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# Solitonic structures in KdV-based higher-order systems

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**Abstract**

Wave propagation in microstructured materials is studied using a Korteweg–de Vries (KdV)-type nonlinear evolution equation. Due to the microstructure, nonlinear effects are described by a quartic elastic potential and dispersive effects — by both the third- and the fifth-order space derivatives. The problem is solved numerically under harmonic initial condition. For nondispersive materials, the quartic elastic potential, compared with that of the second-order (KdV) one, leads to the formation of two additional discontinuities in the harmonic initial wave profile. This together with the additional dispersive effect is the reason for emerging complicated solitonic structures (train of solitons, train of negative solitons and multiple solitons) depending on the values of dispersion parameters. Chaotic motion results if both the third- and the fifth-order dispersion parameters take the small possible values. © 2001 Elsevier Science B.V. All rights reserved.

*Keywords:* Solitonic structures; Wave propagation; KdV-type equation

## 1. Introduction

The classical wave theories usually describe solids as homogeneous materials. Modern physics and technology, however, cannot take the concept of homogeneity as a sole basis because of short wavelengths, coupled fields, phase changes, etc. where the microstructure of solids should be accounted for. Some recent results on mechanical behaviour of microstructured materials are described by Fremond and Miyazaki [1] and Suquet [2].

The characteristic lengths in a solid with microstructure may vary in a large range. For example, metal–matrix composites are divided into three groups [3]: dispersion-strengthened composites (characteristic length of 0.01–0.1 μm); particle-reinforced composites (characteristic length about 0.1 μm); fibre-reinforced composites (characteristic length of 0.1–250 μm or more). In geomaterials the scales are much different, the block structure of the Earth's crust may be characterised by kms in comparison with wavelengths two or three orders larger [4]. It is clear that a field-theoretic approach used to derive the governing equations of wave motion should take such a microstructure into account. A detailed description of modelling the influence of material inhomogeneities is given by Maugin [5] (see also [6,7]).

From the physical viewpoint the most interesting question is the influence of the microstructure on the dispersion of waves. We are inclined to take nonlinear behaviour of a material as a basic notion, especially in such complicated

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cases when many accompanying effects should be accounted for [8]. Then we confront immediately the question of the combined effect of nonlinearity and dispersion.

A large variety of nonlinear wave processes has been described by means of generic equations obtained in different asymptotic limits. These generic equations are actually the evolution equations written in a moving frame and describing very slow variations of wave profiles. The celebrated Korteweg–de Vries (KdV) equation, especially since the seminal paper of Zabusky and Kruskal [9] is often used for describing the basic effects combining quadratic nonlinearity and cubic dispersion. This is a rather simple integrable equation that stems from hydrodynamics and plasma physics and later its modifications are used in many cases. In the general case, a KdV-like equation may be presented in the following form:

$$u_t + [P(u)]_x + D(u) = 0. \tag{1.1}$$

Here and below, the indices denote differentiation. For a purely dispersive structure ($P(u) = 0$ or $P(u) = k_1 u$, $k_1 =$ const.), function $D(u)$ contains odd derivatives of $u$ only:

$$D(u) = k_3 u_{xxx} + k_5 u_{5x} + k_7 u_{7x} + \cdots . \tag{1.2}$$

The case $k_3 \neq 0$, other $k_i = 0$, $P(u) = (1/2)u^2$ corresponds to the KdV equation, modelling waves in shallow water. If in addition to that also $k_5 \neq 0$, then Eq. (1.1) corresponds to shallow water again but with special values of Froude and Bond numbers [10]. This equation has been the topic of many studies, starting from early Kawahara paper [11] up to more recent ones [12,13] (see also the list of references therein). The cases $k_3 \neq 0$, $k_5 \neq 0$, $k_7 \neq 0$ and other $k_i = 0$, $P(u) = (1/2)u^2$ has been studied in [14]. However, the nonlinearity may also be quite different from a quadratic function. The cases studied intensively, are characterised by $P(u) \sim u^p$, $p > 2$ and $k_3 \neq 0$ and other $k_i = 0$ that leads to the so-called modified KdV (mKdV) equation [15]. If waves in shallow-water are described more accurately over the topography of the bottom then [13]

$$P(u) = \tfrac{1}{2}u^2 + \tfrac{1}{3}\alpha u^3 + \alpha u u_{xx}, \quad \alpha = \text{const.}, \tag{1.3}$$

with $k_3 \neq 0$, $k_5 \neq 0$ and other $k_i = 0$. It is clear that a complicated function $P(u)$ like Eq. (1.3) leads to additional physical effects including radiation.

