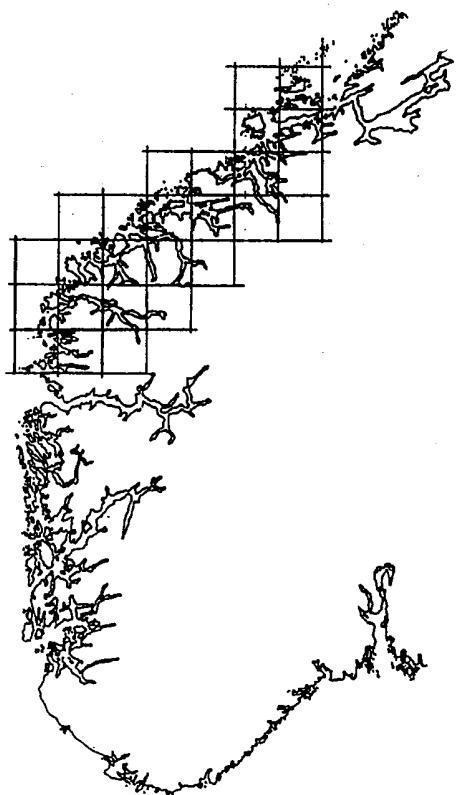


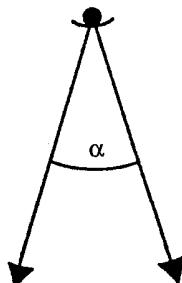
6. FRAKTALID

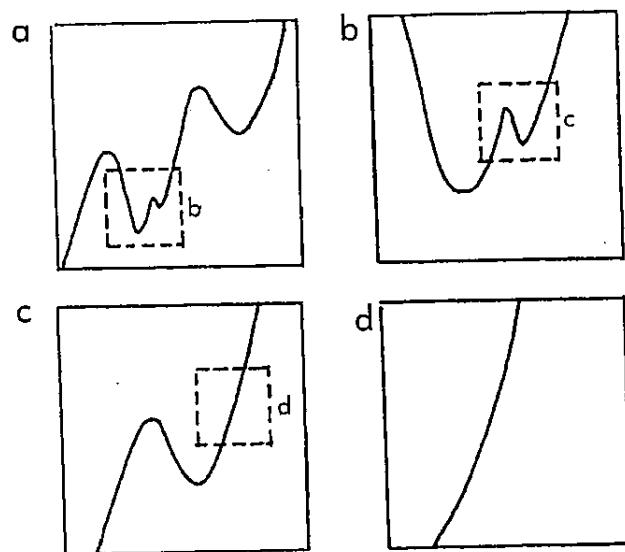
6.1 Üldteooria



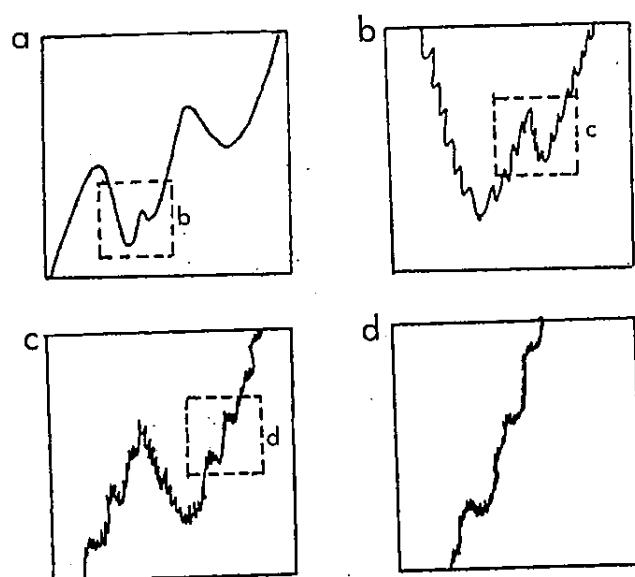
Joonis 7.1. Norra läänerannik. *Feder, 1988.*

Joonis 7.2. Mõõtesirkel
avaga α , millele kaardil
vastab teatud δ km.

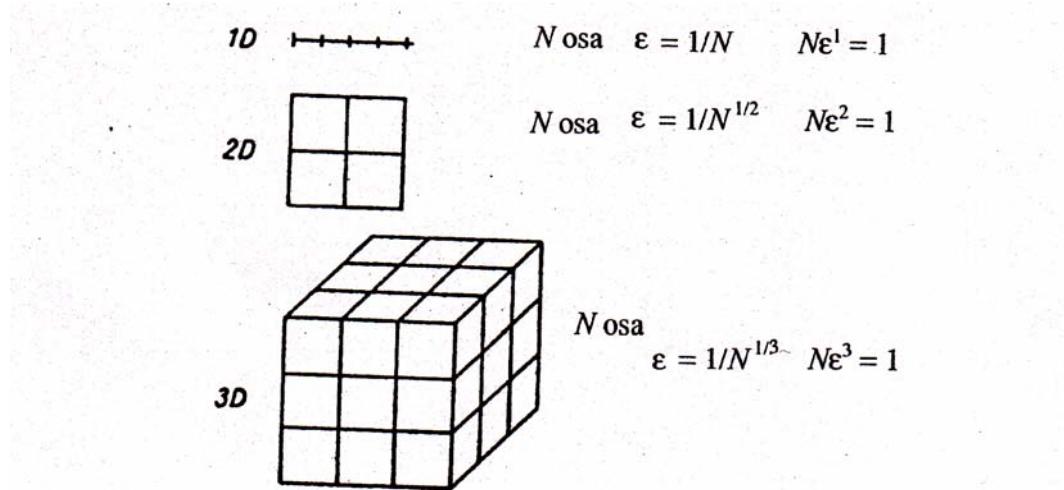




Joonis 7.4. Sile joon ja selle suurendused.



Joonis 7.5. Fraktaalne joon ja selle suurendused.



Joonis 7.16. Korrapärasid struktuurid, mille topoloogiline dimensioon on alati täisarv.

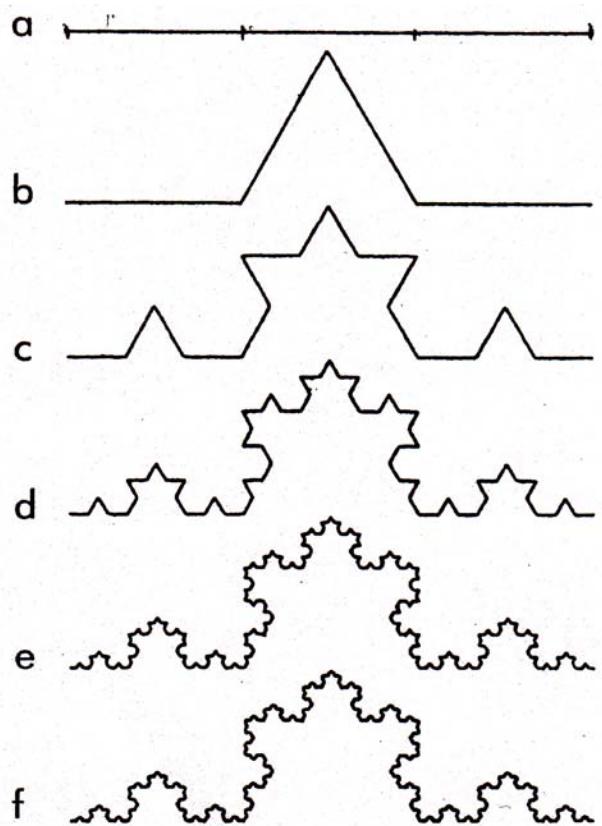
Mis saab, kui $N \epsilon^D = 1$

$D \neq$ täisarv

$$N = \frac{1}{\epsilon^D} = \left(\frac{1}{\epsilon}\right)^D$$

$$\log N = D \log\left(\frac{1}{\epsilon}\right)$$

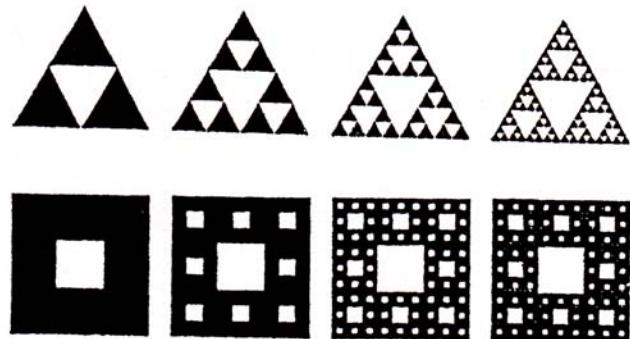
$$D = \frac{\log N}{\log \frac{1}{\epsilon}}$$



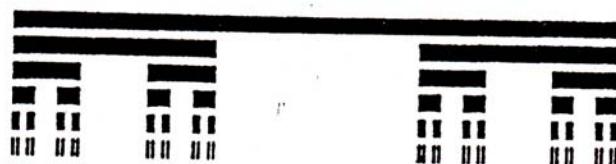
Joonis 7.17. Kochi lumehelbekese konstrueerimine.

$$N = 4, \quad \varepsilon = \frac{1}{3}$$

$$D = \frac{\log 4}{\log \sqrt[3]{1/3}} = \frac{\log 4}{\log 3} = 1.26185$$



Joonis 7.6. Kas need on fraktalid?



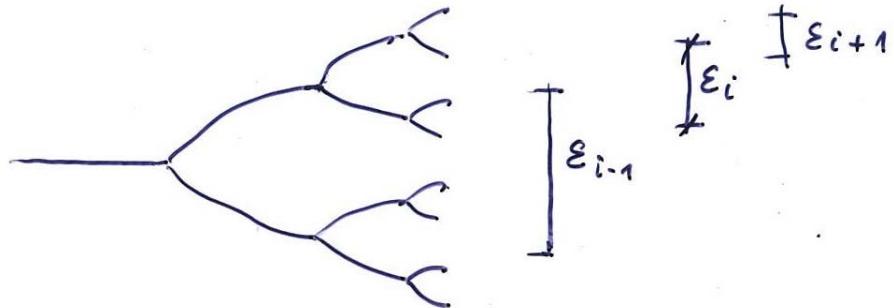
Joonis 7.7. Cantori triaad.

Cantor:

$$N = 2, \quad \varepsilon = \frac{1}{3}$$

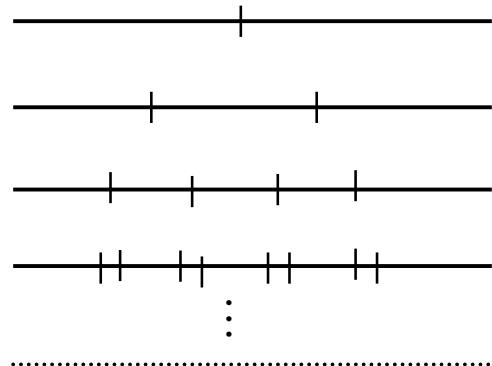
$$D = \frac{\log 2}{\log \frac{1}{3}} = \frac{\log 2}{\log 3} = 0.63092$$

