7. KAOOTILISTE PROTSESSIDE UURIMISMEETODID

1. Renormaliseerimine
2. Melnikovi meetod

    Duffingi võrrand

    dipooli võrrand

    matemaatilise pendli võrrand (vt. rmt)
3. Tširikovi kattumismeetod

    näited (vt. rmt)
4. Ljapunovi eksponendid / numbrid

    näited (vt. rmt)

    Ljapunovi spekter

    tabel näidetega

    seos fraktaaldimensiooniga
5. Kolmogorovi entroopia
6. KAM'i teoreem

    (Kolmogorov –Arnold – Moser)
7. Spektraalanalüüs

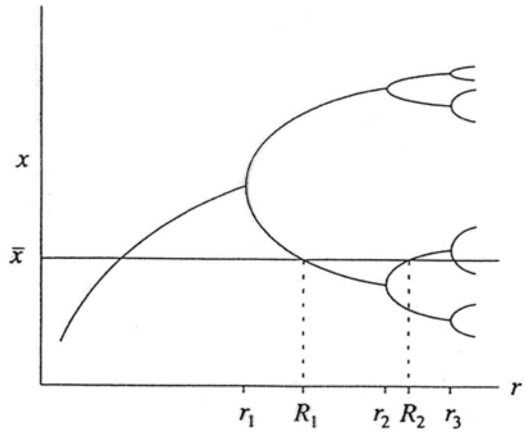
Renormaliseerimine

Funktsionaalanalüüsi teooria, mille kohaselt võrrandisüsteemi omadused antud mastaabis seotakse omadustega teisel mastaabil.

Vt. Lepik - Engelbrecht 8.1 lk.154

Perioodi kahendumine

$$x_{n+1} = f(x_n)$$
$$f(x) = \lambda x (1 - x)$$
- logistiline võrrand vt. Feigenbaumi diagramm



Joonis 8.1 Feigenbaumi diagramm

Periood 1 - . . . . . püsipunkt

$$x_0 = \lambda x_0 (1 - x_0)$$

$$\begin{cases} x_0 = 0 \\ x_0 = \dfrac{\lambda - 1}{\lambda} \end{cases}$$

Kui $\left[\frac{d f(x_0)}{dx}\right] < 1,$ siis stabiilne

Kui $\left[\frac{d f(x_0)}{dx}\right] > 1,$ siis ebastabiilne

Meil $\frac{d f(x)}{d x} = f' = \lambda(1-2x)$

piir $\lambda(1-2x) = 1$

$\lambda_1 = \frac{\pm 1}{|1-2x_0|}$ → tekib periood 2

<u>Periood 2</u> - . . . . . püsipunktid

$$x_2 = (f(x_2))$$

$$x_2 = \lambda^2 x_2(1-x_2)\left[1-\lambda x_2(1-x_2)\right]$$

$$x_0 \to x^+, x^-$$

$$x^+ = \lambda x^-(1-x^-)$$

$$x^- = \lambda x^+(1-x^+)$$

Millal periood 4 ?

st. milline on λ periood 4 tekkimisel, $\lambda = \lambda_2$?

Teisendus:

Olgu $x_n = x^{\pm} + \eta_n$

Asetame x^+, x^- võrranditesse

$$\begin{cases} \eta_{n+1} = \lambda\eta_n\left[(1-2x^+) - \eta_n\right] \\ \eta_{n+2} = \lambda\eta_{n+1}\left[(1-2x^-) - \eta_{n+1}\right] \end{cases}$$

Lahendame η_{n+2} suhtes , täpsus $\vartheta(\eta_n^2)$

$$\eta_{n+2} = \lambda^2 \eta_n [A - B\eta_n]$$

$$A = A(x^+, x^-, \lambda)$$

$$B = B(x^+, x^-, \lambda)$$

Alustasime

$$x_{n+1} = \lambda x_n (1 - x_n)$$

Saime

$$\eta_{n+2} = \lambda^2 \eta_n (A - B\eta_n)$$

Muutujate vahetus , parameetrite vahetus

$$\bar{x} = \alpha \eta, \quad \bar{\lambda} = \lambda^2 A, \quad \alpha = B/A$$

$$\bar{x}_{n+2} = \bar{\lambda} \bar{x}_n (1 - \bar{x}_n)$$

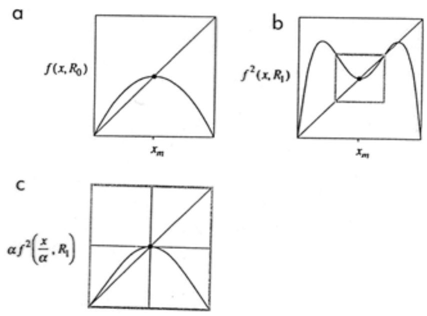
Seega sama seadus

Renormaliseerimine üldiselt

$$f(x) = f_1(x)$$

$$f_2(x) = f_1(f_1(x))$$

$$\ldots$$



Joonis 8.2 Renormaliseerimise idee

1. Arvutame $f_2(x)$
2. Muudame mastaapi tegur $|\alpha| > 1$
3. Pöörame telgi $(-x, -y)$

$$f_1(x, R_0) \sim \alpha f_2\left(\frac{x}{\alpha}, R_1\right)$$

Periood 2 $\rightarrow$ Periood 4

$$f_2\left(\frac{x}{\alpha}, R_1\right) \sim \alpha f_4\left(\frac{x}{\alpha^2}, R_2\right)$$

$$f_1(x, R_0) \sim \alpha^2 f_4\left(\frac{x}{\alpha^2}, R_2\right)$$

$$\ldots\ldots$$

$$f_1(x, R_0) \sim \alpha^n f_{2^n}\left(\frac{x}{\alpha^n}, R_n\right)$$

Feigenbaum (1978):

$$\lim \alpha^n f_{2^n}\left(\frac{x}{\alpha^n}, R_n\right) = g_0(x)$$

$$\alpha = -2{,}502907...$$

Melnikovi meetod

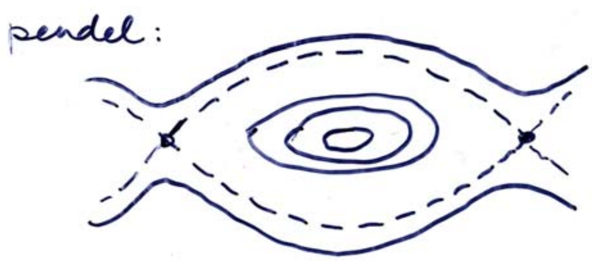
Probleem:

Olgu meil mittelineaarne süsteem

$$\dot{x} = f(x)$$

Kuidas analüütiliselt leida kriteerium, kas süsteem käitub kaootiliselt või mitte?

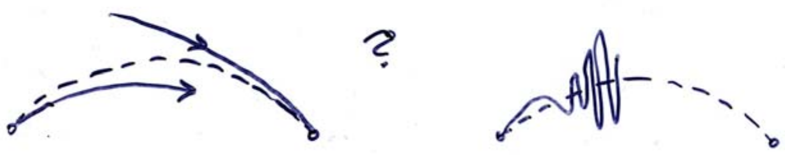
Vaatleme süsteeme, kus eksisteerib sadulpunkt



Lisame välisjõu

$$\dot{x} = f(x) + \varepsilon g(x,t)$$

perioodiline, ω



Kas on hinnang välisjõu parameetritele (ε, ω), mis teevad süsteemi kaootiliseks?

Vrdl: Hüdrodünaamikas Reynoldsi arv R

Olgu

$$\dot{x} = f(x) + \varepsilon g(x,t)$$

$x \in R^2$, g perioodiline perioodiga T (sagedus ω)

mitteautonoomne süsteem → autonoomne, kui

$$x_3 = t \quad , \quad x \in R^3$$

Olgu $f \in C^1(R^2)$, $g \in C^1(R^2 \times R)$ diferentseeritavus

Oletused: ✧ Olgu $\varepsilon = 0$ puhul süsteemil homokliiniline orbiit $q_0 = (t)$ sadulpunktiga x_0

✧ homokliinilise orbiidi sees on perioodilised $q_\alpha(t)$ perioodiga T_α

✧ tegemist on Hamiltoni süsteemiga $\varepsilon = 0$ puhul

Defineerime <u>Melnikovi</u> funktsiooni

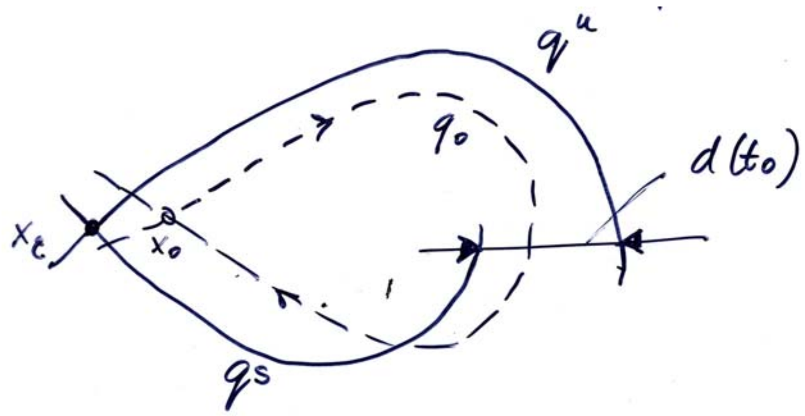
$$M(t_0) = \int_{-\infty}^{\infty} f\left(q^0(t-t_0)\right) \wedge g\left(q^0(t-t_0), t\right) dt$$

$\wedge$ wedge produkt

$$a \wedge b = a_1 b_2 - a_2 b_1$$

Märkus: Kui pole tegemist Hamiltoni süsteemiga, on Melnikovi funktsioon keerulisem

Melnikovi funktsioon määrab häiritud orbiitidel teatud kauguse



q^u - mittestabiilne
q^s - stabiilne

$$d(t_0) = \frac{\varepsilon M(t_0)}{\left| f\left(q^0(0)\right) \right|} + \vartheta\left(\varepsilon^2\right)$$

Teoreem (lihtsustatud)

Kui Melnikovi funktsioonil $M(t_0)$ on lihtsad nullkohad ja ta ei sõltu ε'st, siis väikeste $\varepsilon > 0$ puhul q^u ja q^s lõikuvad.