This is also evident in solids with microstructure. First, there are several generic systems, like sine-Gordon–d' Alembert system for describing the propagation of magnetoacoustic domain walls in elastic ferromagnets [16], or generalised Zakharov systems for describing the propagation of surface solitary waves on a thin film over a nonlinear substrate [17]. The generic equations describe also solitary waves in shape–memory alloys [18]. It has been shown that for martensitic–austenitic shape–memory alloys the governing equation is the sixth-order Boussinesq equation [19] leading to the evolution equation (1.1) with

$$P(u) = -\tfrac{1}{2}u^2 + \tfrac{1}{4}u^4, \tag{1.4}$$

$$D(u) = d u_{xxx} + b u_{5x}, \tag{1.5}$$

where $d$ and $b$ are constants. Here the higher-order dispersion is caused by dislocations in the crystal structure [5].

Coming to the asymptotic limits on which the above mentioned cases are based then actually there is no special difference between possible scales characterising microstructure. If micro- and macrostructure are described by various field equations [6,7] then the outcome is a wave hierarchy in the sense of Whitham [20]. Giovine and Oliveri [21] have applied such an approach to granular materials and derived a generic equation

$$u_t + [P(u)]_x + D_1(u) + \frac{\partial^m}{\partial x^m}[u_t + [P(u)]_x + D_2(u)] = 0. \tag{1.6}$$

In this case $P(u) = (1/2)u^2$, $D_1 = k_{31} u_{xxx}$, $D_2 = k_{32} u_{xxx}$, $m = 2$ and $k_{31} \neq k_{32}$. Engelbrecht et al. [22] have found a wave hierarchy for a dissipative case in microstructured materials modelled following [6,7]. Then instead of dispersive operators $D_i(u)$ dissipative operators $Q(u)$ are involved that include even derivatives ($u_{xx}$) and $m = 1$.

This brief overview shows the range of model equations including higher-order dispersive effects together with complicated nonlinearities. The significant changes in wave profiles may take place due these effects. The questions studied include such like: do solitary waves exist at all in these systems, do solitary waves if they exist interact elastically or not, do they exist up to a certain time only emitting the radiation and possibly end up in an oscillating field, does the existence of oscillatory "tails" of solitary waves lead to chaotic fields, etc. To answer these questions is not only for theoretical interest, the light must be cast over wave phenomena in practical applications of microstructured solids.

The aim of this paper is to study the formation process of solitons and solitary waves in martensitic–austenitic alloys modelled by the governing evolution equation (1.1) with structure expressed by Eq. (1.4). The preliminary results have been reported in [23] and the numerical procedures in [24]. Here we focus our attention to the detailed analysis of dispersive and nonlinear effects that govern the formation process of solitary waves. Section 2 gives the statement of the problem, in Section 3 the influence of both nonlinear and dispersive effects is analysed separately. This analysis shows explicitly how the complicated mathematical model differs from the classical ones. Section 4 involves the discussion and the analysis of numerical results. Balance between the dispersive and nonlinear effects depends upon the intensity and ratio of dispersion parameters. This is the reason for emerging complicated solitonic structures. In Section 5 conclusions and a summary are presented.