Feigenbaum



$$\alpha_i = \frac{\varepsilon_i}{\varepsilon_{i-1}}$$

$$\lim \alpha_i \rightarrow \alpha$$



$$D = \frac{\log 2}{\log(1/\varepsilon)} , \quad \varepsilon = \frac{1}{2} \left(\frac{1}{\alpha} + \frac{1}{\alpha^2} \right)$$

$$D = 0.5438\dots$$

Eksperiment D = 0.5438\dots

Grassberger, 1981

Nüüd on aeg tutvustada ka *Hausdorffi-Besikovitši dimensiooni*, mille Hausdorff määras matemaatilise rangusega juba 1919. a. Lühdalt on siin mõttetäik järgmine. Antud on ülesanne määräata ruumis asetseva punktide hulga S dimensioon. Selleks võtame mõõdufunktsiooni $h(d) = \gamma(\varepsilon)\varepsilon^d$, mis kirjeldab kas sirglöiku, ruutu, kuupi, kera või mingit muud kujundit. Sirglöigu korral geomeetriline tegur $\gamma(\varepsilon) = 1$, kera puhul $\gamma(\varepsilon) = \pi/6$ jne. Katame hulga S mõõdufunktsiooniga $h(d)$ ning summeerime tema mõõduks

$$M_d = \sum h(d).$$

Kui $\varepsilon \rightarrow 0$, siis M_d on kas null või lõpmatus, sõltuvalt d valikust. Hulga S Hausdorffi-Besikovitši dimensioon D_{HB} on kriitiline dimensioon, mille puhul mõõt M_d muutub nullist lõpmatuseks:

$$M_d = \sum \gamma(\varepsilon)\varepsilon^d = \gamma(\varepsilon) \cdot N(\varepsilon)\varepsilon_{\varepsilon \rightarrow 0}^d \begin{cases} 0, & d > D_{HB} \\ \infty, & d < D_{HB} \end{cases},$$

kus $N(\varepsilon)$ on mõõdufunktsioonide arv. On võimalik näidata, et enese-sarnaste fraktalite korral

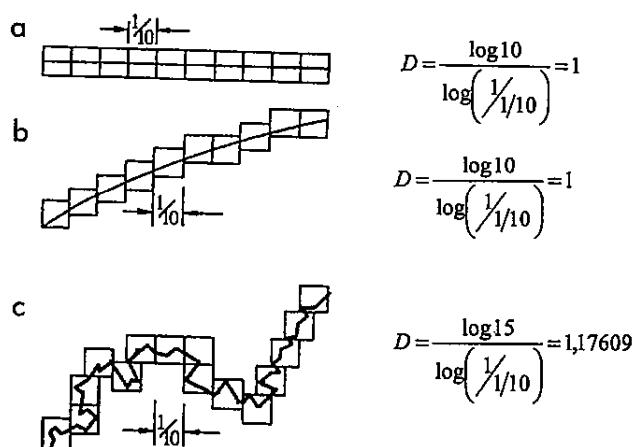
$$D_S = D_{HB}$$

ning praktikas kasutatakse tihti vaid esimest. Nii või teisiti, klassikaline fraktali definitsioon kõlab järgmiselt:

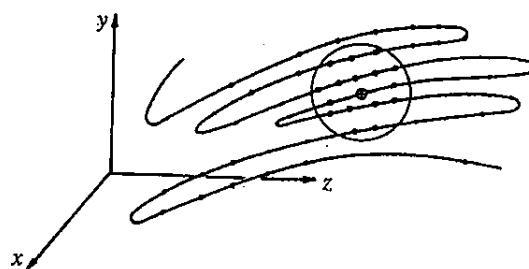
Definitsioon 7.1. *Fraktal on struktuur, mille Hausdorffi-Besikovitši dimensioon on rangelt suurem täisarvulisest topoloogilisest dimensioonist.*

Nii sarnasus- kui Hausdorffi-Besikovitši dimensioon eeldavad, et tegemist on enesesarnasusega. Matemaatiliselt on see igati mõistlik eeldus, kuid füüsikaliseks eksisteerivad piirangud. Leidub alati selline ε_0 , milles väiksemat objekti lihtsalt ei eksisteeri (liivatera ja rannajoon!), seetõttu tuleb dimensioon määrata eksperimentaalselt. Sellest oli juttu juba peatüki alguses – tuleb valida mõõtfunktsioon, katta uuritav struktuur mõõtfunktsiooniga ja loendada nende koguarv N .

Joonisel 7.18 on näidatud sirgjoone, kõvera ja fraktaalse joone dimensiooni määramine, kus mõõtfunktsiooniks on valitud ruut $\varepsilon \times \varepsilon$. Joonisel toodud näites on valitud $\varepsilon = 1/10$. Et aga fraktaalse kõvera (joon. 7.18) dimensiooni täpsemalt määrata, tuleb ε muuta $\varepsilon = 1/20$, $\varepsilon = 1/50$ jne. ning leida dimensiooni piirväärust. Nii talitati ka Norra rannajoone pikkuse arvutamisel. Praktikas nimetatakse sel teel saadud dimensiooni ka *kastdimensiooniks* (*box dimension*).



Joonis 7.18. Dimensiooni määramine: (a) sirglõigul; (b) kõveral; (c) fraktaalsel joonel.



Joonis 7.19. Keeruliste trajektooridega faasiruum. Moon, 1987.

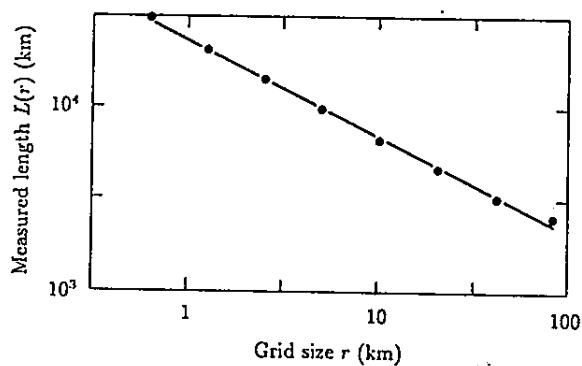


Figure 2 Measured length of Norwegian coast against grid size r . The slope of the straight line gives the fractal dimension of the coast $D \approx 1.52$ [Fed 88].

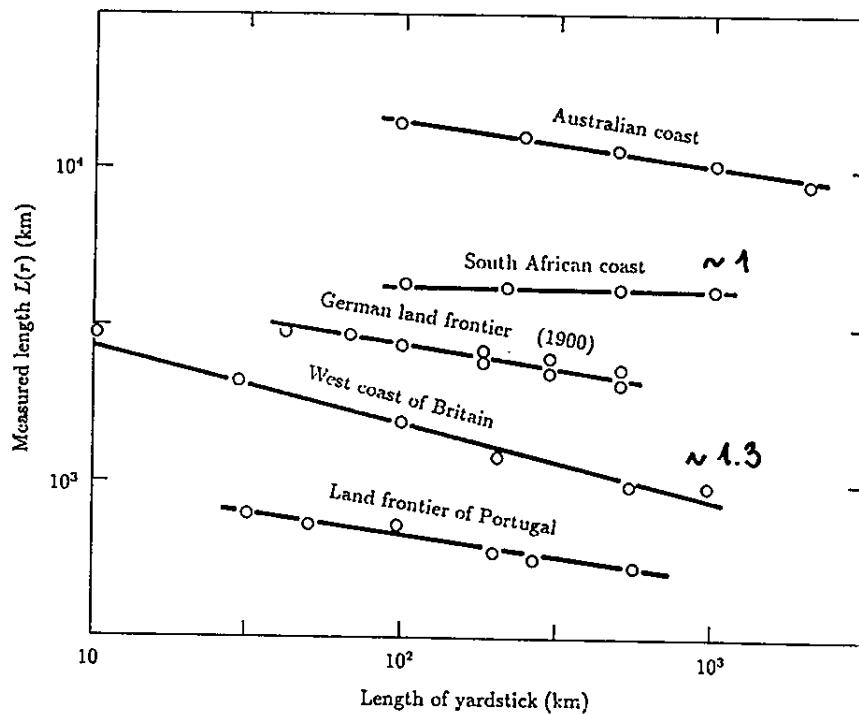


Figure 3 Apparent lengths and fractal dimension of different coasts and land frontiers [Man 83].

Punktdimensioon (pointwise dimension). Olgu meil mingi aja jooksul kogutud teatud arv punkte. Asetame faasiruumi kera (või kuubi) raadiusega r ning loeme ära punktid, mis satuvad selle kera hulka. Olgu see arv $N(r)$. Selles kelas meeleväldset punkti leida lubav töenäoses on

$$P(r) = \frac{N(r)}{N_0},$$

kus N_0 on kogu punktide arv. Punktdimensioon on arvutatav valemiga

$$D_p = \lim_{r \rightarrow 0} \frac{\log P(r, x_i)}{\log r},$$

kus x_i on kera keskpunkti kohavektor. Mõnel juhul pole D_p sõltuv kohavektorist, mõnel juhul on see sõltuvus aga oluline. Siis tuleb tulemust keskmistada üle juhuslikult valitud M kohavektori, mis annab

$$\hat{D}_p = \frac{1}{M} \sum_{i=1}^M D_p(x_i).$$

Korrelatsionidimensioon (correlation dimension). Ka siin on aluseks diskreetsete punktide hulk $\{x_i\}$ faasiruumis, punktide koguarv on N . Alustuseks tuleb arvutada paariviisi punktide vahedistants

$$s_{ij} = |x_i - x_j|.$$

Korrelatsioonifunktsioon $C(r)$, kus r on samuti mõõtühik (kera raadius), on määratud valemiga

$$C(r) = \lim_{N \rightarrow \infty} \frac{1}{N^2} \times (\text{paaride } (i, j) \text{ arv, mille puhul } s_{ij} < r).$$

Korrelatsionidimensioon D_G on määratud valemiga

$$D_G = \lim_{r \rightarrow 0} \frac{\log C(r)}{\log r}.$$

Informatsioonidimensioon (information dimension) arvestab sage-dust, millega trajektoor satub igasse mõõtkuupi (või mõõtkerasse). Taas kaetakse punktihulk, mille dimensiooni soovime mõõta N kuubiga, mille küljepikkus on ε (N keraga, mille raadius on ε).

Et arvutada informatsioonidimensiooni, loeme kokku punktide arvu N_i igas keras, mida on N tükki ja tõenäosus leida punkti just selles keras on P_i , kus

$$P_i = \frac{N_i}{N_0}, \quad \sum P_i = 1.$$

Sin N_0 on punktide koguarv ($N_0 \neq N$). Informatsiooni entroopiat defineeritakse valemiga

$$I(\varepsilon) = - \sum P_i \log P_i.$$

Väikese ε korral

$$I \approx D_I \log(1/\varepsilon).$$

Seega väikese ε korral saame defineerida dimensiooni

$$D_I = \lim_{\varepsilon \rightarrow 0} \frac{I(\varepsilon)}{\log(1/\varepsilon)} = \lim \frac{\sum P_i \log P_i}{\log \varepsilon}.$$

Tegelikult annavad kõik need protseduurid võrdlemisi sarnase tulemuse. On siiski teada (Moon, 1987), et kehtib

$$D_G \leq D_I \leq D_C.$$

Ilmselt pakub nüüd huvi neid teadmisi rakendada meile juba tuntud atraktoritele. Tulemused on toodud tabelis 7.1.