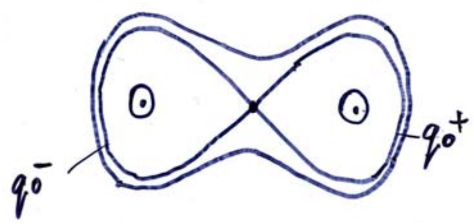
Järeldus: Kui häiritud orbiitide hulgad lõikuvad, toimub segamine ja seega võib tekkida kaos.

Näide: Duffingi võrrand

$$\dot{x} = y$$
$$\dot{y} = x - x^3 + \varepsilon(\underbrace{\mu}_{\text{välisjõu amplituud}} \cos t - 2.5\, y)$$

$\varepsilon = 0$ puhul Hamiltoni süsteem

$$H(x,y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4}$$
$$\dot{x} = \frac{\partial H}{\partial y} \qquad \dot{y} = -\frac{\partial H}{\partial x}$$



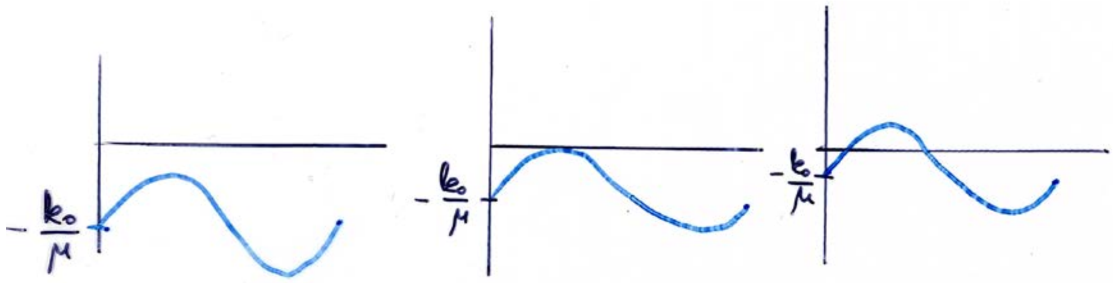
sadul
2 tsentrit

$\varepsilon = 0$

$$q_0^{\pm}(t) = \pm \begin{vmatrix} \sqrt{2}\ \mathrm{sech}\, t \\ -\sqrt{2}\ \mathrm{sech}\, t \tanh t \end{vmatrix}$$

$$M_0(t) = \int_{-\infty}^{+\infty} y_0(t)\left[\mu \cos(t + t_0) - 2.5\, y_0(t)\right] dt$$
$$= -\sqrt{2}\ \mu \int_{-\infty}^{+\infty} \mathrm{sech}\, t\ \tanh t\ \cos\omega(t + t_0)\, dt$$
$$-2\gamma \int_{-\infty}^{+\infty} \mathrm{sech}^2\, t\ \tanh^2 t\ dt$$
$$M_0(t) = \sqrt{2}\mu\, \pi \mathrm{sech}\left(\frac{\pi}{2}\right)\left[\sin t_0 - \frac{k_0}{\mu}\right]$$
$$k_0 = 10 \cosh\left(\frac{\pi}{2}\right) \Big/ \left(3\sqrt{2}\,\pi\right) \sim 1.88$$

$\sin t_0 - \frac{k_0}{\mu}$



$k_0 > \mu > 0$ $\quad M(t_0)$ pole nullkohti

$k_0 = \mu$ $\quad M(t_0)$ puutub horisontaaltelge

$\mu > k_0 > 0$ $\quad M(t_0)$ on nullkoht

$\rightarrow \quad q^u, \quad q^s$ lõikuvad

eeldus kaosele

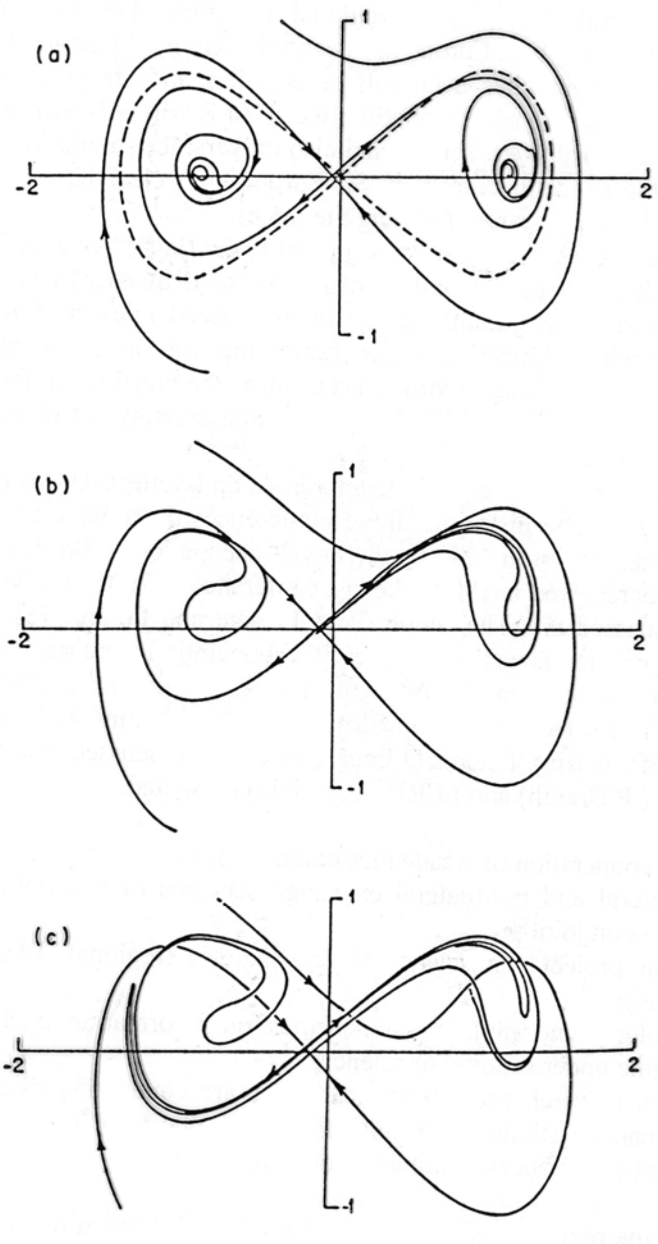
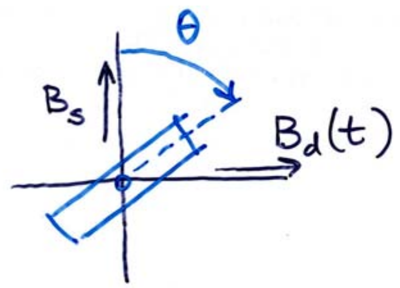


Figure 5. Poincaré maps for the perturbed Duffing equation in Example 1 with $\varepsilon = .1$ and (a) $\mu = 1.1$, (b) $\mu = 1.9$, and (c) $\mu = 3.0$. Reprinted with permission from Guckenheimer and Holmes (Ref. [G/H]).

Näide:

Dipool statsionaarses + ajas muutuvas magnetväljas



Normaliseeritud liikumisvõrrand

$$\ddot{\theta} + \gamma\dot{\theta} + \sin\theta = f_1 \cos\theta \cos\omega t + f_0$$

statsionaarne väli ajas muutuv väli

Hamiltoniaan

$$H = \frac{1}{2}v^2 + (1 - \cos\theta)$$

$q = \theta$

$p \equiv v = \dot{\theta}$

$H = \mathrm{const} \quad (H = 2)$ homokliinilisel orbiidil sadulpunktiga $\theta = v = 0$

Selle orbiidi võrrand

$$\theta^* = 2\,\mathrm{arctg}\,(\sinh t)$$

$$v^* = 2\,\mathrm{sech}\, t$$

Meie algvõrrand Hamiltoniaani abil

$$\dot{q} = \frac{\partial H}{\partial p} + \varepsilon g_1$$

$$\dot{p} = -\frac{\partial H}{\partial q} + \varepsilon g_2$$

$$g_1 = 0$$
$$g_2 = f_1 \cos\theta \cos\omega t + f_0$$
$$\gamma = \varepsilon\bar{\gamma}, \; f_0 = \varepsilon \bar{f}_0, \quad f_1 = \varepsilon \bar{f}_1, \; 0 << \varepsilon < 1$$

Melnikovi funktsioon (vt. Guckenheimer, Holmes, 1983, Moon 1987)

$$M(t_0) = -8\bar{\gamma} + 2\pi \bar{f}_0 + 2\pi \bar{f}_1 \, \omega^2 \operatorname{sech}\left(\frac{\pi\omega}{2}\right) \cos\omega t_0$$

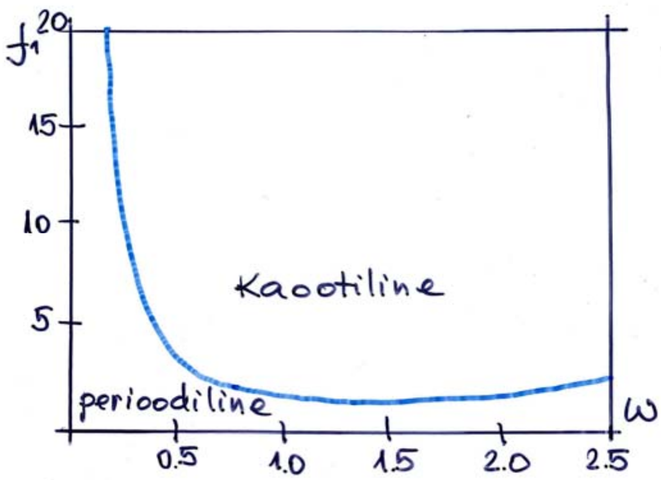
Nullkoht, kui

$$f_1 > \left|\frac{4\gamma}{\pi} - f_0\right| \frac{\cosh(\pi\omega/2)}{\omega^2}$$

Kui $f_0 = 0$, siis

$$f_{1c} = \frac{4\gamma}{\pi\omega^2} \cosh\left(\frac{\pi\omega}{2}\right)$$

Kriitiline



<u>Tširikovi kattumiskriteerium</u>

Chirikov, overlapping criterion

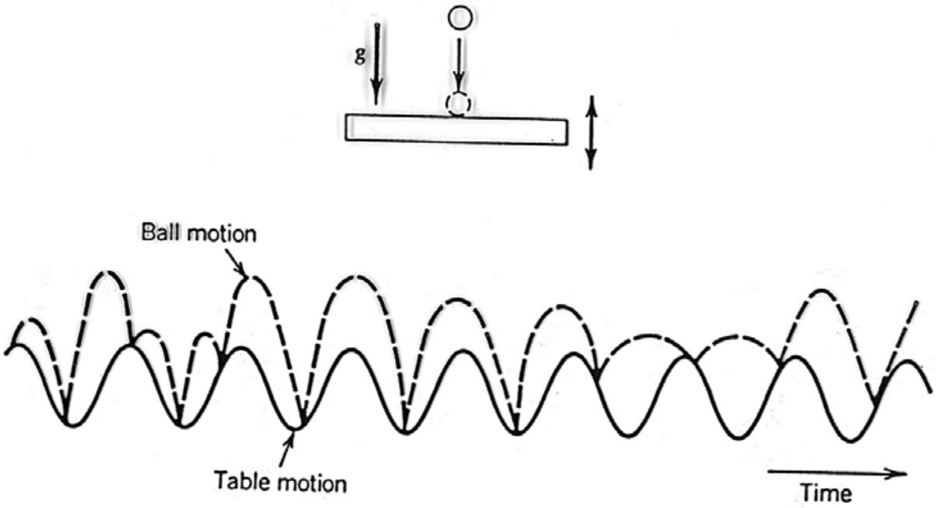
Idee: võnkuvad süsteemid, regulaarne liikumine ümber tsentrite.