## 2. Mathematical model

In the present paper the evolution equations (1.1) and (1.4) are studied. We write them as follows:

$$u_t + [P(u)]_x + du_{xxx} + bu_{5x} = 0, \tag{2.1}$$

where $u(x,t)$ is the strain, and $d$ and $b$ are the third- and the fifth-order dispersion parameters, respectively. The fourth-order (quartic) elastic potential is

$$P(u) = -\tfrac{1}{2}u^2 + \tfrac{1}{4}u^4. \tag{2.2}$$

As said in Section 1, this is a mathematical model for waves in shape–memory alloys [19,23]. We are interested in the formation of solitary waves from an harmonic excitation. Boundary conditions are given in the periodic form

$$u(x,t) = u(x + 2n\pi, t), \quad n = \pm 1, \pm 2, \ldots \tag{2.3}$$

and the initial excitation by

$$u(x,0) = \sin x, \quad 0 \le x \le 2\pi. \tag{2.4}$$

Later for the sake of convenience, the notations

$$d_l = -\log d, \qquad b_l = -\log b \tag{2.5}$$

are introduced.

Eq. (2.1) with Eq. (2.2) is nonintegrable and for the solutions (2.1)–(2.4) the pseudospectral method and the corresponding code [23,24] is used. The numerical analysis is carried out for the following range of parameters:

$$0 \le d_l \le 2.4, \qquad 0.4 \le b_l \le 4.4. \tag{2.6}$$

It should be noted that $d_l \sim 2.32$ corresponds to the Zabusky–Kruskal case [9] with quadratic nonlinearity and soliton train.

The accuracy of the code and the time-step constants have been analysed earlier [24].

## 3. Nonlinearity and dispersion

### 3.1. Effects of nonlinearity

The quartic nonlinearity (2.2) describes the fourth-order elastic potential possessing two minima that is characteristic to shape–memory alloys [18,19]. Clearly then

$$[P(u)]_x = [P(u)]_u u_x = (-u + u^3)u_x, \tag{3.1}$$

while the ordinary KdV equation possesses the quadratic nonlinear terms $\pm uu_x$ only. The potential $P(u)$ together with the corresponding potentials for the KdV and mKdV (quartic term only) equations is shown in Fig. 1. Although the amplitude of the initial excitation (2.4) is taken equal to the unity, the values $|u| > 1$ appear in the process of formation of solitary waves. The derivative $[P(u)]_u$ is shown in Fig. 2 for our case (3.1) and the KdV case. An essential difference between these two cases is clearly observed. The KdV equation involves the potential $u^2/2$ resulting in the $uu_x$ term. Now, at $u = u_{\mathrm{cr}}$, the influence of the potential (2.2) and the standard KdV one is the same. It is easily calculated that $u_{\mathrm{cr}} = \pm\sqrt{2}$. At $u = u_{\mathrm{q}} = \pm 1$, the character of the quartic potential (2.2) is changed. For $0 < |u| < |u_{\mathrm{q}}| < |u_{\mathrm{cr}}|$, the nonlinear effects due to potential (2.2) are qualitatively different from the KdV case. For $|u_{\mathrm{q}}| < |u| < |u_{\mathrm{cr}}|$, the nonlinear effects due to Eq. (2.2) are weaker than for the KdV case while for $|u| > |u_{\mathrm{cr}}|$ they are stronger.

Next, the influence of nonlinearities is crucial in the formation process of solitary waves. Since the pioneering study of Zabusky and Kruskal [9] it is known that the train of solitons starts to form in a region where there is a tendency for a shock wave formation in the purely nonlinear case (dispersion neglected). In Fig. 3 shock wave profiles are presented for three potentials. In the KdV-type of the quadratic nonlinearity (case (a) of Fig. 3), the shock wave from an initial harmonic excitation (2.4) has the well-known $N$-form [20]. The case (b) of Fig. 3 shows two discontinuities formed at the same region of the wave profile (sign correspondence). Finally, the case (c) of the