Tabel 7.1. Mõne dünaamilise süsteemi fraktaaldimensioonid

Struktuur	Dimensioon
<i>Hénoni kujutis</i>	$D_C = 1,26$
$a = 1,4$ $b = 0,3$	$D_G = 1,21$
<i>Feigenbaumi diagramm</i>	$D_C = 0,538$
$r = 3,5699456$	
<i>Lorenzi atraktor</i>	$D_C = 2,06$
	$D_G = 2,05$

Mandelbroti ja Julia hulgad

Ruutkujutis

$$Z_{n+1} = Z_n^2 + C$$

$$Z_n = x_n + y_n i$$

$$C = p + q i$$

Komplekstasandil

$$x_{n+1} = x_n^2 - y_n^2 + p$$

$$y_{n+1} = 2x_n y_n + q$$

1) fikseerime C

otsime x_n, y_n koonduvuspiirkonda

Julia hulgad

2) otsime C koonduvuspiirkonda

Mandelbroti hulk M

(vt. manpwin)

Kui $C \in M$, siis on Julia hulk sidus

Kui $C \notin M$, siis on Julia hulk mittesidus

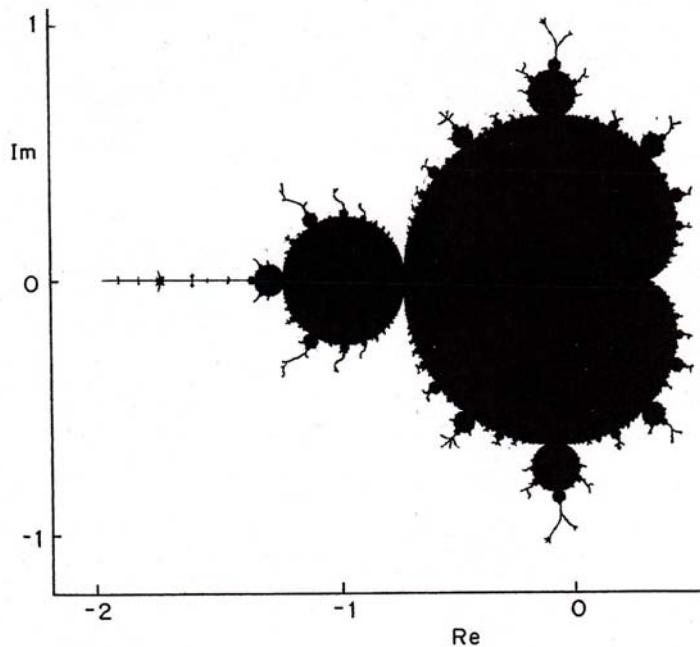
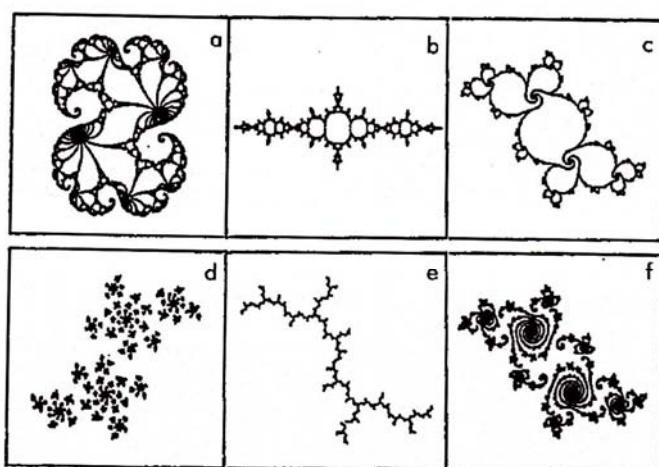


Figure 14.2 The Mandelbrot set M in the complex plane



Joonis 7.13. Tüüpilised Julia hulgad (a) $C = 0,27334 + 0,00742i$; (b) $C = -1,25$; (c) sidus Julia hulk hetk enne lagunemist Cantori hulgaks $C = -0,111 + 0,6557i$; (d) Fatou tolm, $C = 0,11031 - 0,67037i$; (e) $C = i$; (f) Cantori hulk, mis areneb parameetri C väikesel varieerimisel kujundist (c). *Peitgen, Richter, 1986.*

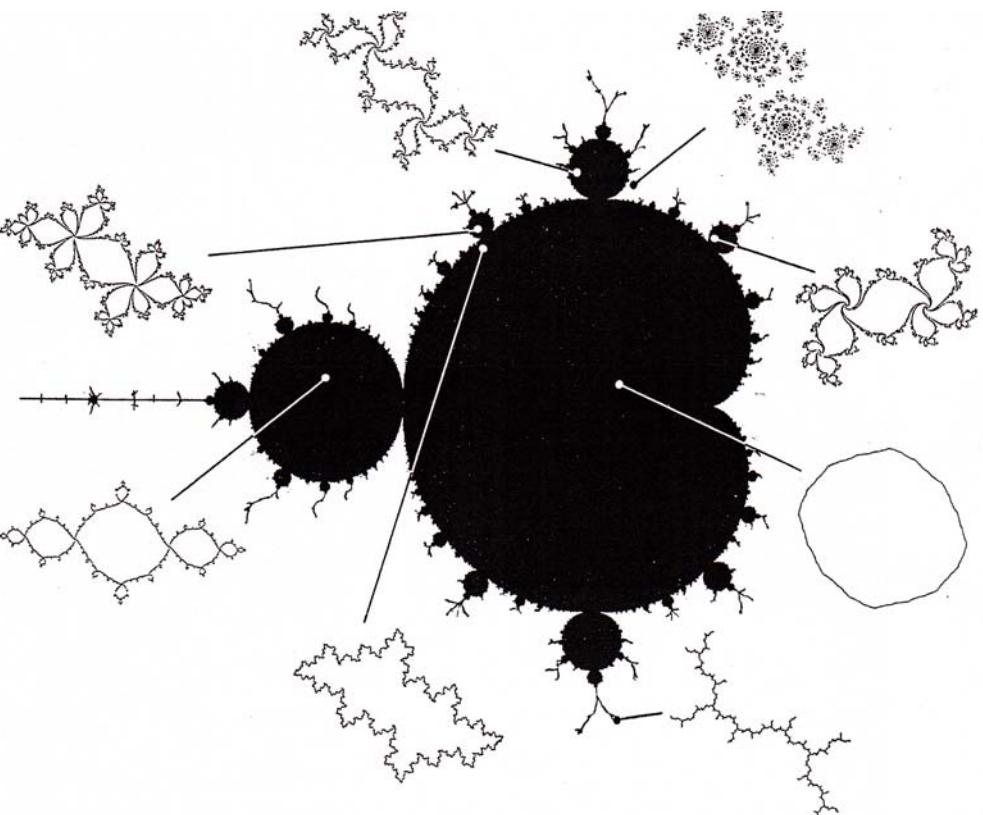
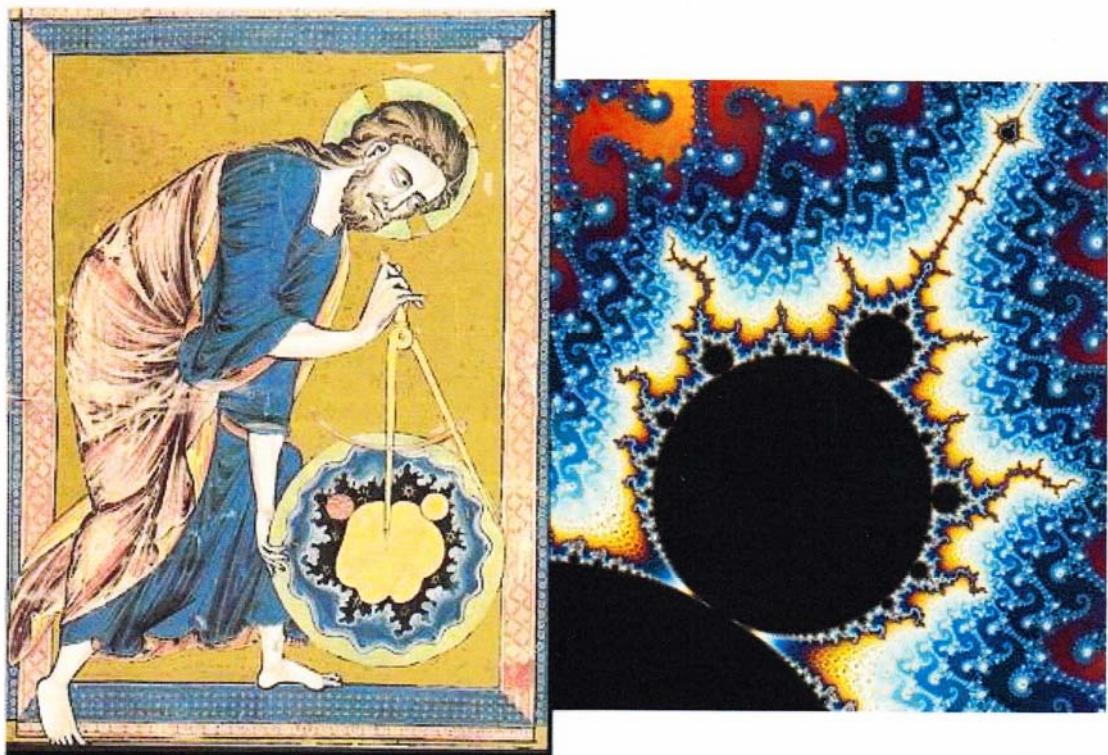


Figure 14.6 Julia sets $J(f_c)$ for c at various points in the Mandelbrot set. The Julia sets are displayed in more detail in figure 14.7

Frontispiece of a Bible Moralisee (ca. 1250)



Order from Chaos: Frontispiece of a Gothic Bible Moralisee (13th century)

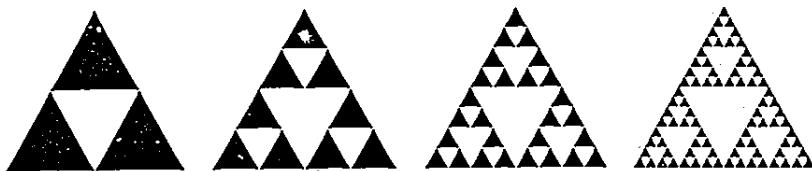


FIGURE 2.15: Construction of the triangular Sierpinski gasket. The initiator is a filled triangle. The generator eliminates a central triangle as shown. The fourth generation of the prefractal is shown to the right. The fractal curve obtained in the limit of an infinite number of generations has the fractal dimension $D = \ln 3 / \ln 2 = 1.58\dots$.

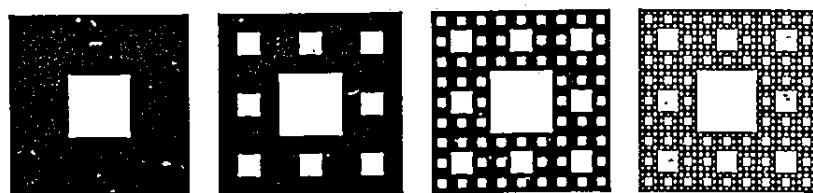


FIGURE 2.16: Construction of the Sierpinski carpet. The initiator is a square and the generator (shown on the left-hand side) is made of $N = 8$ squares. They are obtained by contractions of ratio $r = 1/3$. The right-hand side of the figure shows the fourth construction stage. The similarity dimension is $D = \ln 8 / \ln 3 = 1.89\dots$