Põhitrajektoor – orbiit ümber tsentri

Keeruline süsteem → mitu tsentrit

Häiritud keeruline süsteem → eri tsentrite ümber eksisteerivad orbiidid võivad lõikuda!

Füüsikaliselt võimatu.

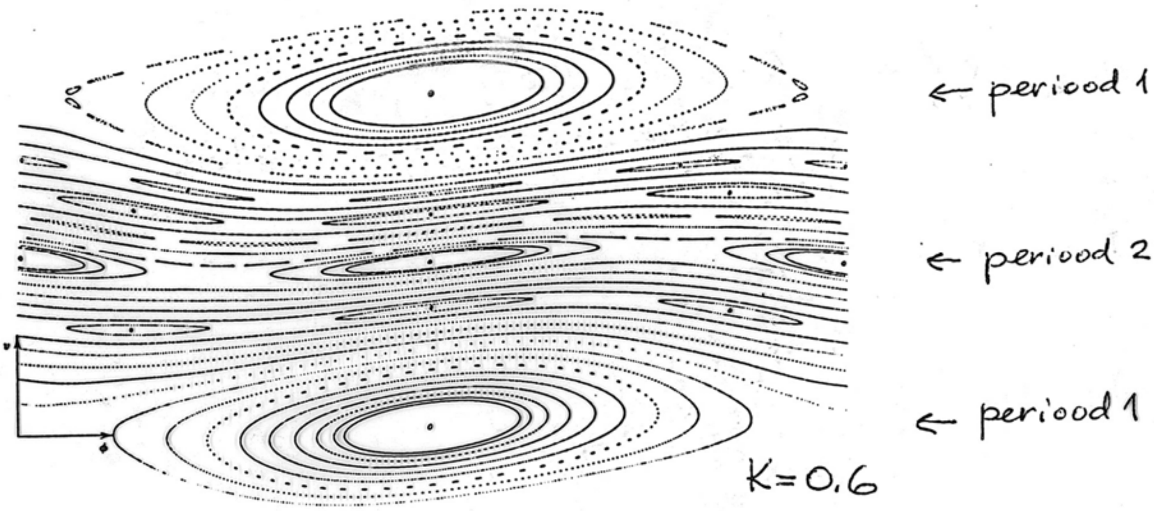


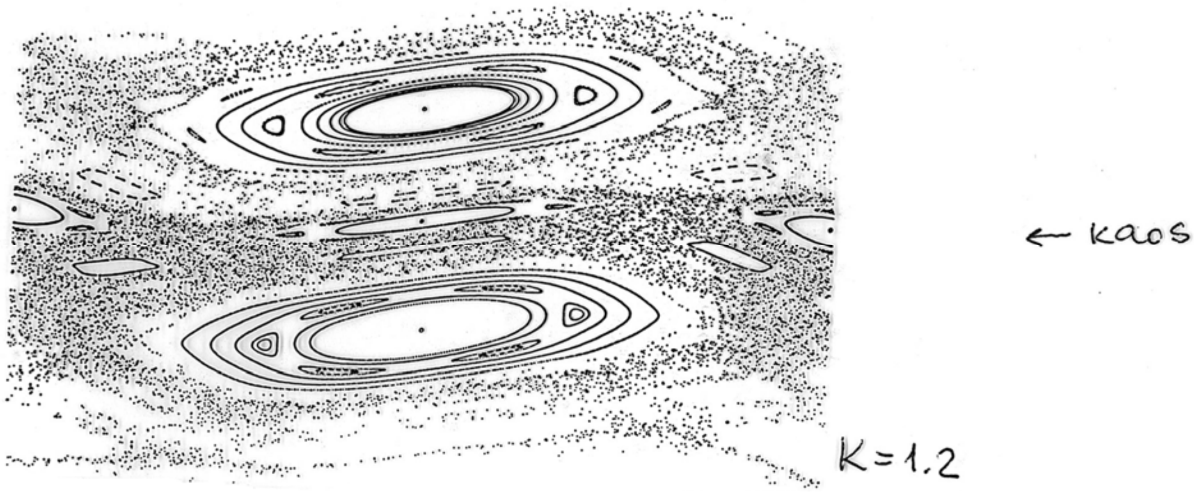
$$v_{n+1} = v_n + K \sin \varphi_n$$

$$\varphi_{n+1} = \varphi_n + v_{n+1}$$

v_n – kiirus enne põrget

φ_n – põrke aeg


← periood 1
← periood 2
← periood 1
K=0.6


← kaos
K=1.2

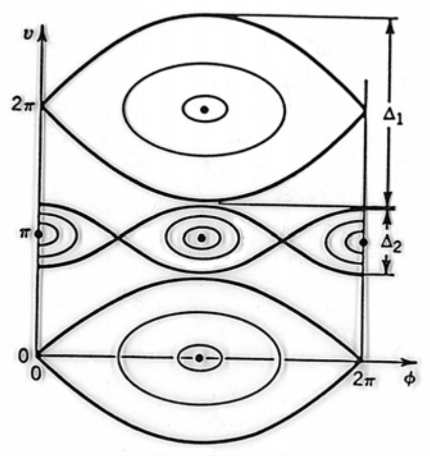


Figure 5-22 Sketch of period-1 and period-2 orbits and concomitant quasiperiodic orbits for the standard map used in the derivation of Chirikov's criterion.

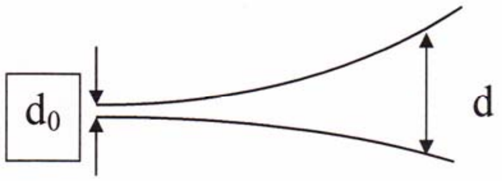
$$\Delta_1 = 4\,K^{1/2}$$

$$\Delta_2 = K$$

$$\Delta_1 + \Delta_2 = 2\pi \qquad \rightarrow \qquad 4\,K^{1/2} + K_c = 2\pi$$

kattuvuse tingimus $\quad K_c$ – kriitiline K

<u>Lyapunov exponents</u> , entropy



$$d(t) = d_0 e^{\lambda t}$$

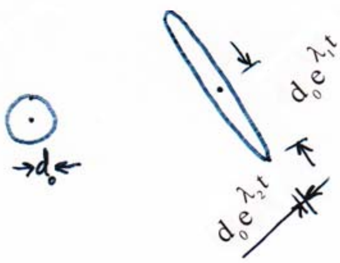
$$d_n = d_0 e^{\Lambda n}$$

$$\lambda = \frac{1}{t_N - t_0} \sum_{k=1}^{N} \log \frac{d(t_k)}{d_0(t_{k-1})}$$

average over N points of a trajectory

$$\boxed{\begin{array}{l} \lambda > 0 \rightarrow \text{chaos} \\ \lambda \leq 0 \rightarrow \text{regular} \end{array}}$$

Spectrum



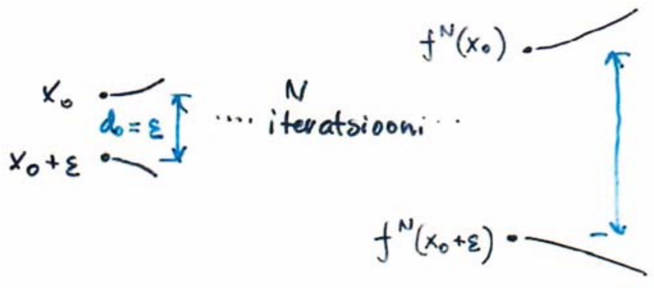
$\boxed{\lambda_1, \lambda_2, \lambda_3, \ldots}$

regular motion $\lambda_i \leq 0$

chaotic motion $\boxed{\lambda_i > 0}$

$\boxed{\lambda_i \longleftrightarrow d}$

$$x_{n+1} = f(x_n)$$



$$d_1 = \varepsilon 2^{N\Lambda(x_0)}$$

$$\text{or } \varepsilon e^{N\lambda(x_0)}$$

$$\varepsilon 2^{N\Lambda(x_0)} = \left| f^N(x_0+\varepsilon) - f^N(x_0) \right|$$

$$N\Lambda(x_0) = \log_2 \left| \frac{f^N(x_0+\varepsilon) - f^N(x_0)}{\varepsilon} \right|$$

$$N \to \infty \quad , \quad \varepsilon \to 0$$

$$\Lambda(x_0) = \lim_{N\to\infty} \frac{1}{N} \underbrace{\lim_{\varepsilon\to 0} \log_2 \left| \frac{f^N(x_0+\varepsilon) - f^N(x_0)}{\varepsilon} \right|}_{\text{tuletis}}$$

$$\Lambda \text{ või } \lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} \log_2 \left| \frac{d\, f^N(x_n)}{dx} \right|$$

tähistus

λ – diferentsiaalvõrrand

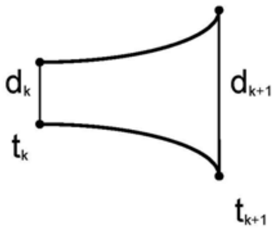
Λ – kujutis

Numbriline arvutus

$$\dot{x} = f(x, c)$$

x – muutujate vektor

c – parameetrite vektor



$$\frac{d(t_{k+1})}{d(t_k)} \quad ?$$

1. Integreerime $\dot{x} = f(x, c)$ mingil ajasammul t_k

   tulemus $x^*(t_k)$

2. Kasutame algtingimust

   $$x^*(t_k) - \eta$$

   kus η määratakse võrrandist

   $$\dot{\eta} = A \cdot \eta$$

   $$A = Df\left(x^*(t_k)\right)$$ s.o. tuletis

3. Võtame $\eta(0) = 1$

4. Leiame

   $$\tau = t_{k+1} - t_k$$

   $$\frac{d(t_{k+1})}{d(t_k)} = \frac{|\eta(\tau; t_k)|}{|\eta(0; t_k)|}$$

   $$\lambda = \frac{1}{t_N - t_0} \sum_{k=1}^{N} \log_2 \frac{d(t_{k+1})}{d(t_k)};$$