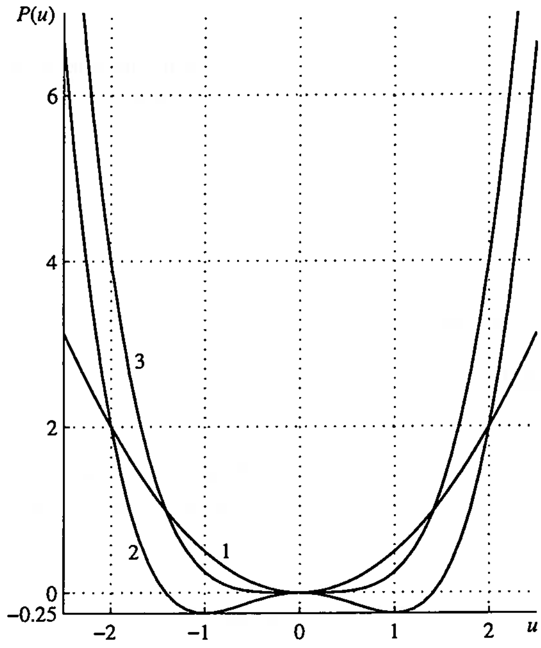


Fig. 1. Potentials $P(u)$ vs. $u$: (1) KdV equation; (2) Eq. (2.1) with Eq. (2.2); (3) mKdV equation.

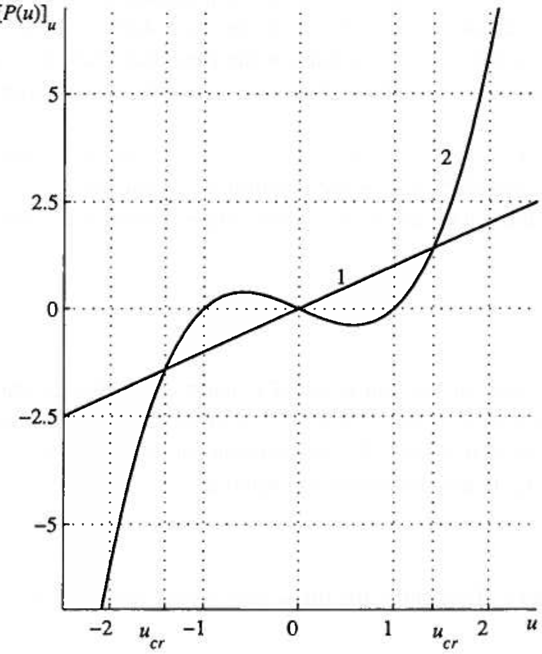


Fig. 2. Derivative of the potential $P(u)$ vs. $u$. For legend see Fig. 1.

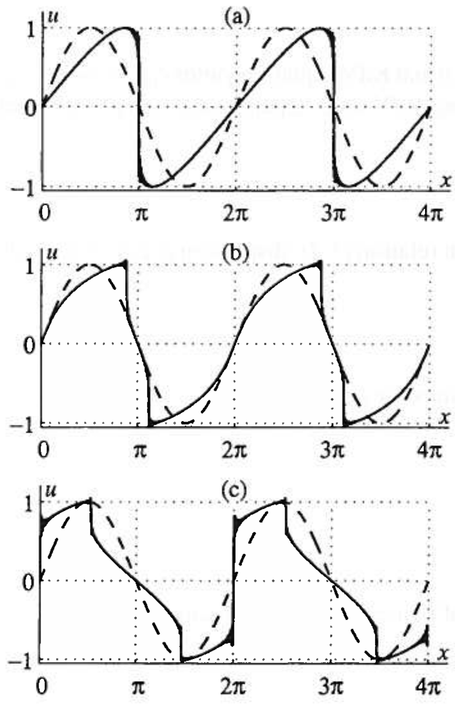


Fig. 3. Shock wave profiles: (a) quadratic potential of the KdV-type; (b) potential involving only the quartic term (the mKdV-type); (c) quartic potential (2.2).

potential (2.2) has three discontinuities for a period, two corresponding to the quartic term and one to the quadratic term (note the influence of the sign difference). For both the KdV-type and mKdV-type nonlinearities the wave profiles shown in Fig. 3 correspond to $t = 1.1$, while for the potential (2.2) the wave profile is shown at $t = 2.0$. Therefore, in the last case the formation of a discontinuous wave profile lasts about two times longer than the other cases.