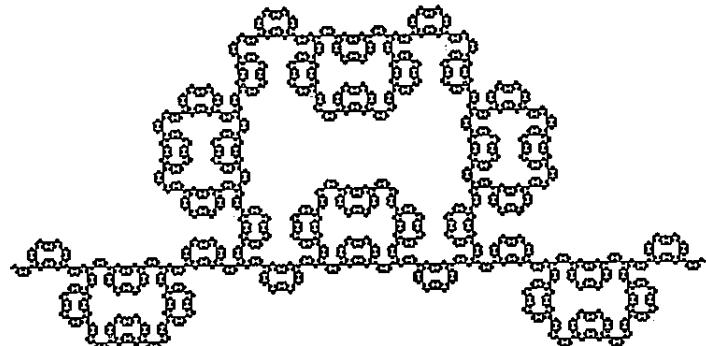
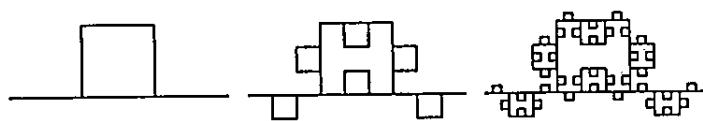
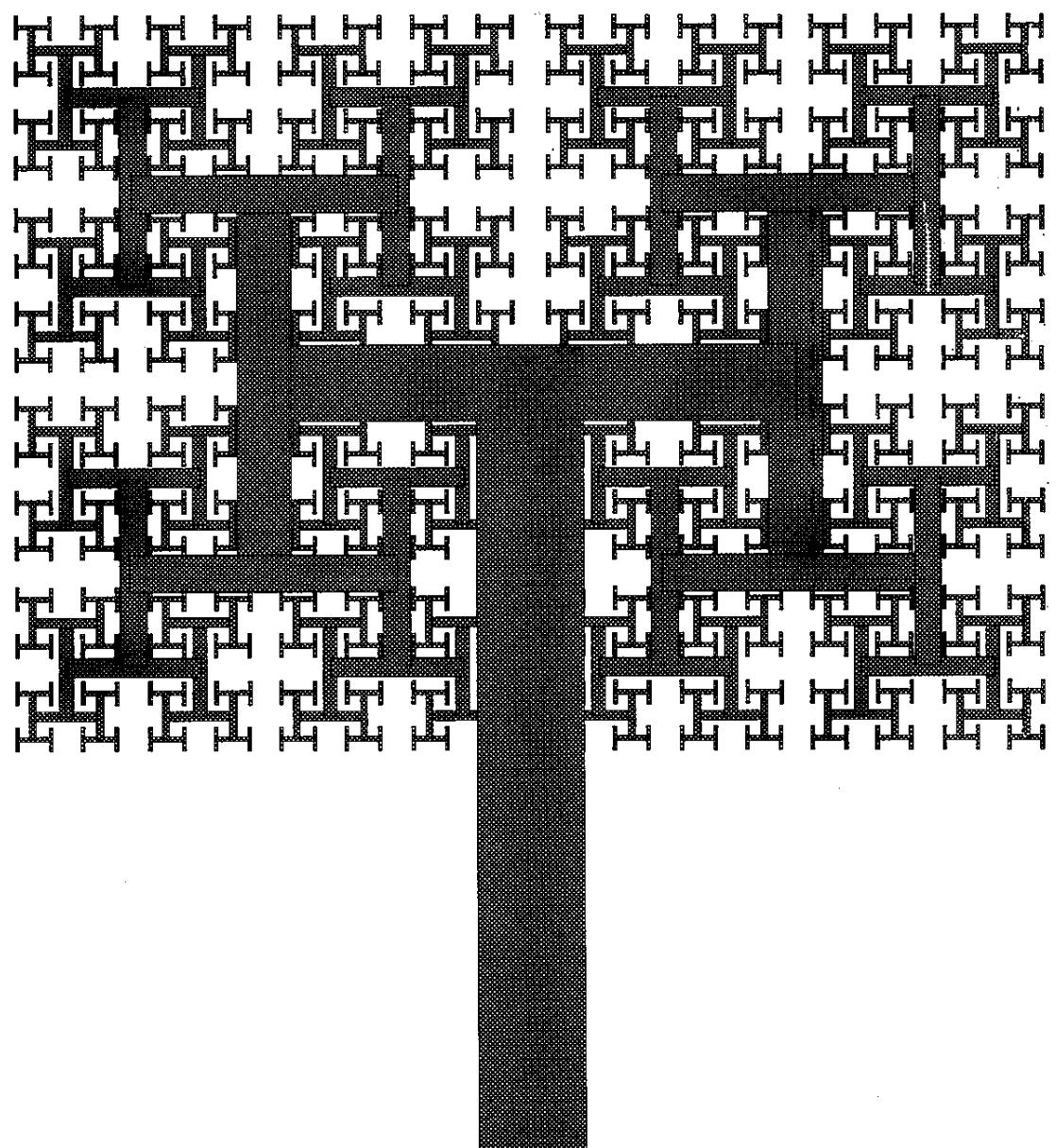
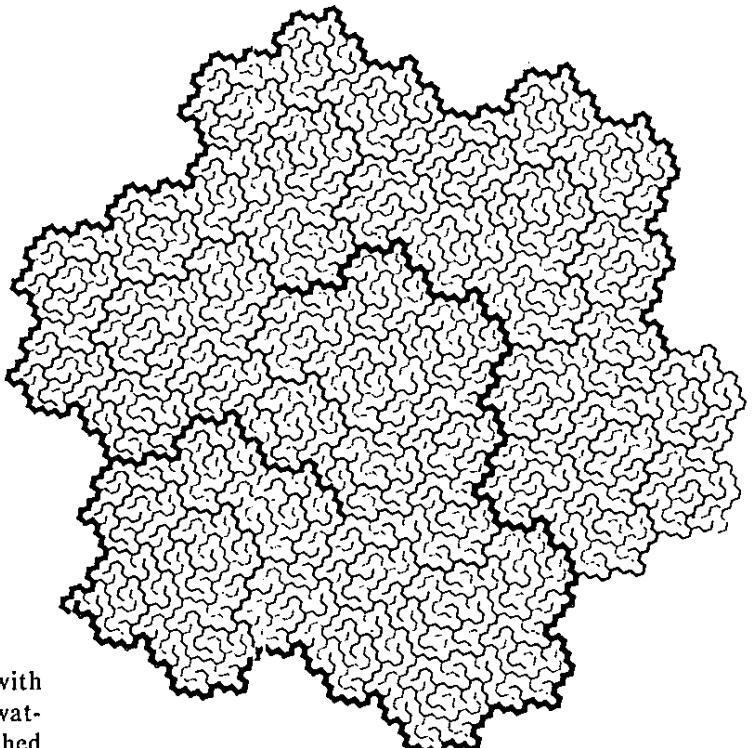


FIGURE 2.14: Construction of a Mandelbrot-Given curve without branches. This curve is obtained using a generator with a single loop. The fractal dimension is $D_B = \ln 6 / \ln 3 = 1.63\dots$

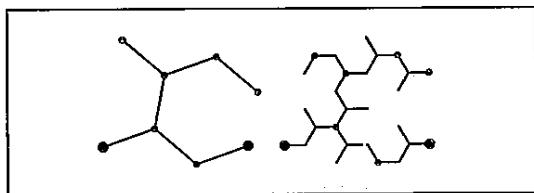




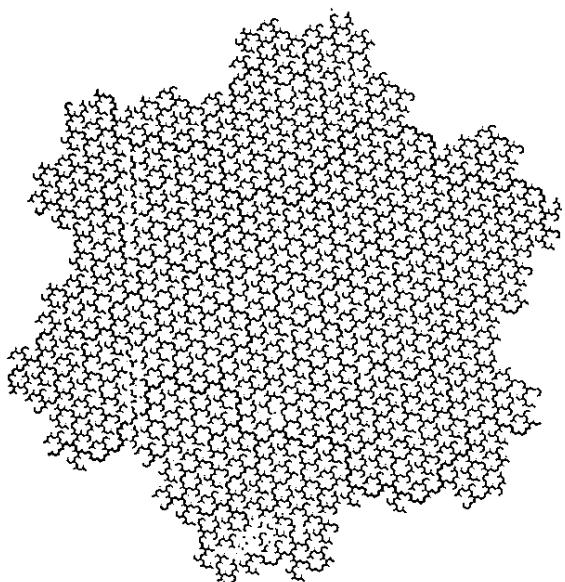
flake curve. Here it is a fractal curve with $D \sim 1.1291$, which is part river and part watershed. And every other river and watershed also converges to a curve of fractal dimension $D \sim 1.1291$.

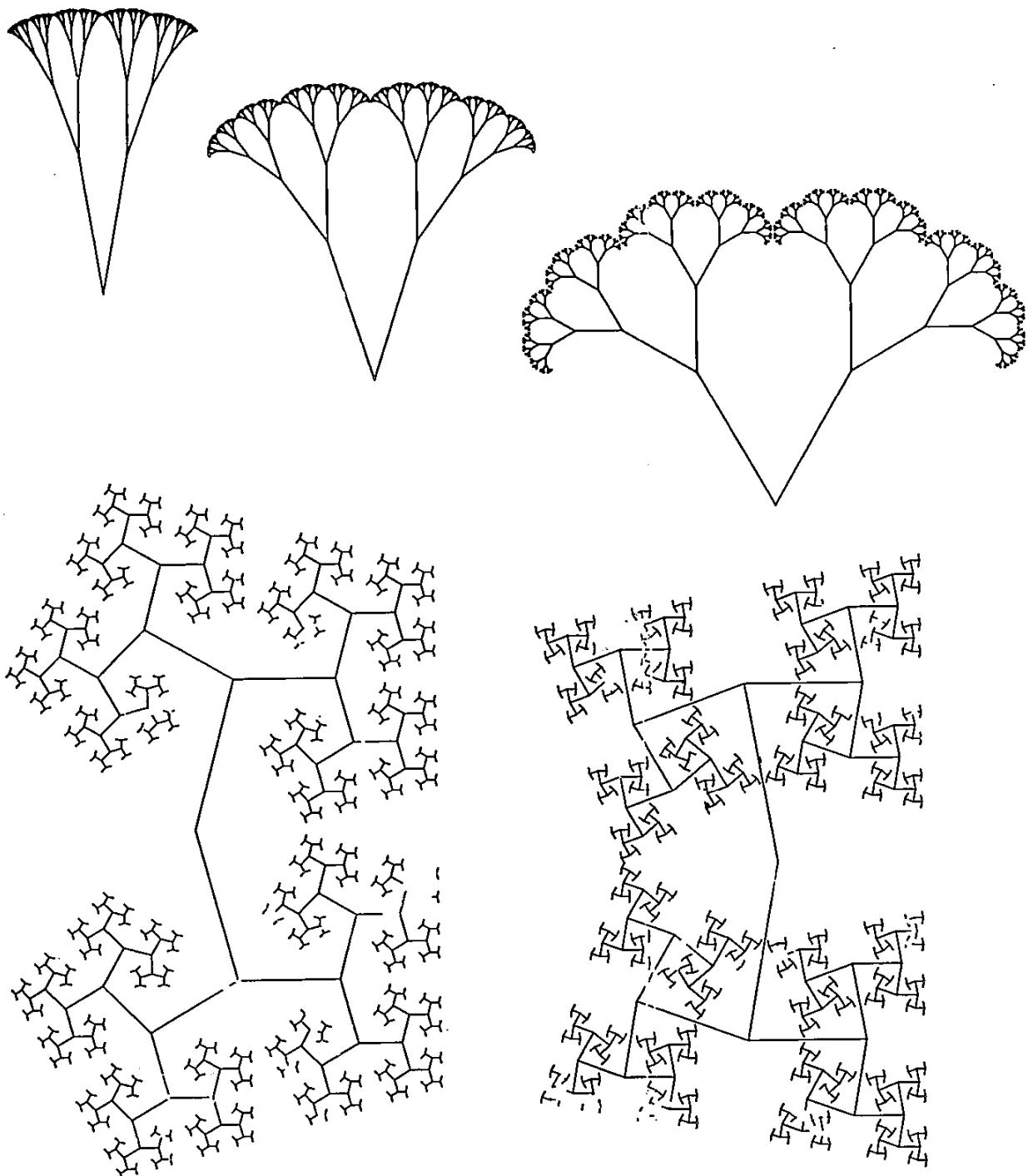
FRANCE. One who as a schoolboy often gazed on a map showing the rivers Loire and Garonne does not feel far from home.