5. Järgmine algväärtus

   $$\eta(0; t_{k+1}) = \frac{\eta(\tau; t_k)}{|\eta(\tau; t_k)|}$$

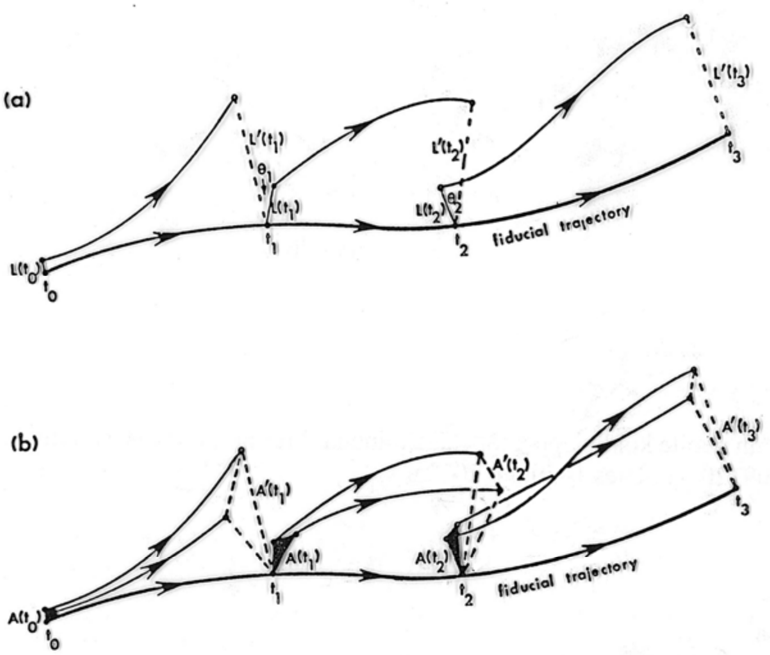


Fig. 4. A schematic representation of the evolution and replacement procedure used to estimate Lyapunov exponents from experimental data. a) The largest Lyapunov exponent is computed from the growth of length elements. When the length of the vector between two points becomes large, a new point is chosen near the reference trajectory, minimizing both the replacement length L and the orientation change ϑ. b) A similar procedure is followed to calculate the sum of the two largest Lyapunov exponents from the growth of area elements. When an area element becomes too large or too skewed, two new points are chosen near the reference trajectory, minimizing the replacement area A and the change in phase space orientation between the original and replacement area elements.

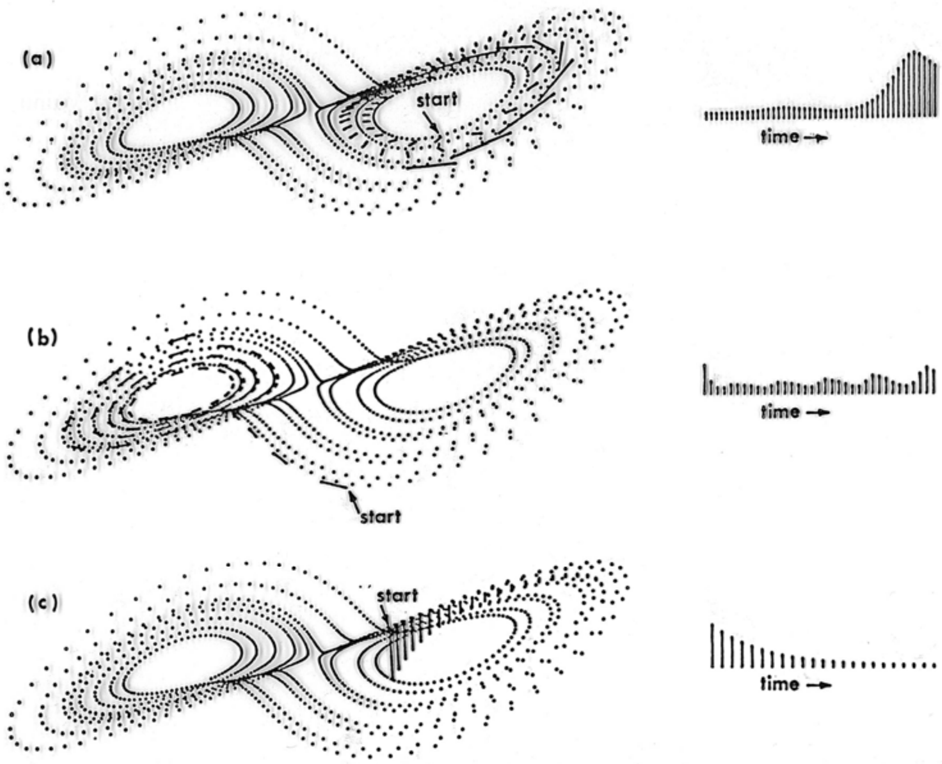
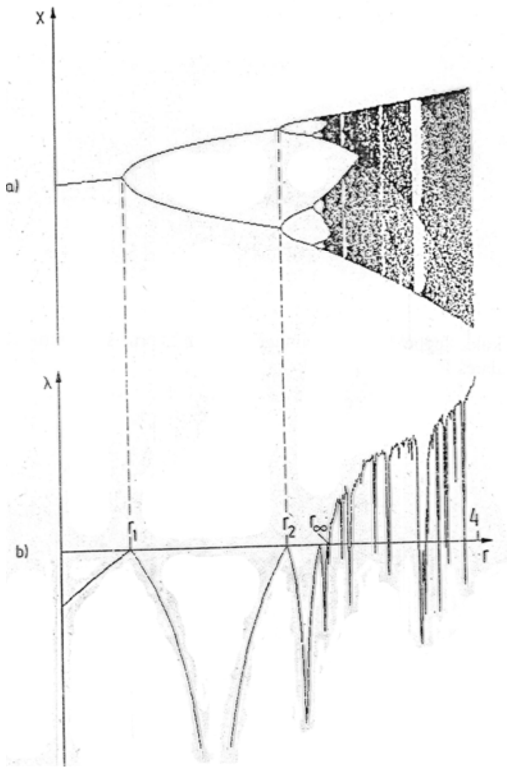
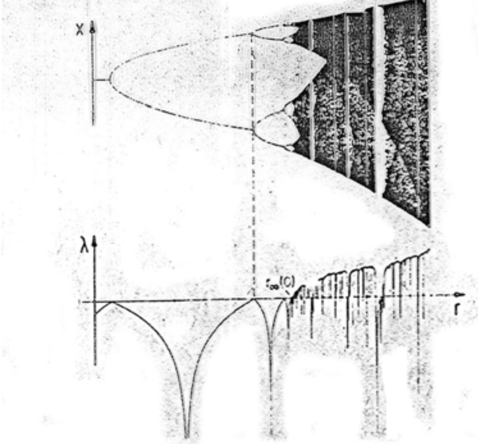


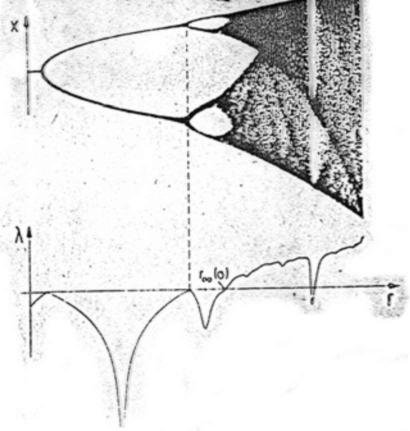
Fig. 1. The short term evolution of the separation vector between three carefully chosen pairs of nearby points is shown for the Lorenz attractor. a) An expanding direction ($\lambda_1 > 0$); b) a "slower than exponential" direction ($\lambda_2 = 0$); c) a contracting direction ($\lambda_3 < 0$).

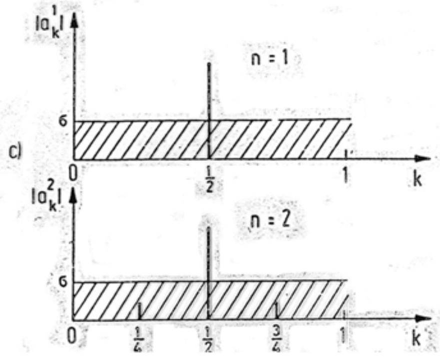


$x_{n+1} = f(x_n)$



$x_{n+1} = f(x_n) + \xi_n$





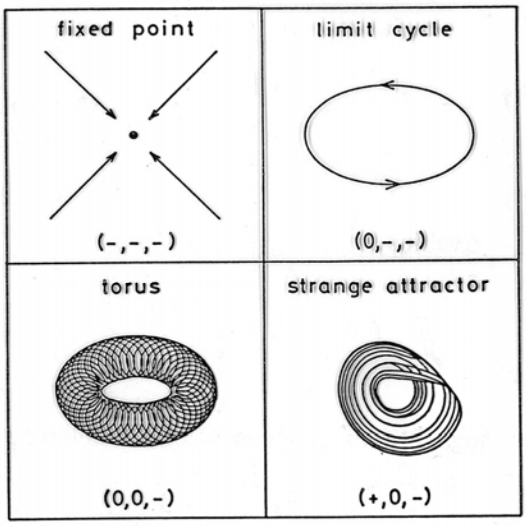


Fig. 70: Conncetion between the dimensions of simple attractors embedded in three-dimensional phase space and the signs of their three Liapunov exponents given in the brackets. (Zero means that the Liapunov exponent has this value.) (After Shaw, 1981.)