It is well-known that in the KdV case the dispersive term balances the influence of the nonlinear term in such a way that the train of solitons is formed. Here, for the potential (2.2), the process of balancing starts at three regions over a period instead of one in the KdV case. Clearly the outcome should be more complicated and needs also a full analysis of dispersive effects.

### 3.2. *Dispersive effects*

The influence of the higher-order dispersion is usually analysed on the example of the KdV-like normalised equation with the fifth-order derivative added [25,26]. Here instead of the quadratic nonlinearity we have a more complicated potential and are forced to keep different constants at dispersive terms like in [27,28].

The linearised version of Eq. (2.1) has the dispersion relation

$$\omega = k^3(bk^2 - d) \tag{3.2}$$

for frequency $\omega$ and wavenumber $k$. Evidently, the phase and group velocities are

$$c_{\mathrm{ph}} = \frac{\omega}{k} = k^2(bk^2 - d), \tag{3.3}$$

$$c_{\mathrm{gr}} = \frac{\mathrm{d}\omega}{\mathrm{d}t} = k^2(5bk^2 - 3d), \tag{3.4}$$

respectively. For the comparison, the usual KdV-equation yields $c_{\mathrm{ph}} = -dk^2$, $c_{\mathrm{gr}} = -3dk^2$. In Fig. 4, the curves (3.3) and (3.4) together with corresponding KdV-case dependencies are plotted against $k$. For the KdV-case dispersion is normal [29], i.e.

$$c_{\mathrm{gr}} < c_{\mathrm{ph}} < 0, \tag{3.5}$$

for Eq. (2.1), following its dispersion relation (3.2), dispersion is either normal (see inequality (3.5)) or anomalous, satisfying inequality

$$c_{\mathrm{gr}} > c_{\mathrm{ph}}. \tag{3.6}$$

The zeros of phase and group velocities for our case are

$$k_0 = \sqrt{\frac{d}{b}} = 10^{(1/2)(b_1 - d_1)}, \tag{3.7}$$

$$k_0^* = \sqrt{\tfrac{3}{5}}k_0, \tag{3.8}$$

respectively. The minimum values of phase and group velocities occur at

$$k_{\mathrm{m}} = \frac{1}{\sqrt{2}}k_0, \tag{3.9}$$

$$k_{\mathrm{m}}^* = \sqrt{\tfrac{3}{10}}k_0, \tag{3.10}$$

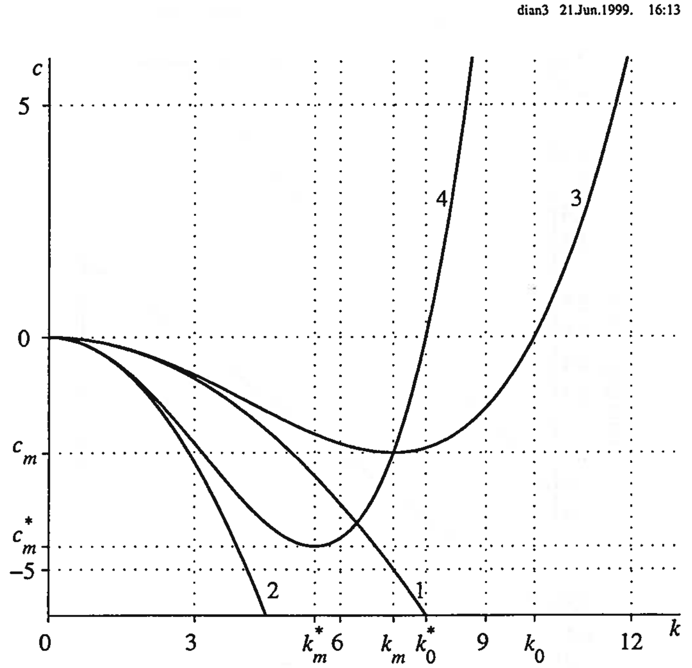


Fig. 4. Phase and group velocities: (1) $c_{\rm ph}$ for the KdV; (2) $c_{\rm gr}$ for the KdV; (3) $c_{\rm ph}$ for Eq. (2.1); (4) $c_{\rm gr}$ for Eq. (2.1). Here $d_{\rm l} = 1$, $b_{\rm l} = 3$.