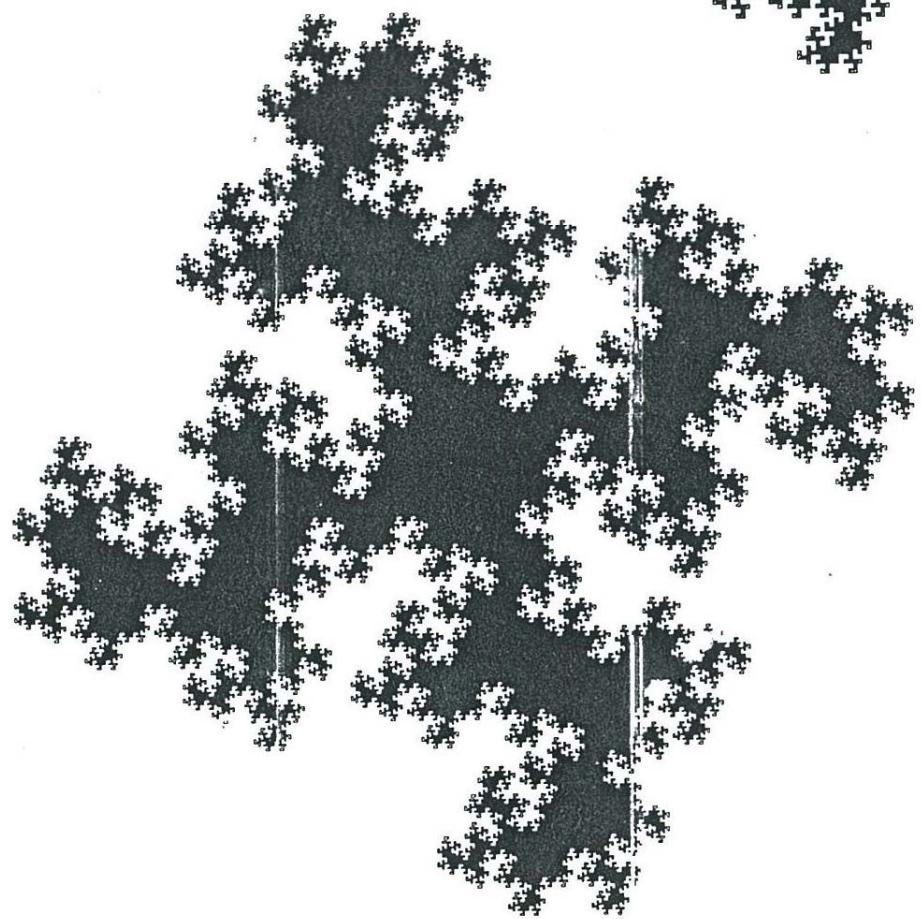
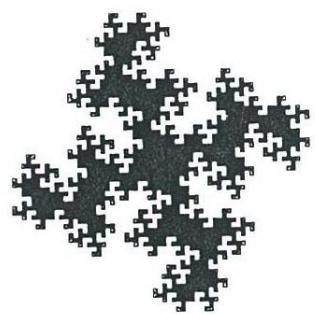
BOTTOM OF PLATE 71. A RIVER TREE CONSTRUCTED DIRECTLY BY A KOCH CASCADE. When the generator is itself tree-shaped, it generates a tree. For example, let the generator be

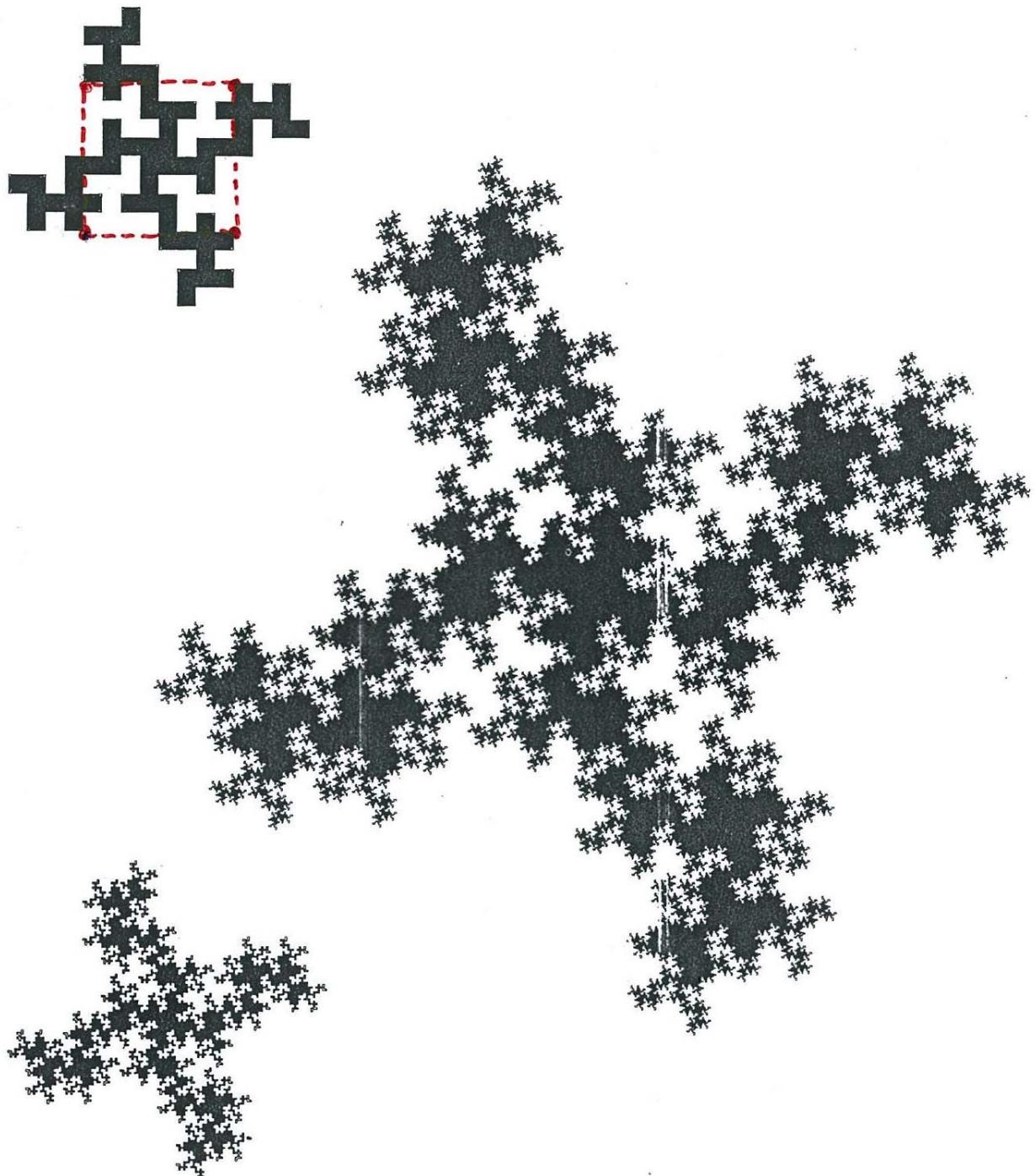


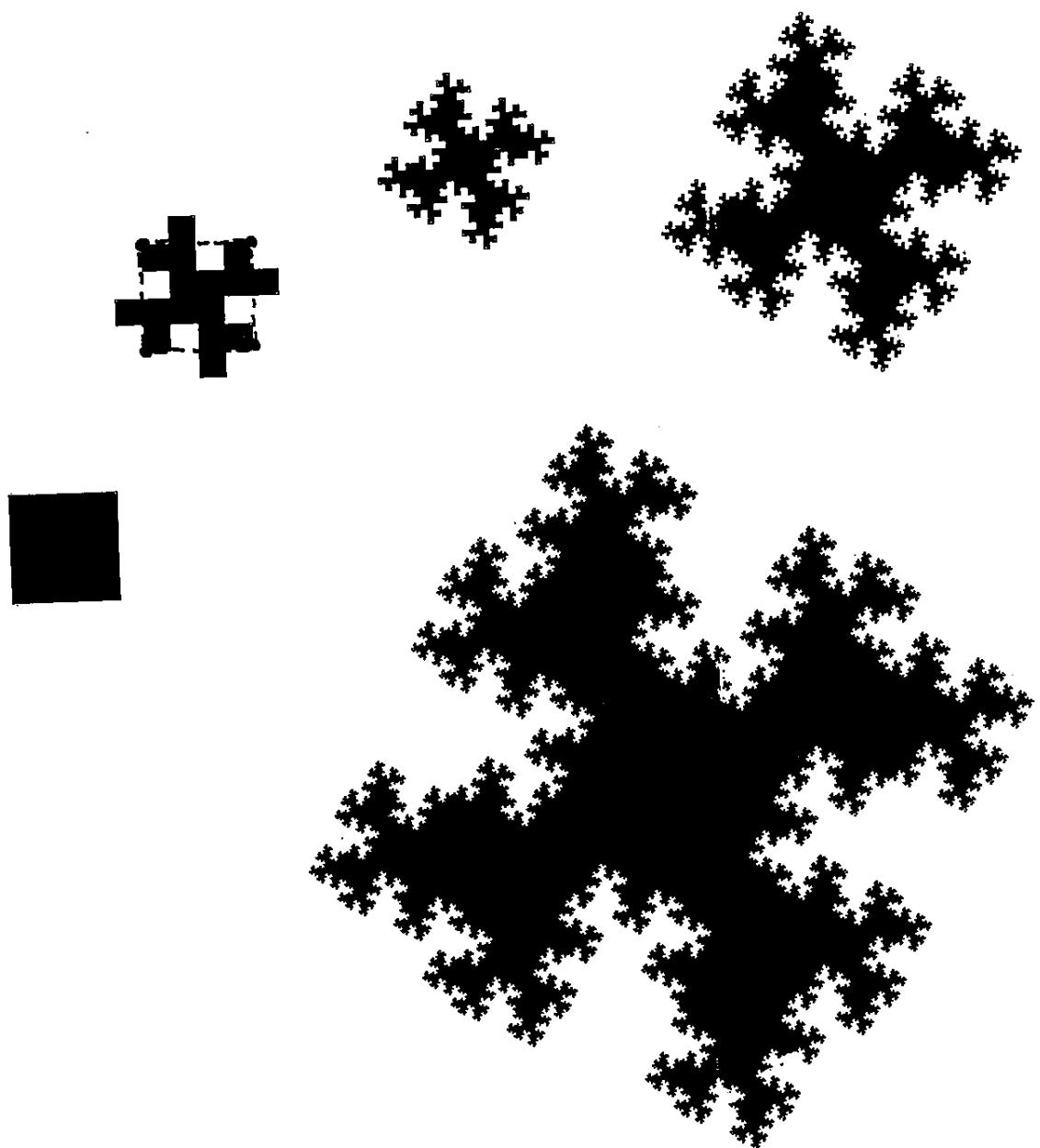
Here we have an alternative method of draining the Koch curve of Plate 46. (The last branches near the "sources" have been clipped off.) ■

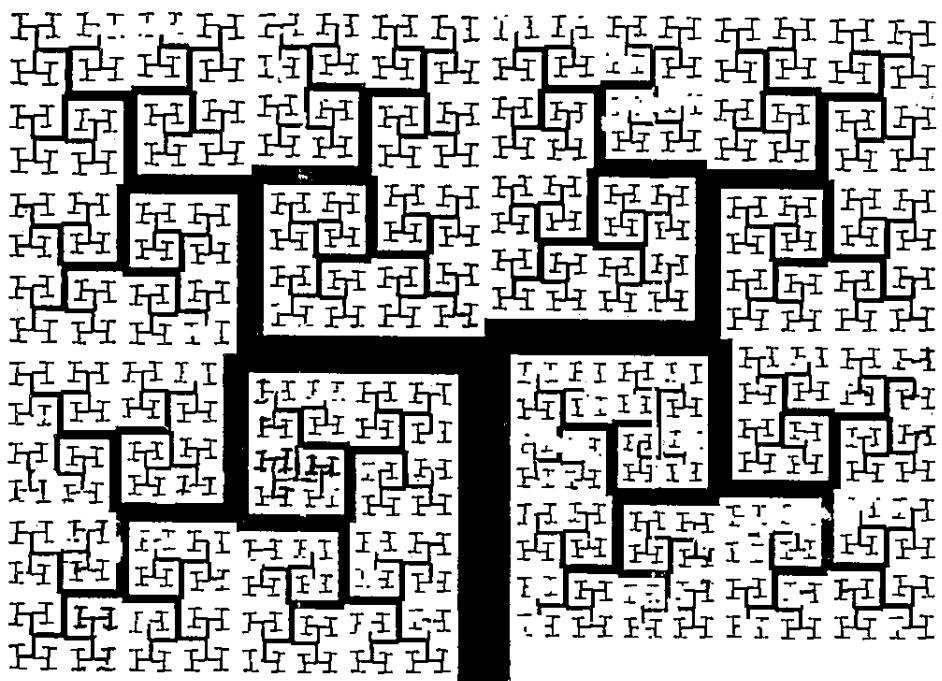
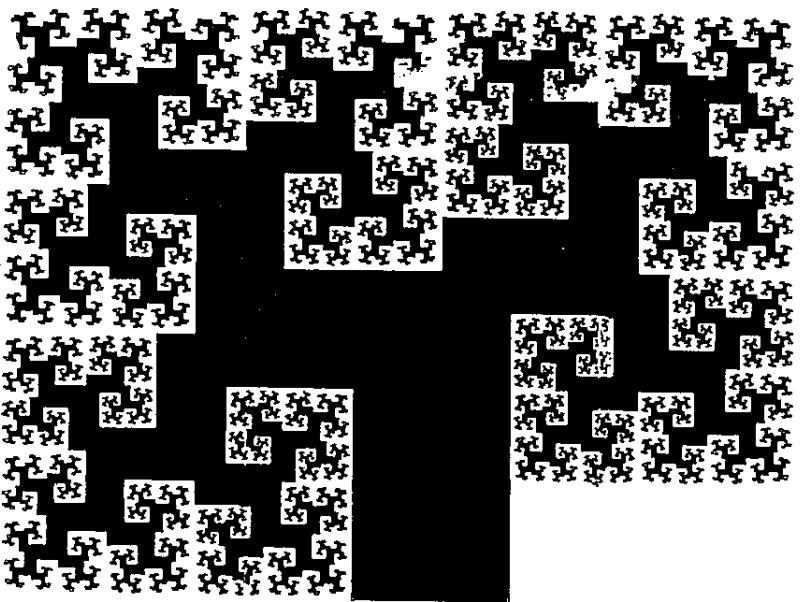