Table I
The model systems considered in this paper and their Lyapunov spectra and dimensions as computed from the equations of motion

System	Parameter values	Lyapunov spectrum (bits/s)†	Lyapunov dimension‡
Hénon: [25] $X_{n+1} = 1 - aX_n^2 + Y_n$ $Y_{n+1} = bX_n$	$a = 1.4$ $b = 0.3$	$\lambda_1 = 0.603$ $\lambda_2 = -2.34$ (*bits/iter.*)	1.26
Rossler–chaos: [26] $\dot{X} = -(Y+Z)$ $\dot{Y} = X + aY$ $\dot{Z} = b + Z(X-c)$	$a = 0.15$ $b = 0.20$ $c = 10.0$	$\lambda_1 = 0.13$ $\lambda_2 = 0.00$ $\lambda_3 = -14.1$	2.01
Lorenz: [23] $\dot{X} = \sigma(Y-X)$ $\dot{Y} = X(R-Z) - Y$ $\dot{Z} = XY - bZ$	$\sigma = 16.0$ $R = 45.92$ $b = 4.0$	$\lambda_1 = 2.16$ $\lambda_2 = 0.00$ $\lambda_3 = -32.4$	2.07
Rossler–hyperchaos: [24] $\dot{X} = -(Y+Z)$ $\dot{Y} = X + aY + W$ $\dot{Z} = b + XZ$ $\dot{W} = cW - dZ$	$a = 0.25$ $b = 3.0$ $c = 0.05$ $d = 0.5$	$\lambda_1 = 0.16$ $\lambda_2 = 0.03$ $\lambda_3 = 0.00$ $\lambda_4 = -39.0$	3.005
Mackey–Glass: [27] $\dot{X} = \dfrac{aX(t+s)}{1+[X(t+s)]^c} - bX(t)$	$a = 0.2$ $b = 0.1$ $c = 10.0$ $s = 31.8$	$\lambda_1 = 6.30E{-3}$ $\lambda_2 = 2.62E{-3}$ $\lvert\lambda_3\rvert < 8.0E{-6}$ $\lambda_4 = -1.39E{-2}$	3.64

†A mean orbital period is well defined for Rossler chaos (6.07 seconds) and for hyperchaos (5.16 seconds) for the parameter values used here. For the Lorenz attractor a characteristic time (see footnote – section 3) is about 0.5 seconds. Spectra were computed for each system with the code in appendix A.
‡As defined in eq. (2).

Fraktaaldimensioon ja Ljapunovi eksponendid

Sfäär $r = \delta$ $\rightarrow$ deformeerub ellipsoidiks

$$\mu_1^n \delta, \ \mu_2^n \delta, \mu_3^n \delta$$

$$\lambda_i = \log \mu_i$$

$\{\lambda_1\}$ Ljapunovi eksponendid $\rightarrow$ spekter

$\{\mu_1\}$ Ljapunovi numbrid

Kui $\{\lambda_1, \lambda_2\}$ siis

$$D_L = 1 + \frac{\log \mu_1}{\log\left(\frac{1}{\mu_2}\right)} = 1 - \frac{\lambda_1}{\lambda_2}$$

$$D_L \leq D_C$$

Kui $\{\lambda_1, \lambda_2, \lambda_3, ...\}$, st mitmemõõtmeline

Järjestame Ljapunovi numbrid

$$\mu_1 > \mu_2 > \mu_3 > \mu_4 \ ... > \mu_N$$

Leiame μ_k nii, et

$$\mu_1 \mu_2 \dots \mu_k \geq 1$$

$$D_L = k + \frac{\log(\mu_1, \mu_2 \dots \mu_k)}{\log(1/\mu k_{+1})}$$

Kui $\mu_1 > 1, \ \mu_2 = 1, \ \mu_3 < 1$

siis $\mu_1 \mu_2 \geq 1, \quad k = 2$

$$D_L = 2 + \frac{\log(\mu_1, u_2)}{\log(1/\mu_3)} = 2 + \frac{\log \mu_1}{\log(1/\mu_3)} = 2 + \frac{\lambda_1}{|\lambda_3|}$$

Kolmogorovi entroopia (K entroopia)

Informatsiooniteooria kohaselt on entroopia S

informatsiooni mõõt mis vaja teatud seisundis i^*

oleva süsteemi asukoha määramiseks

$$S = -\sum_{i=1}^{n} P_i \log P_i$$

P_i seisundis i oleva süsteemi leidmise tõenäosus

Olgu meil trajektoor d-mõõtmelises ruumis

$x_1(t), \ldots x_d(t)$, osakesed l^d (d=3 $\rightarrow$ kuup.)

ajahetked intervalliga τ

$P_{i_0 \ldots i_n}$ – ühendtõenäosus, et

$x(t=0)$ on osas i_0

$x(t=\tau)$ on osas i_1

$\ldots\ldots$

$x(t=n\tau)$ on osas i_n

$$K_n = -\sum P_{i_0 \ldots i_n} \log P_{i_0 \ldots i_n}$$

On võrdeline informatsiooniga, mida vaja süsteemi leidmiseks trajektooril

$i_1^* \ldots i_n^*$ täpsusega l

$K_{n+1} - K_n$ on lisainfo, mis vajalik selleks et ennustada millises osas

i^*_{n+1} on süsteem kui teada on $i_1^* \ldots i_n^*$.

$K_{n+1} - K_n$ mõõdab informatsiooni kadu $\rightarrow$ K entroopia

$$K = \lim_{\tau\to 0} \lim_{i\to 0} \lim_{N\to\infty} \frac{1}{N\tau} \sum_{n=0}^{N-1} \left(K_{n+1} - K_n\right)$$
$$= - \lim_{\tau\to 0} \lim_{i\to 0} \lim_{N\to\infty} \frac{1}{N\tau} \sum P_{i_0 \ldots i_N} \log P_{i_0 \ldots i_N}$$

vt joonis

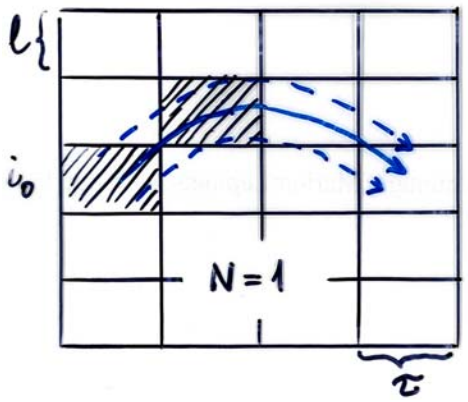
Seos Ljapunovi eksponentidega

$$K = + \sum_i \lambda_i^+ \qquad \text{(üle positiivsete väärtuste)}$$

Seega: K entroopia

1. mõõdab keskmist infokadu
2. on võrdne positiivsete Ljapunovi eksponentide summaga

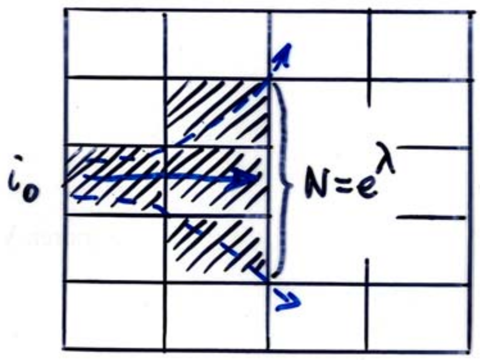
Regular motion



$P_{i_0} = l \quad , \quad P_{i_0 i_1} = l \cdot 1$

$K = 0$

Chaotic motion

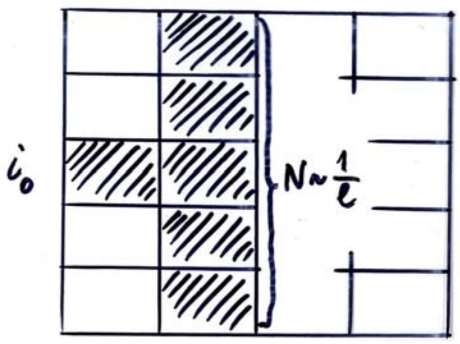


$P_{i_0} = l \quad , \quad P_{i_0 i_1} = l e^{-\lambda}$

$K = \lambda > 0$

$P_{i_0 i_1} \sim P_{i_0} \left(\frac{1}{N} \right)$

Random motion



$P_{i_0} = l \quad , \quad P_{i_0 i_1} \cong l^2$

$K \sim -\log l \to \infty$

$P_{i_0 i_1} \sim P_{i_0} \left(\frac{1}{N} \right)$

KAM teoreem

Kolmogorov – Arnold – Moser

Kolmogorovi idee – kuidas vedeliku segamisel energia jaguneb ja dissipeerub: tekivad keeridvoolud, siis väiksemad, siis veel väiksemad...

Meil – faasiruum, milles liikumine. Mis juhtub, kui need osad lähevad üha keerulisemaks nii nagu segamisel?

1D Hamiltoni süsteem:

$$\frac{\partial p}{\partial t} = -\frac{\partial H}{\partial q} \quad , \quad \frac{\partial q}{\partial t} = \frac{\partial H}{\partial p}$$

q – üldistatud koordinaat

p – üldistatud kiirus

prototüüp – pendel (vt. joonis)

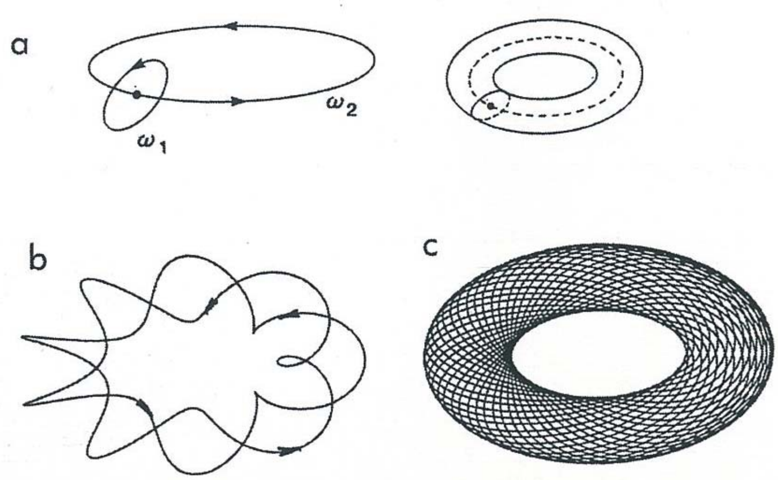
häirituse mõju (vt. joonis)

Perioodiline Hamiltoni süsteem (periood T). Integralkõverad faasiruumis asetsevad tooril (vt. joonis)

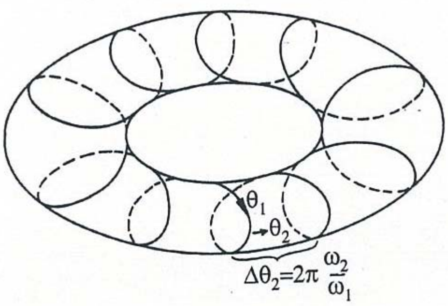
Toori ristlõige (Poincare lõige) võib olla nn. Integreeritav väändekujutis (twist mapping)

Trajektoor „väändub” ümber toori (vt. joonis). Sõltuvalt sagedusest ω võib ta iseenda jälle kohtuda või mitte.