respectively. For these wavenumbers, the respective values of phase and group velocities are (see also [28])

$$c_{\rm m} = -\frac{1}{4}\frac{d^2}{b}, \tag{3.11}$$

$$c_{\rm m}^* = -\frac{9}{20}\frac{d^2}{b}. \tag{3.12}$$

Dispersion relation (3.2) describes normal dispersion in a certain interval $0 < k < k_{\rm e}$. The interval can be determined using the condition $c_{\rm ph} = c_{\rm gr}$ that yields $k_{\rm e} = k_{\rm m}$. Indeed,

$$c_{\rm gr} = \frac{{\rm d}\omega}{{\rm d}k} = c_{\rm ph} + k\frac{{\rm d}c_{\rm ph}}{{\rm d}k} \tag{3.13}$$

and $c_{\rm gr} = c_{\rm ph}$ is satisfied only at ${\rm d}c_{\rm ph}/{\rm d}k = 0$. In terms of harmonics, condition $k_{\rm e} = k_{\rm m}$ means that the behaviour of harmonics with wavelength $\lambda < 2\pi k_{\rm m}^{-1}$ corresponds to the anomalous dispersion and with wavelength $\lambda > 2\pi k_{\rm m}^{-1}$ to the normal dispersion.

## 4. Discussion

Equipped with the knowledge about the separated influence of dispersive effects and nonlinearity in model (2.1) and (2.2) the question about the existence of solitary waves in the full model has to be answered. Here we touch very briefly the question about single solitons and their properties because this analysis has been given in [30] and focus our attention on the process of emerging of solitonic structures from a harmonic excitation.

It is expected that the higher-order dispersive effects together with complicated quartic nonlinearity will cause significant changes (both qualitative and quantitative) in the formation of solitary waves compared with the process

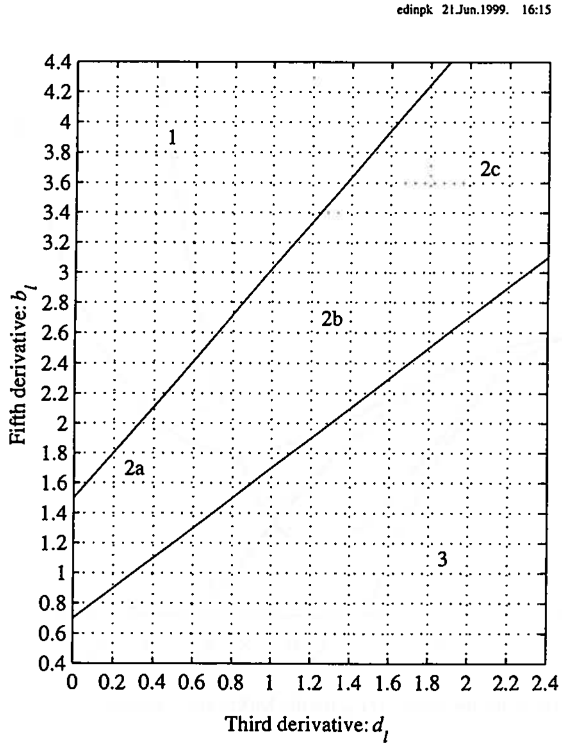


Fig. 5. Regions with different dispersive properties in the $d_l$–$b_l$ plane.