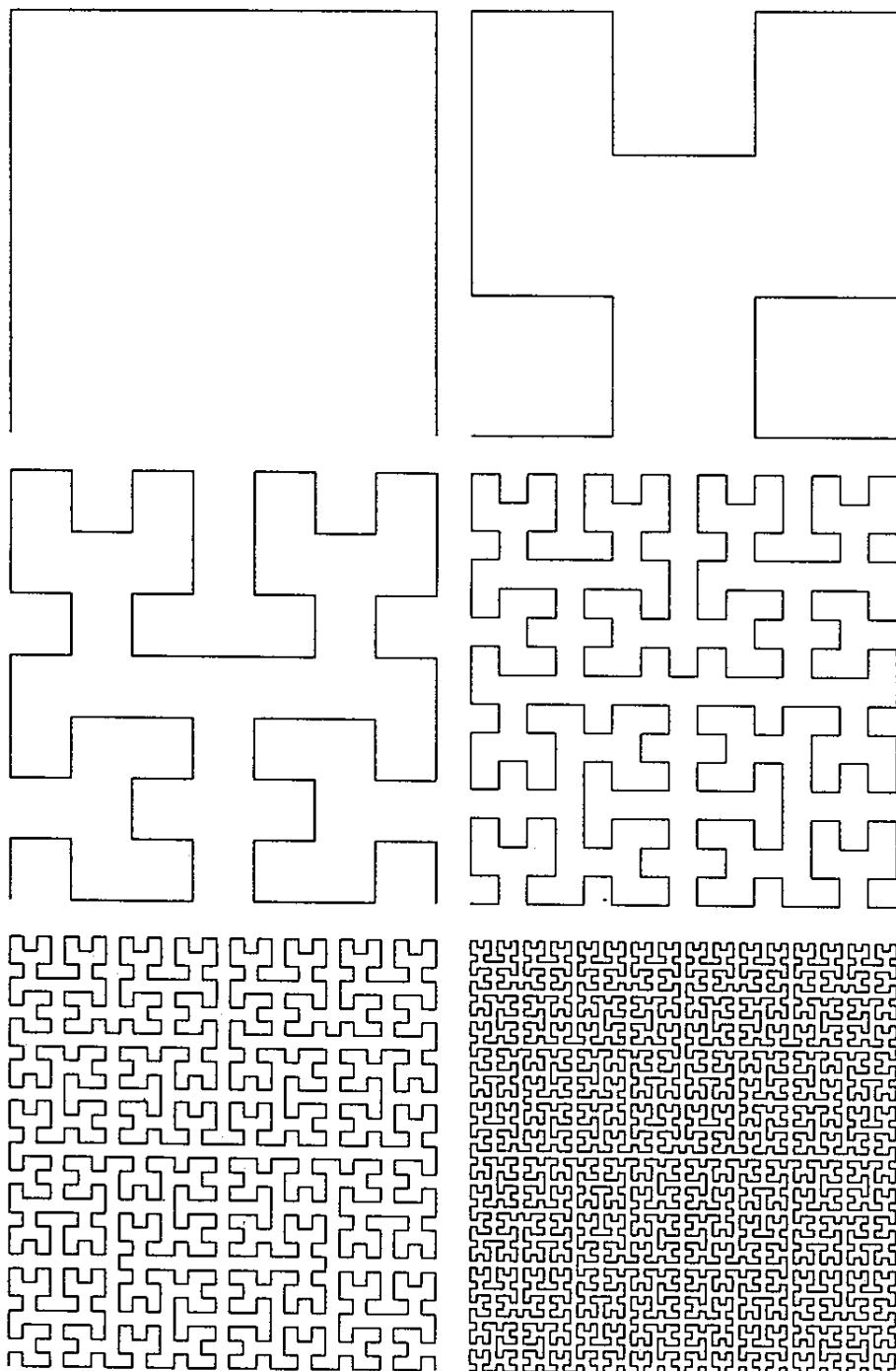


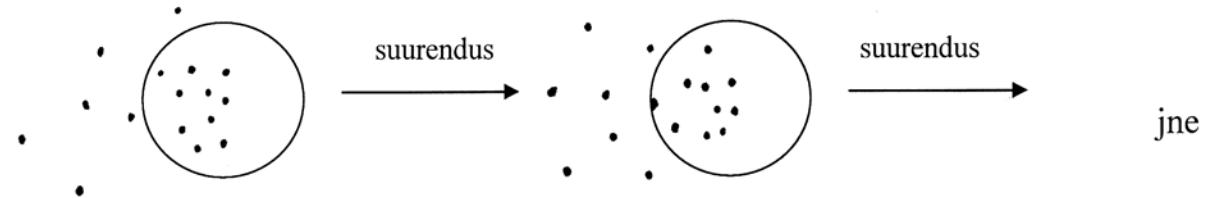
Fig. C.2: The first six stages in the generation of the space filling Hilbert curve. The OL-system is given by the axiom "X", the angle $\delta = \frac{\pi}{2}$ and the production rules $X \rightarrow -YF+XFX+FY-$, $Y \rightarrow +XF-YFY-FX+$.

6.2 Fraktaalgeomeetria

Def.1. Punktide jada $\{x_n\}_{n=1}^{\infty}$ meetrilises ruumis (X, d) on Cauchy jada, kui iga antud numbril $\varepsilon > 0$ jaoks eksisteerib täisarv $N > 0$ nii, et

$$d(x_n, x_m) < \varepsilon \text{ kõigi } n, m > N \text{ jaoks.}$$

S.o. mida kaugemale jadas, seda tihedamalt punktid



Cauchy jada koonduvuse korral
nimetatakse meetrilist ruumi täielikuks

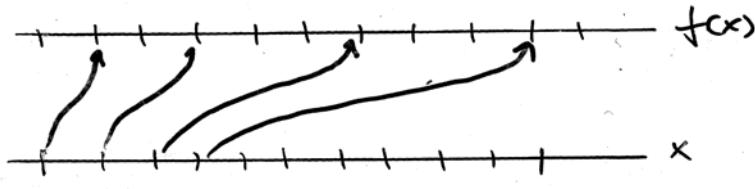
Def. 2. Olgu (X, d) täielik meetriline ruum.

Siis $H(X)$ tähistab hulka, mille punktid on ruumi X kompaktsed mittetühjad alamhulgad.

$H(X)$ on fraktal

Def. 3. Olgu (X, d) meetriline ruum. Teisendus X' l on funktsioon $f : X \rightarrow X'$, mis seab vastavusse täpselt iga punkti $f(x) \in X'$ igale punktile $x \in X$.

Def. 4. Teisendus $f : R \rightarrow R$ kujul
 $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_Nx^N$
 on polünomiaalne teisendus; N – teisendusaste
 $a_i, i = 0, 1, \dots, N$ – reaalarvud, $a_N \neq 0$, N – mitte-negatiivne täisarv.

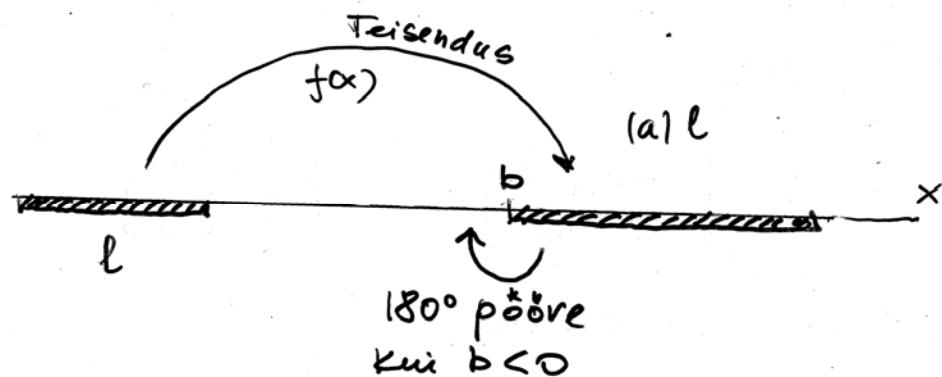


Def. 5. Teisendus $f : R \rightarrow R$ kujul

$$f(x) = \frac{ax+b}{cx+d}, \quad a, b, c, d \in R, \quad ad \neq bc$$

on Möbiuse teisendus; (lineaarne murdteisendus).

Def. 6.1 Teisendus $f : \mathbb{R}^1 \rightarrow \mathbb{R}^1$ kujul
 $f(x) = ax + b$
on affiinne teisendus.



Def. 6.2 Teisendus $w : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ kujul
 $w(x_1, x_2) = ax_1 + bx_2 + e, \quad cx_1 + dx_2 + f$
on affiinne teisendus.

Def. 6.3 Teisendus $v : \mathbb{R}^3 \rightarrow \mathbb{R}^3$
jne.

$$\mathbb{R}^2 \times \mathbb{R}^2$$

$$w(x) = w \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$$

$$= \mathbf{A}x + \mathbf{t}$$

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \mathbf{t} = \begin{bmatrix} e \\ f \end{bmatrix}$$

Polaarkoordinaatides

$$A = \begin{bmatrix} r_1 \cos \theta_1 & -r_2 \sin \theta_2 \\ r_1 \sin \theta_1 & r_2 \cos \theta_2 \end{bmatrix}$$

$r_1, \theta_1 \rightarrow$ punkti a, c polaarkoordinaadid
 $r_2, \theta_2 \rightarrow$ punkti b, d polaarkoordinaadid

$$r_1 = \sqrt{a^2 + c^2}$$

$$r_2 = \sqrt{b^2 + d^2}$$

$$\theta_1 = \arccos \frac{a}{\sqrt{a^2 + c^2}}$$

$$\theta_2 = \arccos \frac{b}{\sqrt{b^2 + d^2}}$$

Sarnasus

Affiinne teisendus $w : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ on sarnasusteisendus kui

$$w \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} r \cos \theta & -r \sin \theta \\ r \sin \theta & r \cos \theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$$

θ – pöördenurk

r – mastaabitegur (scaling factor)

Kui

$$w \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

siis pööre (rotation)

Kui

$$w \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

siis peegeldus (reflection)

Kui $0 < r < 1$ – teisendus on ahenduv

r – ahendustegur

Unfolding the Blueprint

	rotate	rotate & scale	rotate & scale & reflect
—			

Figure 5.6 : The images of the first two rows do not allow the precise determination of the transformation.