Def. Kui trajektoor tuleb tagasi punkti a peale m korda käimist mööda toori, siis a on m - järku perioodiline punkt.



Joonis 2.20. Toori pinnal asetsev trajektoor: (a) võnkumiste liitmine; (b) suhe T_1/T_2 on ratsionaalarv; (c) suhe T_1/T_2 on irratsionaalarv. *Parker, Chua, 1989.*



Joonis 8.17. Toor faasiruumis, suhe ν_1/ν_2 on ratsionaalne. *Schuster, 1984.*

Kuidas käituvad trajektoorid / orbiidid kui liikumine on häiritud mingi väikese parameetriga μ?

m – ratsionaalne arv

Poincaré – Birkhoffi teoreem

Iga ratsionaalse orbiidi μ häiritus tekitab hüperboolsed ja elliptilised punktid

hüperboolsed – omaväärtused reaalsed – sõlm, sadul

elliptilised – omaväärtused kompleksed – tsentrid

Kuna süsteem on perioodiline ja süsteem konservatiivne, siis toori maht peab säilima ja tekkida saab k hüperboolset ja k elliptilist punkti (vt. joonis).

m -- irratsionaalne arv

Alguseks näide : 2D Hamiltoni süsteem

$$\mathbf{p} = p_1, p_2$$
$$\mathbf{q} = q_1, q_2$$

Olgu

$$H(p, q) = H^0(p) \quad \text{-- sõltub ainult}$$

$$\frac{\partial p_k}{\partial t} = 0 \qquad \frac{\partial q_k}{\partial t} = \frac{\partial H(p)}{\partial p_k}$$

$\uparrow$ 0, sest $\partial H / \partial q_k = 0$

Integreerimisel

$$p_k = \text{const.}; \quad q_k = \left(\partial H(p) / \partial p_k\right) t + q_k^0$$

int. konst.

2D toor

2 sagedust: $\omega_1 = \dfrac{\partial H(p)}{\partial p_1}, \quad \omega_2 = \dfrac{\partial H(p)}{\partial p_2}$

Kui sagedused pole ratsionaalsed, siis pole seda ka nende suhe, ning tingimust

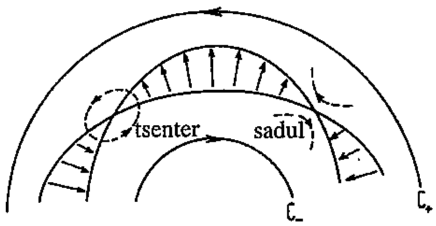
$$k_1 \omega_1 + k_2 \omega_2 = 0,$$

k_1, k_2 - täisarvud, pole võimalik täita.

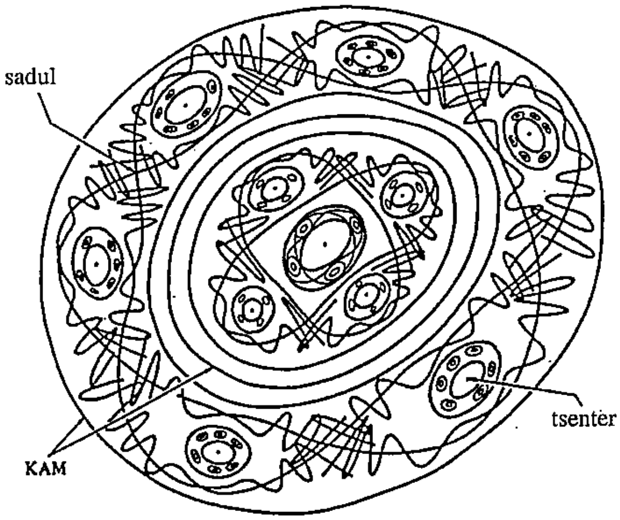
Kuidas käitub süsteem, mille Hamiltoniaan

$$H(\mathbf{p}, \mathbf{q}, \mu) = H^0(\mathbf{p}) + \mu H^1(\mathbf{p}, \mathbf{q})$$

μ - väike parameeter



Joonis 8.18. Häiritud toori kujutisel tekivad uued püsipunktid, suhe ν_1/ν_2 on ratsionaalne. *Schuster, 1984*.



Joonis 8.19. Häiritud toor laguneb väiksemateks toorideks, uute püsipunktide struktuur on enesesarnane.

KAM teoreem (lihtsustatud)

Olgu süsteem kirjeldatud Hamiltoniaaniga

$$H(p,q,\mu) = H^0(p) + \mu H^1(p,q).$$

Iga $\mu > 0$ puhul eksisteerib $\delta > 0$, nii et tingimuse $|\mu| < \delta$ rahuldamisel eksisteerib häiritud süsteemi faasiruum, mis sisaldab algselt faasiruumi (toori), välja arvatud teatud hulgad mõõduga vähem kui μ

vt. joonis

teoreemi täpne esitus -- Guckenheimer, Holmes, 1983

J. Guckenheimer, P. Holmes. Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields. Springer New York et al., 1983.

Spektraalanalüüs ja korrelatsioonifunktsioon

Dünaamiliste protsesside analüüsil on tihti kasulik rääkida sageduste keeles. See on hõlpsalt tehtav, kui kasutada Fourier' teisendust, mis seob signaali $x(t)$ ringsagedusega ν. Fourier' teisendus defineeritakse järgmiselt:

$$X(\nu) \sim \int_{-\infty}^{+\infty} e^{-i\nu t} x(t)\, dt, \tag{8.7.1}$$

kus $X(\nu)$ on signaali $x(t)$ teisendus Fourier' ruumis. Fourier' teisendus on kergesti realiseeritav paljudes tuntud programmipakettides. Kasutusel on ka energiaspekter

$$S(\nu) = |X(\nu)|^2. \tag{8.7.2}$$

Loomulikult saab spektrit väljendada ka ringsageduse ν asemel funktsioonina sagedusest f, sel juhul tähistame seda $S(f) = S_f$.

Kui tegemist on kujutisega $x_{n+1} = f(x_n)$, siis saab spektri arvutada korrelatsioonifunktsiooni $C(m)$ abil. Korrelatsioonifunktsioon ise on arvutatav kui

$$C(m) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} x_{n+m} x_n \tag{8.7.3}$$

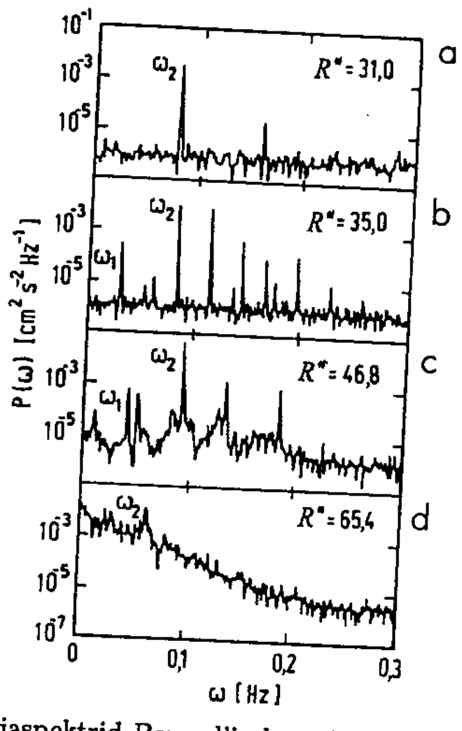
ja spektri jaoks kehtib valem (Schuster, 1984):

$$S_f \propto \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} \cos(2\pi m f) C(m). \tag{8.7.4}$$

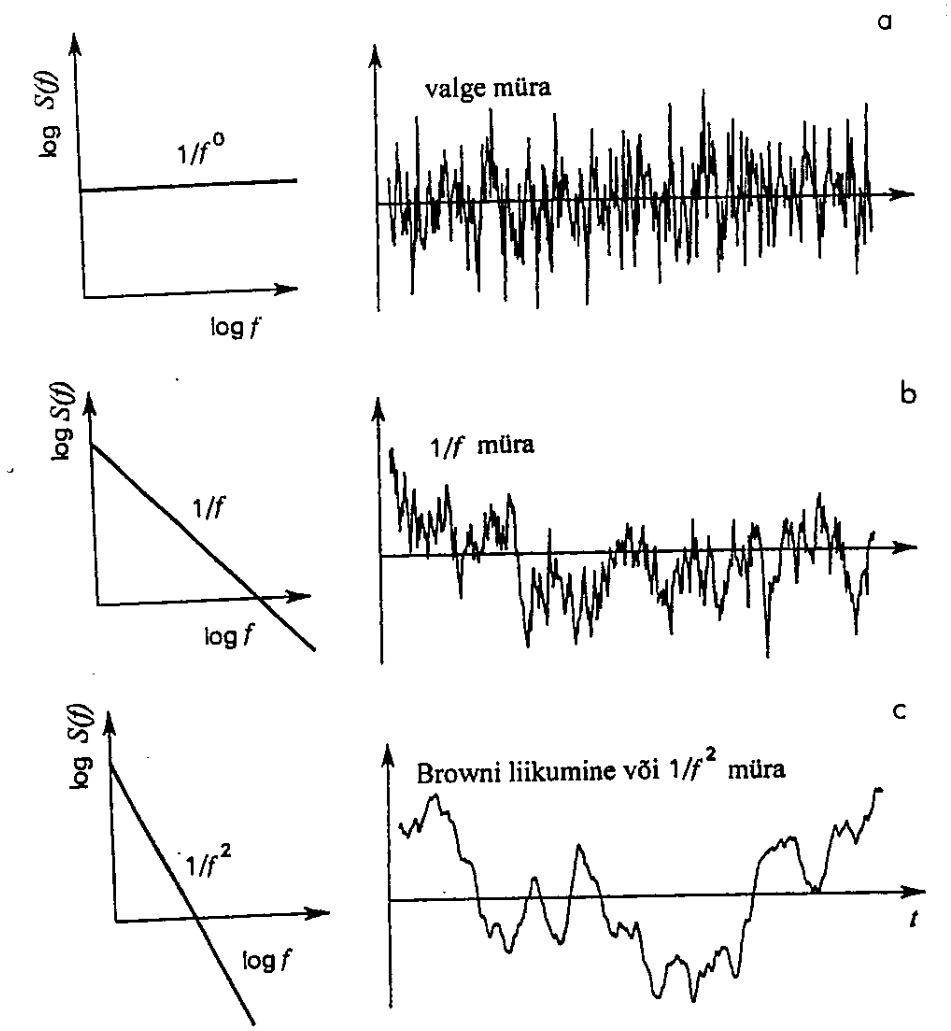
Energiaspektrit on sobiv kasutada kaootiliste protsesside iseloomustamiseks. Perioodiliste protsesside puhul sisaldab spekter iseloomulikke sageduste väärtusi, aperioodiliste protsesside korral on sagedusspekter lai ja temas pole võimalik eristada üksikuid sagedusi. Kerkib loomulik küsimus: kui meil on määratud kaootiline liikumine kui determineerituse kadu igati determineeritud süsteemis, siis milline on sellele liikumisele vastav tüüpiline spekter?