governed by the classical KdV system [31]. The numerical experiment has demonstrated many features of that process (earlier results presented in [23]). Contrary to the analysis of the usual KdV system [31], here the process must be explained in the plane of two control factors — two dispersion parameters $d_l$ and $b_l$. The corresponding $d_l$–$b_l$ plane is shown in Fig. 5. In the lower-right region of the $d_l$–$b_l$ plane the fifth-order dispersion parameter clearly dominates over the third-order one. In the upper-left region of the $d_l$–$b_l$ plane the situation is certainly of the reverse character: the third-order dispersion parameter dominates over that of the fifth-order. The analysis of changes of phase and group velocities (see Fig. 4) shows that the dispersion is anomalous in the first case and normal in the second case. We limit here ourselves to the case $db > 0$ only. Then the conditions of Karpman [28] could be followed which are derived for $[P(u)]_u = u$. He has shown that the positive solitons are unstable emitting radiation while negative solitons are stable provided

$$\varepsilon > \tfrac{1}{2}, \qquad \varepsilon = |da|^{1/2} b^{-1}, \tag{4.1}$$

where $a$ is the amplitude of the soliton. In our case $[P(u)]_u = -u + u^3$ (see Fig. 2). It can be easily concluded that in our case negative solitons should be stable if $a > u_\mathrm{q}$ and positive solitons should be stable if $a < u_\mathrm{q}$. The existence of the radiation raises the question whether solitary wave exists up to a certain limit and due to emitting radiation ends up in an oscillatory field. It is known [26] that oscillatory components propagate with the $c_\mathrm{ph}$ and steady solitary waves occur at $c_\mathrm{ph} = c_\mathrm{gr}$ which means the minimum of $c_\mathrm{ph}$ (see expression (3.13)).

The numerical experiment for what the pseudospectral code [23,24] is used has shown the following. First, the existence of stable positive and negative solitons has been proved satisfying the conservation laws and the criteria of Karpman [28]. The interaction of solitons, however, exhibits more complicated character. It has been shown that negative solitons are interacting elastically while positive solitons start to emit radiation during the interaction

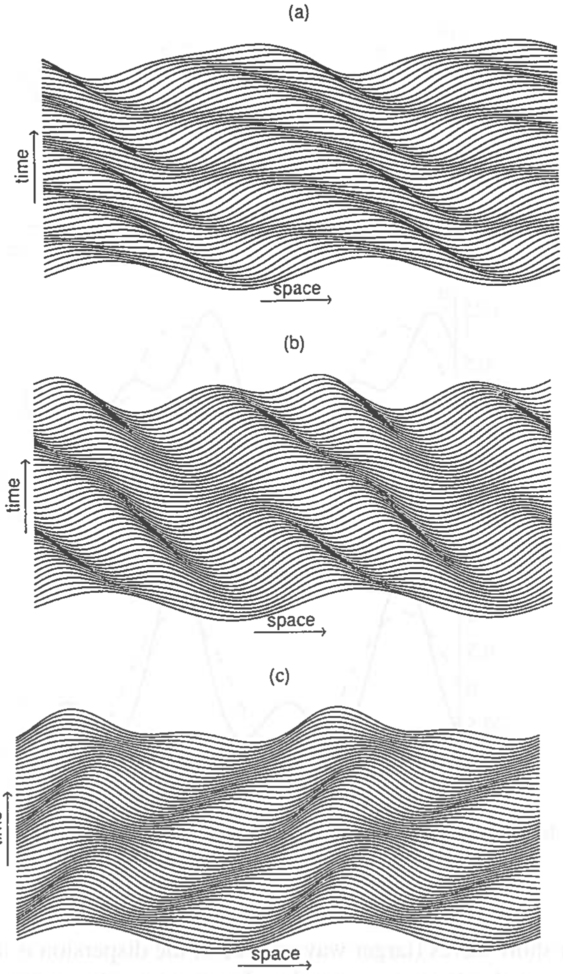


Fig. 6. Timeslice plots over two periods of excitation: (a) a train of negative solitons; (b) a double soliton; (c) a train of position solitons.

process and end up in an oscillating field after several interactions. For details we refer to our earlier paper [30].

Second, we are interested in the emerging solitonic structures from an initial harmonic excitation. In order to demonstrate the importance of the ratio of dispersion parameters for the formation process we refer to the $d_1$–$b_1$ plane in Fig. 5. The typical timeslices and wave profiles are shown in Fig. 6 and 7.