Admissible Transformations

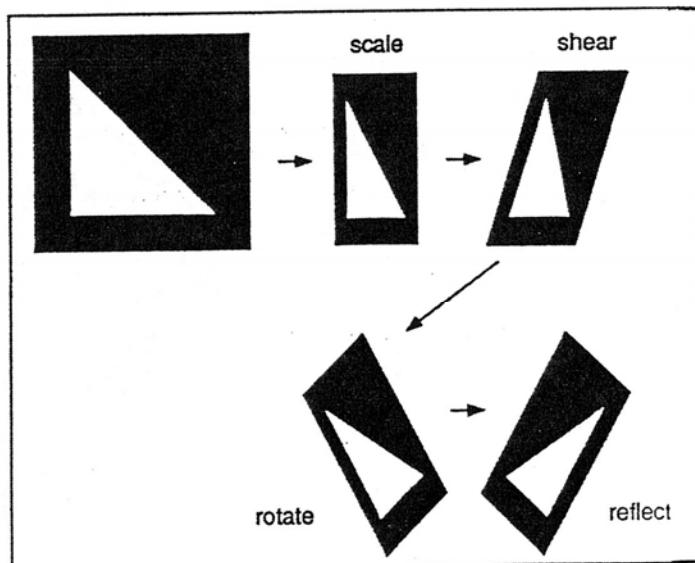


Figure 5.3 : Transformations with scaling, shearing, reflection, rotation and translation (not shown) are admissible in an MRCM.

Sierpinski Gasket Variation

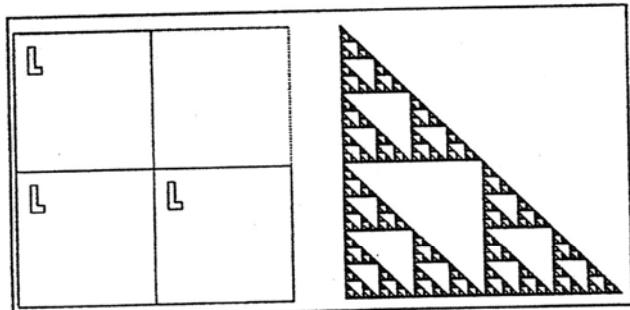


Figure 5.9 : IFS with three similarity transformations with scaling factor 1/2.

The Twin Christmas Tree

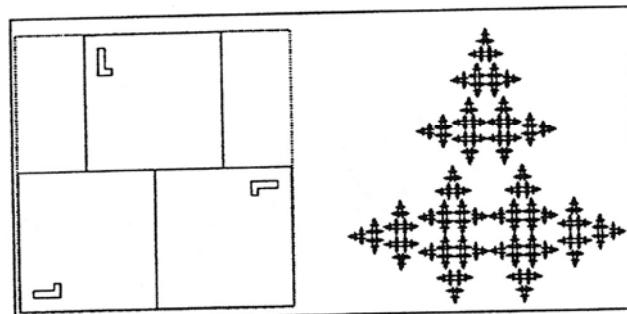


Figure 5.10 : Another IFS with three similarity transformations with scaling factor 1/2.

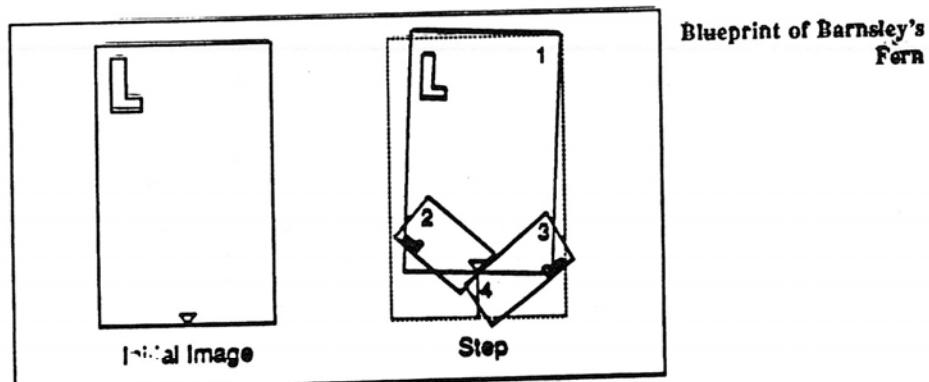


Figure 5.22 : Blueprint of Barnsley's fern.

Barnsley Fern Transformations

	Translations		Rotations		Scalings	
	e	f	θ	ψ	r	s
1	0.0	1.6	-2.5	-2.5	0.85	0.85
2	0.0	1.6	49	49	0.3	0.34
3	0.0	0.44	120	-50	0.3	0.37
4	0.0	0.0	0	0	0.0	0.16

Table 5.23 : Transformations for the Barnsley fern. The angles are given in degrees.

Barnsley's Fern

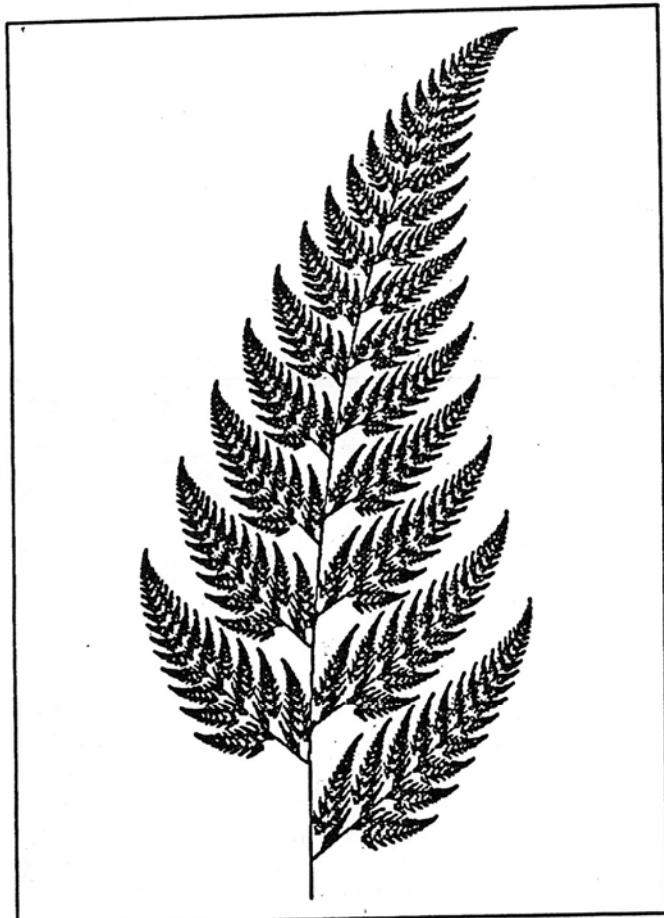
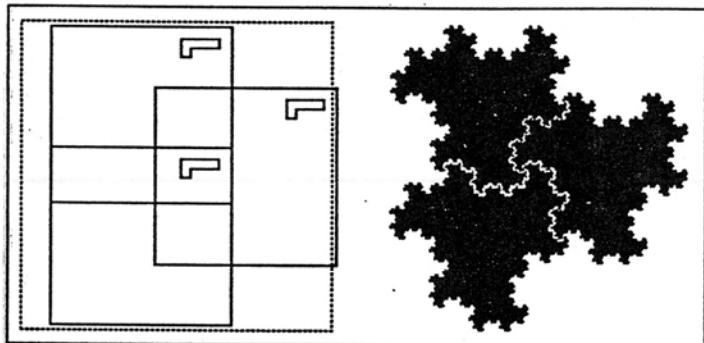
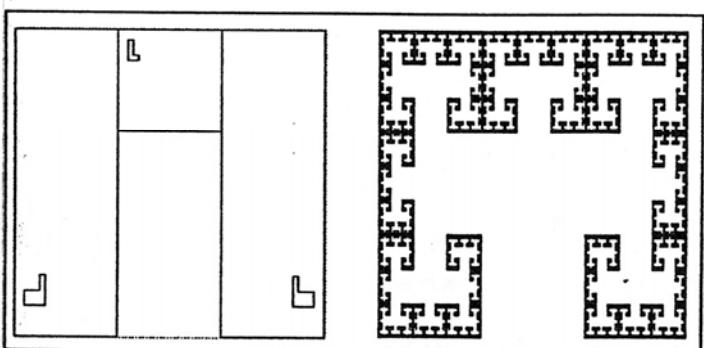


Figure 5.25 : Barnsley's fern generated by an MRCM with only four lens systems.



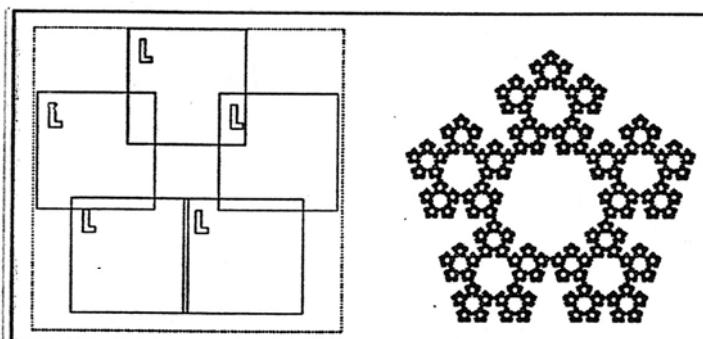
A Dragon With
Threefold Symmetry

Figure 5.11 : The white line is inserted only to show that the figure can be made up from three parts similar to the whole.



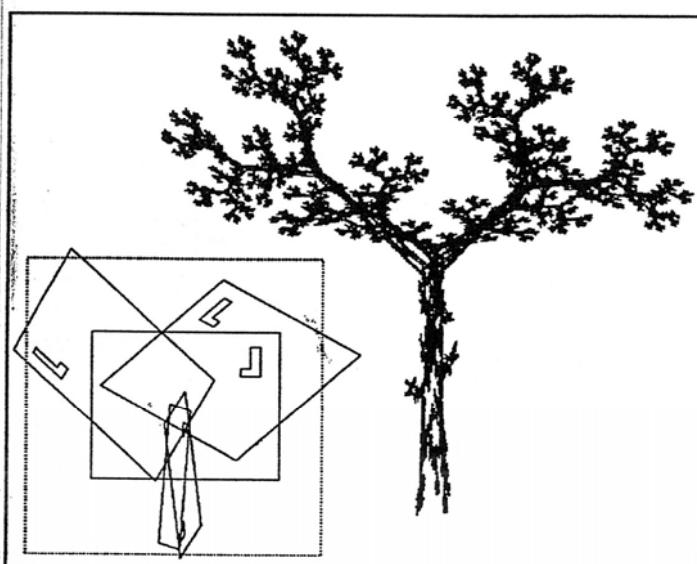
The Cantor Maze

Figure 5.12 : IFS with three transformations, one of which is a similarity. The attractor is related to the Cantor set.



Crystal with Five Transformations

Figure 5.15 : IFS with five similarity transformations. Can you see Koch curves in the attractor?



A Tree

Figure 5.16 : The attractor of an MRCM with five transformations can even resemble the image of a tree (the attractor is shown twice as large as the blue-print indicates).

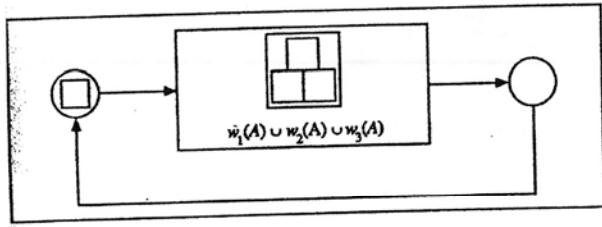


Figure 5.7 : The operation of an MRCM as a feedback system.

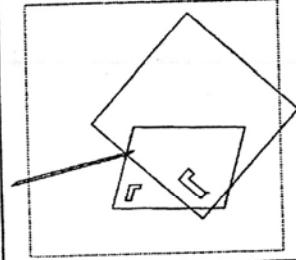


Figure 5.13 : IFS with three affine transformations (no similarities).

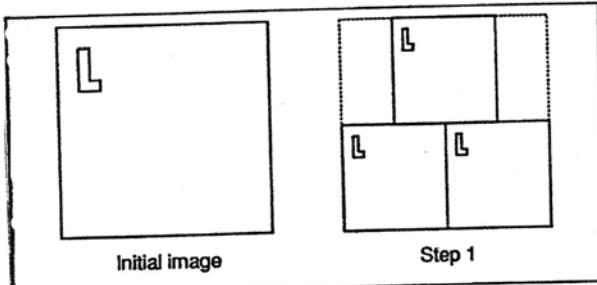


Figure 5.8 : Blueprint of an MRCM using a unit square with an inscribed letter 'L' in the top left corner as an initial image. The purpose of the outline of the initial image on the output on the right is to allow the identification of the relative positioning of the images.