Alustame siingi perioodi kahendumisest kui heast mudelülesandest. Lahend perioodiga 1 olgu meil ringsagedusega ν_0 ja spektris peab siis esinema ainult üks joon, mis märgib ära antud sageduse tema intensiivsusega. Perioodi kahendumisel sagedus muutub ja peab tekkima nn.

subharmooniline sagedus $\nu_0/2$. Iga järgmine kahendumine toob mängu uued subharmoonilised sagedused. Seejuures on j-nda subharmooniku $\nu_0/2^j$ intensiivsus $j+1$-se kahendumise juures amplituudi muutumise mõõt. Kahe teineteisele järgneva subharmooniku intensiivsuse suhe on väljendatav Feigenbaumi arvu $\alpha = 2,50290...$ abil ja on võrdeline suurusega $(1/2\alpha)^2$ (Schuster, 1984). Kaootilise liikumise puhul pole subharmoonikud enam eristatavad. Bénard'i eksperiment (Swinney, Gollub, 1978) näitab ilmekalt, milline on kaootilise liikumise spekter läbi perioodi kahendumise kaskaadi (joon. 8.20). Domineerivad madalad sagedused ja kõrgemate sageduste osa on alla surutud. Seda tüüpi spektrit nimetatakse $1/f$ müraks. Kui üldistada spektri käitumist funktsiooni $1/f^n$ kasutades, siis on olukord järgmine. Kui $n = 0$, on tegemist valge müraga. Kui $n = 1$, siis ongi tegemist meid huvitava



Joonis 8.20. Energiaspektrid Benard'i eksperimendist: (a) harmooniline liikumine; (b) kvaasiperioodiline liikumine; (c) kaootilise liikumise algperiood; (d) täielik kaos. Parameeter R^* on suhteline Rayleigh' arv. *Swinney, Gollub, 1978.*



Joonis 8.21. Vasakul skemaatiline spekter, paremal iseloomulik signaal aja funktsioonina: (a) valge müra; (b) $1/f$ müra; (c) Browni liikumine.

kaootilise protsessiga. Kui aga $n = 2$, siis on iseloomulikuks protsessiks Browni liikumine. Kõigi nimetatud protsesside spektrid ja iseloomulikud signaali kujud on skemaatiliselt esitatud joonisel 8.21.

Järgnevalt vaatame, kuidas kasutada kaose uurimisel korrelatsioonifunktsiooni, eeldades, et andmed on antud aegreana $y = y(t)$.

Mõõtmistulemuste analüüs

$$S(n) = S(t_0 - \eta\, \tau_s)$$

t_0 – alghetk rmt 8.9

τ_s – mõõtmisintervall

1) tuletiste leidmine , $S, \dot{S}$ faasitasand

ebatäpne!

2) viitega mõõtmistulemused faasitasandi

(faasiruumi konstrueerimiseks) – <u>hea meetod!</u>

$$S(n - T) = S(t_0 + (n + t)\, \tau_s)$$

$$\vec{y}(n) = [S(n), S(n + T), S(n + 2T), \ldots, S(n + (D - 1)\, \tau)T]$$

T – ? D – ?

T – koordinaatide S(m) sõltumatus

lineaarne autokorrelatsioonifunktsioon $C_L(\tau)$

T on $C_L(\tau)$ esimene nullkoht

D – esitusdimensioon (embedded dimension)

Mõõdetud aegread

Kaootilisus?

$$S(n) = S(\underbrace{t_0}_{\text{algmoment}} + n \underbrace{\tau_s}_{\text{ajaintervall}})$$

Arvutatud aegrida $\overline{S}(n)$ on võrrandi $\dot{\overline{S}}(n) = f(\overline{s})$

lahend.

Faasiruum, atraktor, dimensioon,

Ljapunovi eksponendid, K – entroopia

Mõõdetud:

Konstrueerime

$$y(n) = [S(n), S(n+T), S(n+2T), \ldots, S(n+(d-1)T)]$$

$T \rightarrow$ täisarv 1, 2, 3, . . . viivis (time delay)

$d \rightarrow$ esitusdimensioon (embedded dimension)

Konstrueerime atraktori $\rightarrow$ dimensioon $\rightarrow$ . . .

Vt. H.D. Abarbanel, R. Brown, J.J. Sidorowich, L.Sh. Tsimring. The analysis of observed chaotic data in physical systems. Rev. Mod. Phys. 1993, 65, N 4, 1331 – 1392.

$$\frac{d^2x}{dt^2} = -bx \qquad \begin{cases} \dfrac{dx}{dt} & = & y \\ \dfrac{dy}{dt} & = & -bx \end{cases}$$

Faasitasand $\{x\ ,\ y\}$

$x(t) = D_t$

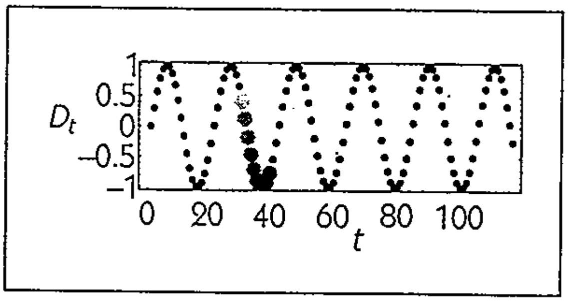


Figure 6.15
The quantity D_t measured from Eq. 6.25.

$$\frac{dx(t)}{dt} = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h}$$

$$\frac{dD_t}{dt} = \frac{D_{t+h} - D_t}{h} = \frac{D_t - D_{t-h}}{h}$$

Faasitasand $\{dD_t, D_{t+h}\}$, $\{D_t, D_{t-h}\}$

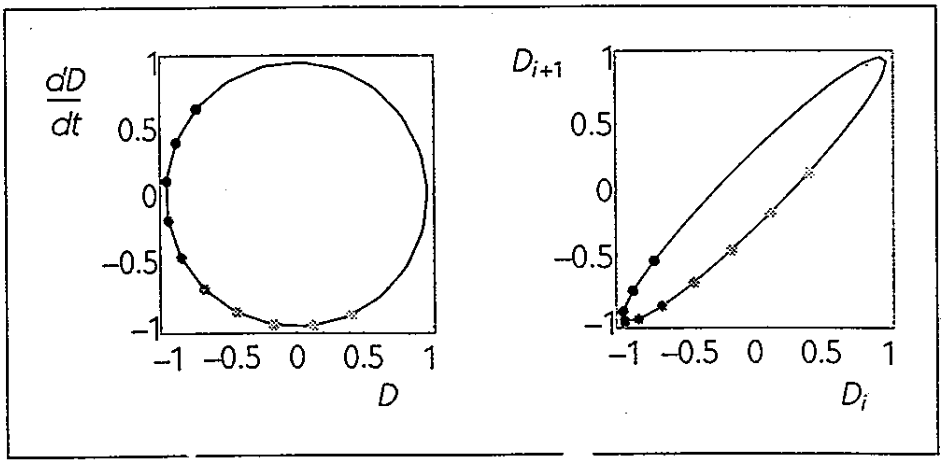


Figure 6.16 Two versions of the reconstructed phase plane for the data generated from Eq. 6.25. The gray dots indicate the position in the phase plane at the corresponding times in Figure 6.15.

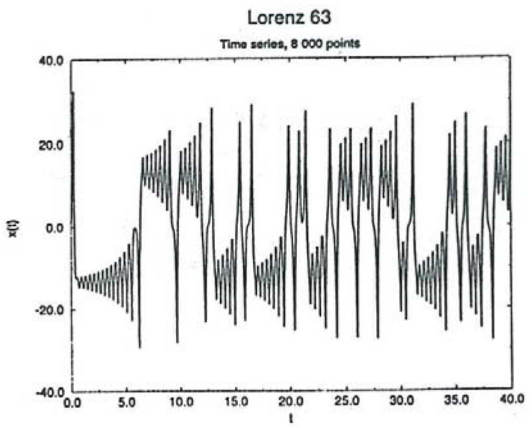


FIG. 1. Chaotic time series $x(t)$ produced by Lorenz (1963) equations (11) with parameter values $r=45.92$, $b=4.0$, $\sigma=16.0$.

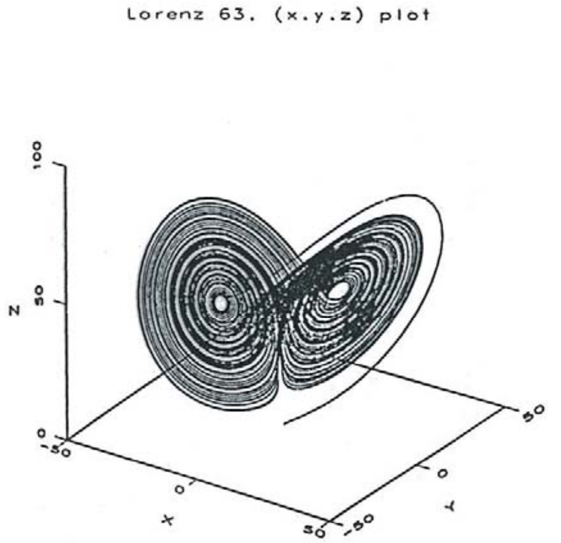


FIG. 2. Lorenz attractor in three-dimensional phase space $(x(t),y(t),z(t))$.