In region 1 (Fig. 5) the normal dispersion dominates, only for very short wavelengths the dispersion is anomalous. Clearly the third-order dispersion effects are more important than those caused by the fifth-order effects. A train of negative solitons forms from the initial harmonic excitation (Figs. 6a and 7a). This situation is similar to that governed by the KdV equation with $P(u) = -(1/2)u^2$, $k_3 \neq 0$ and other $k_i = 0$. The number of solitons in a train depends upon the dispersion parameter $d_1$. In region 3 (Fig. 5) the anomalous dispersion dominates and the fifth-order dispersive effects take over the third-order effects. Now a train of positive solitons forms from the initial harmonic excitation (Figs. 6c and 7c). Obviously the situation is similar to the KdV equation with $P(u) = (1/2)u^2$, $k_3 \neq 0$ and other $k_i = 0$.

Region 2 in Fig. 5 is more complicated and involves several subregions. The main feature is the rivalry between normal and anomalous dispersion that depends also on the wavelength. For long waves (smaller wavenumbers) the

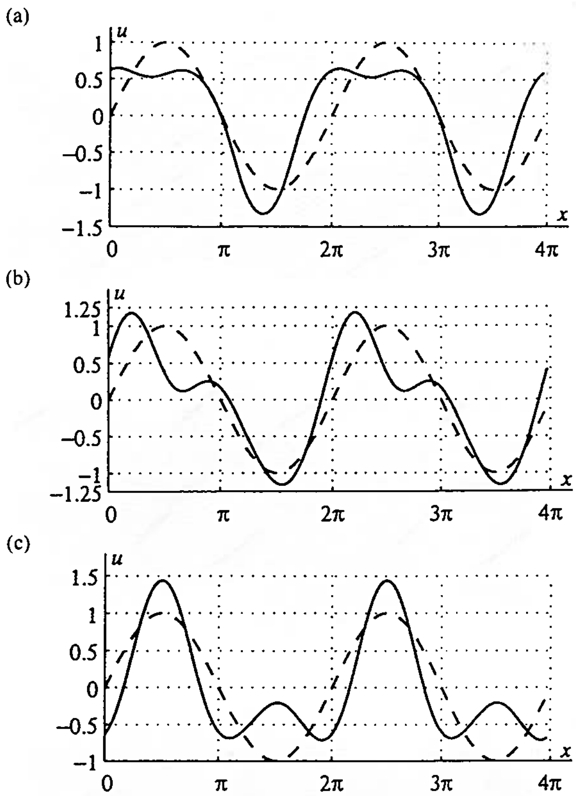


Fig. 7. Initial harmonic wave profile (dashed line) and typical emerging profiles (solid line): (a) a train of negative solitons; (b) a double soliton; (c) a train of positive solitons.

dispersion is normal while for short waves (larger wavenumbers) the dispersion is anomalous. In a train of solitons both situations can occur. The rivalry between normal and anomalous dispersion leads to a situation when both the train of negative and the train of positive solitons might start to form. In subregion 2a (caused by both the third- and fifth-order effects) is stronger that yields in multiple solitons (Figs. 6b and 7b). In subregion 2c dispersion is weaker and spatio-temporal chaos can take place. It means the fluctuations in amplitudes (and spectral densities) are irregular within a certain limit. In subregion 2b interaction of multiple solitons and the train of positive solitons takes place.

## 5. Conclusions

The microstructured material under consideration is characterised by a quartic elastic potential and high-order dispersive effects. The emerging solitonic structures depend on dispersive properties (governed by $d_1$ and $b_1$) and on tendencies to the formation of discontinuities (governed by nonlinearity). This yields several types of stable solitonic structures in most cases but in the case of weak dispersion spatio-temporal chaos has been observed. The mathematical model for the wave propagation — a KdV-type evolution equation is still a relatively simple model used to describe microstructural properties. The influence of various scales of a microstructure may lead to more complicated models where internal variables play decisive role (see, for example, [21,22]). These studies are in progress.

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