_1 = 0.9999k_{res}$ (lower right) in normalised coordinates (x, y). Area $|x| \le 60$, $|y| \le 90$ is shown at each panel.

The fraction of sea surface occupied by extreme elevations or waves propagating in an unexpected direction apparently is small as compared with the area of a wave storm, because extensive areas of appearance of nonlinear effects may occur only if the heights of the incoming waves and their intersection angle are specifically balanced. An important difference should be underlined between specific waves possibly excited by the described mechanism and those arising owing to focusing of transient and directionally spread waves. In the latter case a number waves with different frequencies and propagation directions are focused at one point at a specific time instant to produce a time-varying transient wave group that normally does not propagate far from the focussing area. A wave hump from nonlinear interaction, theoretically, has unlimited life-time and may cross large sea areas in favourable conditions. Thus, one should account for the expected life-time of nonlinear wave humps (additionally to the sea area covered by extreme elevation at a certain time instant) when estimating the probability of occurrence of abnormally high waves.

Fig. 3. Interaction patterns of soliton-like surface waves in very shallow water near Kauksi resort on Lake Peipsi, Estonia (Photo by Lauri Ilison, July 2003).

Acknowledgements

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