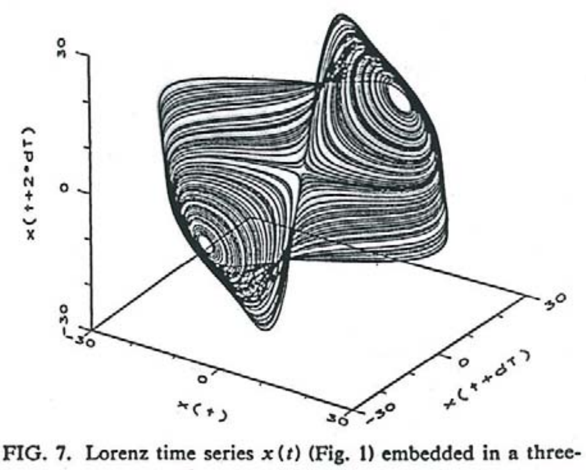


FIG. 7. Lorenz time series $x(t)$ (Fig. 1) embedded in a three-dimensional phase space $(x(t),x(t+dT),x(t+2dT))$, $dT=0.2$.

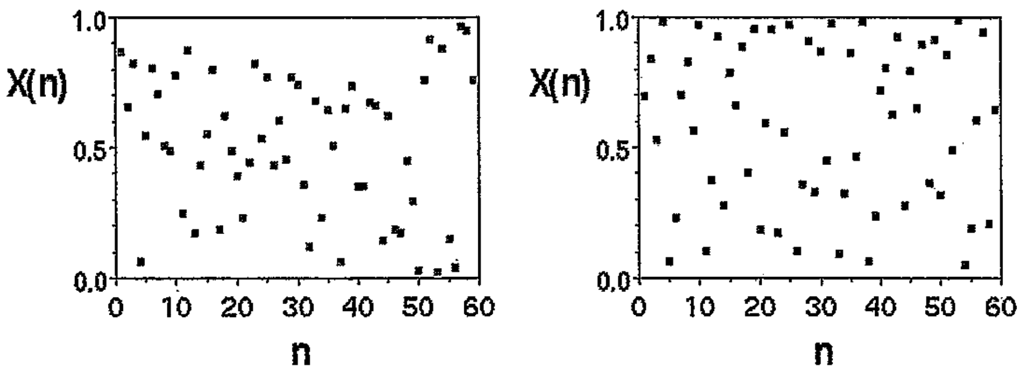
10 An Introduction to the Mathematics and Meaning of Chaos

Larry S. Liebovitch[1]

It is a pleasure for me to be here, both to talk and to listen. I thank Curt Lindberg and the other organizers for this opportunity. It has been a long time since I have been here in Austin. In fact, the last time I came here, from Boston; I got here by flying on Eastern Airlines. So it's been some time.

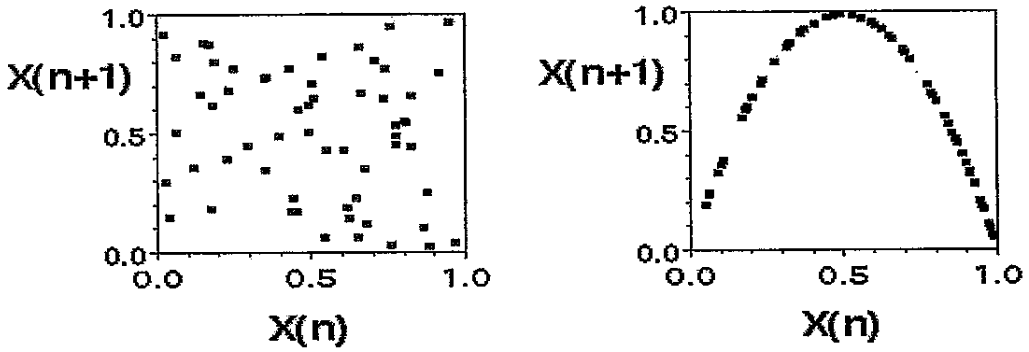
What I would like to do is to try to knit together some of the more mathematical approaches we have heard this morning; and the approaches that we are going to hear later. The question that I want to ask is: What is the nature of the relationship between the mathematics and the metaphorical lessons that we learn from it?

I'll start off by giving some very simple mathematical examples for people who are not familiar with the mathematics. Here I present two different data sets; two different time series. You can think of these, for example, as temperature measurements made each day over a two-month period. Although these data sets are different, they both seem to have the same qualitative feature. They both look as if the temperature each day varies at random.



But what we are learning today is that not everything that looks random is random. You can see this when I transform this data. Instead of plotting the temperature each day I can make a plot of temperature today against the temperature yesterday. If I do that for both these data sets, they look something like this.

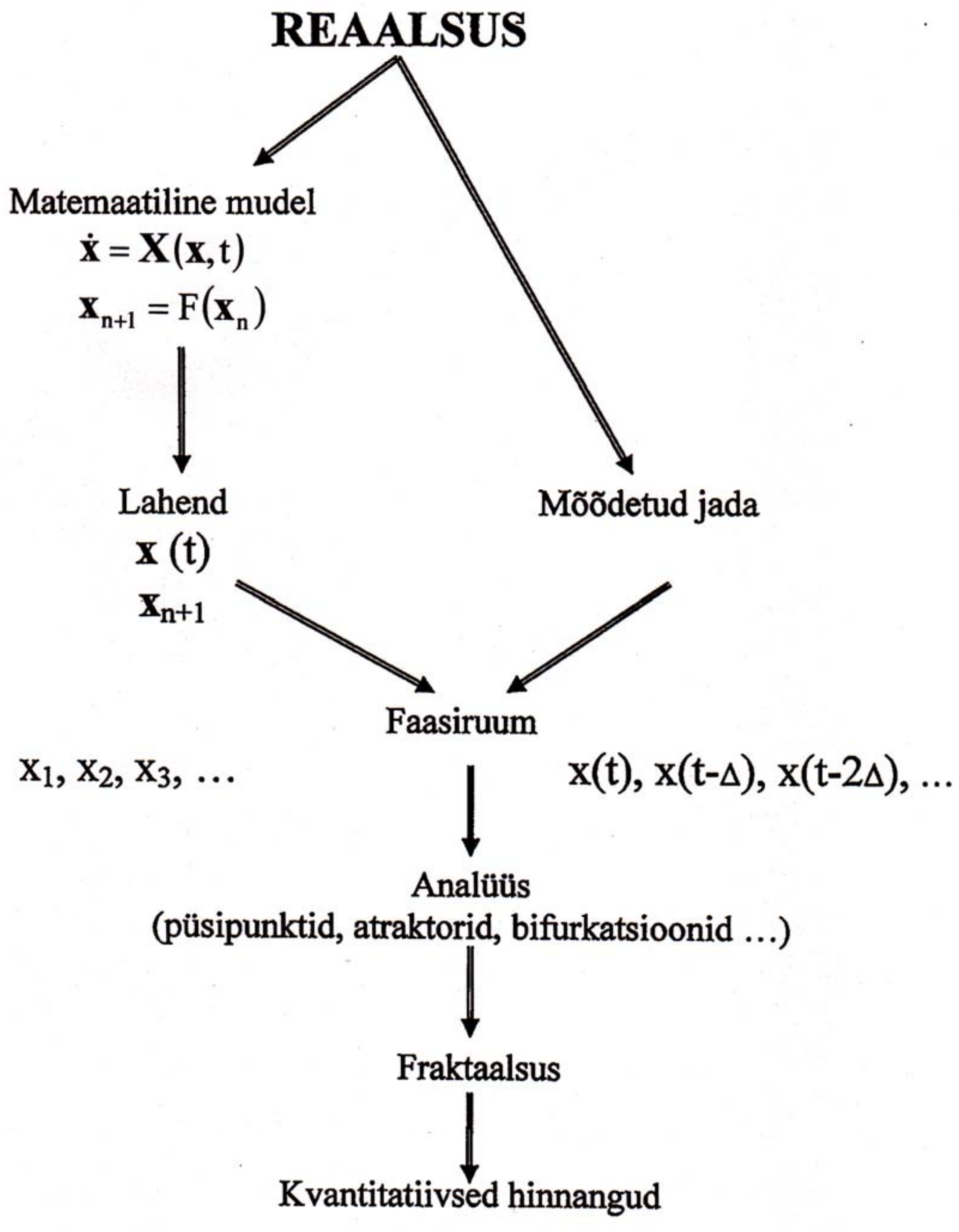
[1] Professor, Center for Complex Systems and Brain Sciences, Center for Molecular Biology and Biotechnology, Department of Psychology, Florida Atlantic University, liebovitch@clifford.ccs.fau.edu.



This first data set really does look random. The temperature today has no particular relationship to the temperature yesterday. It varies all over the place. But if I look at the second data set, we suddenly see that it is not random at all. The temperature today is a very simple function, expressed by this line, of what the temperature was yesterday. Even though we understand how the temperature, one day, affects the temperature the next day, over many days the temperature seems to fluctuate at random. The jargon word for describing this phenomenon is the word "chaos."

We have learned something new and important about the data by transforming the time series of the temperature every day, to this plot of the temperature today versus the temperature yesterday. This plot of the temperature today versus the temperature yesterday is called a phase space.

Let me give you another example. This is one from Lorenz, and it's a model of the motion of the air in the atmosphere. In this model, the air is heated from below; and then cooled from the top. The hot air rises and the cool air falls. The motion of the air forms convective rolls that rotate. The first one on the left starts off rotating counter-clockwise; and it will rotate that way for a while. It slows down and speeds up. It is actually going to stop. When it stops rotating, it may rotate in the opposition direction. It does that for a little while. Then it stops; slows down; and speeds up; and stops. And then it may switch again and rotate in the opposition direction.



REAALSUS
Matemaatiline mudel
$\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x}, t)$
$\mathbf{x}_{n+1} = F(\mathbf{x}_n)$
Lahend
$\mathbf{x}\,(t)$
$\mathbf{x}_{n+1}$
Mõõdetud jada
Faasiruum
$x_1, x_2, x_3, \ldots$
$x(t), x(t-\Delta), x(t-2\Delta), \ldots$
Analüüs
(püsipunktid, atraktorid, bifurkatsioonid …)
Fraktaalsus
Kvantitatiivsed hinnangud