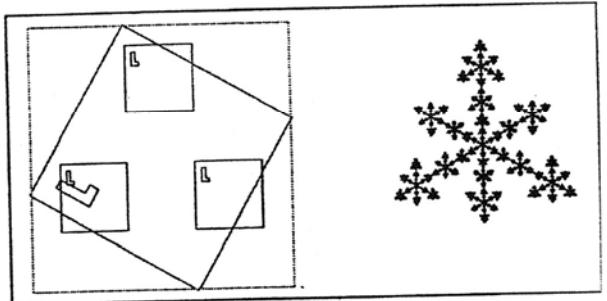


Figure 5.14 : IFS with four similarity transformations.

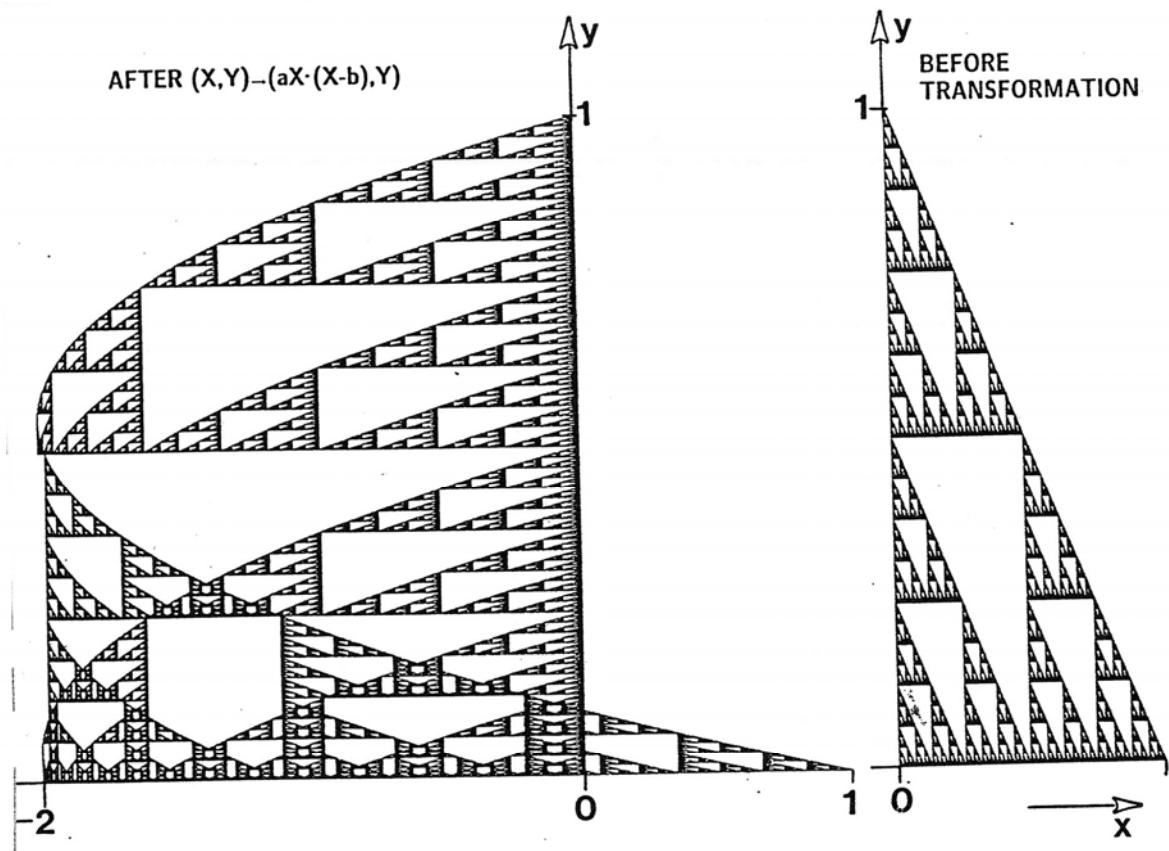
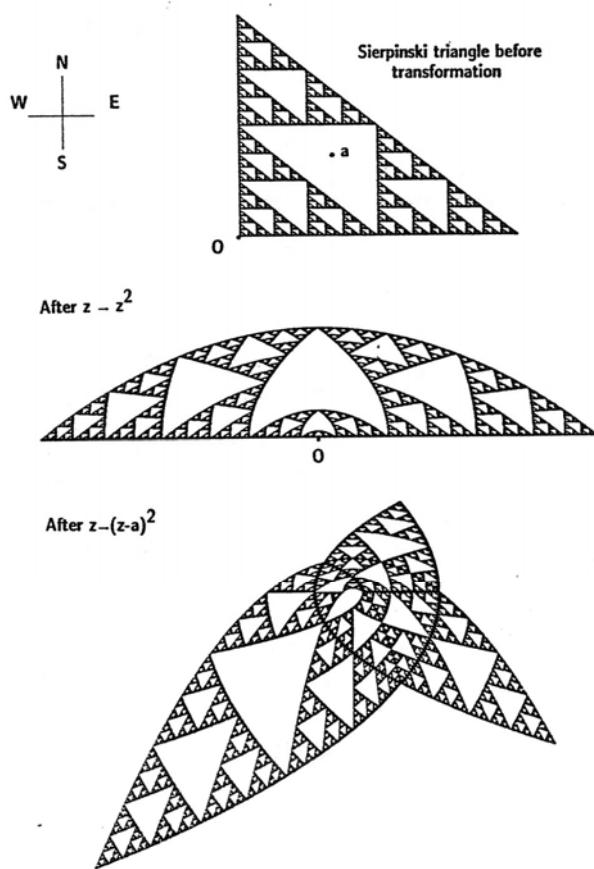


Figure 3.1.11

A Sierpinski triangle before and after the polynomial transformation $x \mapsto ax(x - b)$ is applied to the x-axis. Evaluate the real constants a and b .

Figure 3.4.1

Quadratic transformations are described by showing how they act on a Sierpinski triangle. Use a magnifying glass to check that the transformations behave locally like similitudes.



Rectangle in MRCM

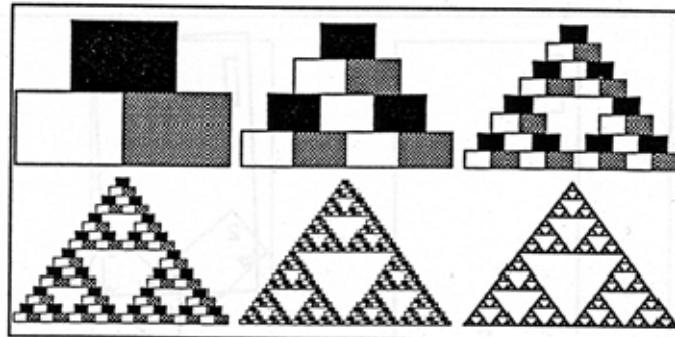


Figure 1.13 : Starting with a rectangle the iteration leads to the Sierpinski gasket. Shown are the first five steps and the result after some more iterations (lower right).

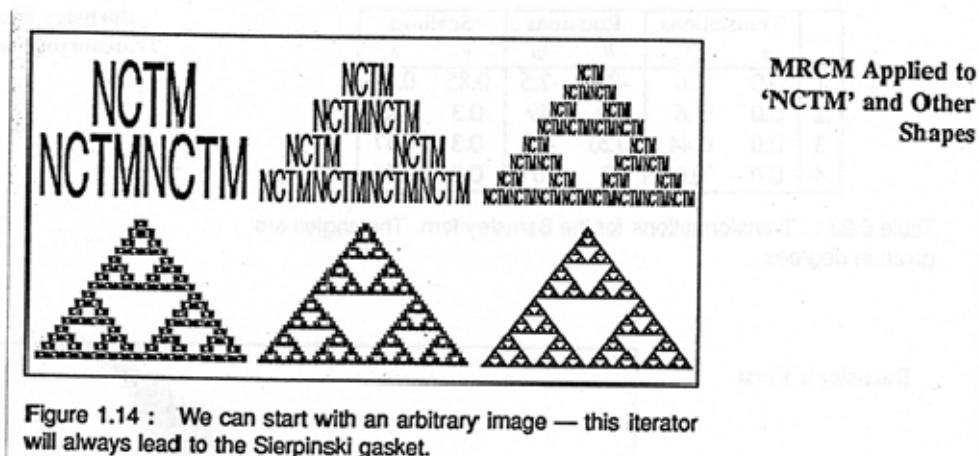


Figure 1.14 : We can start with an arbitrary image — this iterator will always lead to the Sierpinski gasket.

MRCM for the Sierpinski Gasket

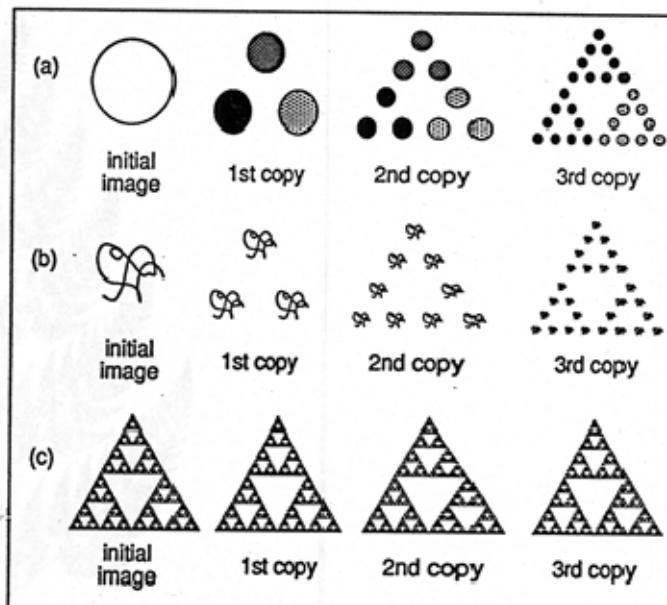


Figure 5.1 : Three iterations of an MRCM with three different initial images.

Milline on saadud kujundite fraktaalne dimensioon?

$$w = Ax + t \quad A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}, \quad t = \begin{vmatrix} e \\ f \end{vmatrix}$$

$$A = \begin{vmatrix} r_1 \cos \theta_1 & -r_2 \sin \theta_2 \\ r_1 \sin \theta_1 & r_2 \cos \theta_2 \end{vmatrix}$$

$$r_1 = r_2 = r \rightarrow \text{sarnasus}$$

$$r_1 = \sqrt{a^2 + c^2} \quad r_2 = \sqrt{b^2 + d^2}$$

Teoreem: Olgu $\{R^m; w_1, w_2, \dots, w_N\}$ hüperboolne IFS ja olgu A tema atraktor. Olgu w_n sarnasusteisendus mastaabiteguriga r_n iga $n \in \{1, \dots, N\}$ puhul. Siis kehtib

$$\sum_{n=1}^N |r_n|^{D(A)} = 1, \quad D(A) \in [0, m]$$

Märkused:

- 1) hüperboolne IFS \rightarrow ahenev ($0 < r < 1$)
- 2) teoreemil on iseärasused erijuhtudel (ülekattuvus, jne).

Vt. M. Barnsley. Fractals Everywhere, Boston, Academic Press, 1988.

Näide: Sierpinski (5.9 tabelis)

Kolm sarnasusteisendust

	a	b	c	d
w_1	0.5	0	0	0.5
w_2	0.5	0	0	0.5
w_3	0.5	0	0	0.5

$$r_i = \sqrt{0.5^2 + 0} = 0.5$$

Teoreemist

$$(0.5)^D + (0.5)^D + (0.5)^D = 1$$

$$3(0.5)^D = 1$$

$$D \ln 0.5 = \ln 1/3$$

$$D = \frac{\ln 1/3}{\ln 0.5} = \frac{\ln 3}{\ln 2} \cong 1.5850$$

Näide: Kristall (5.15 tabelis)

Viis sarnasusteisendust

$$r_i = 0.382$$

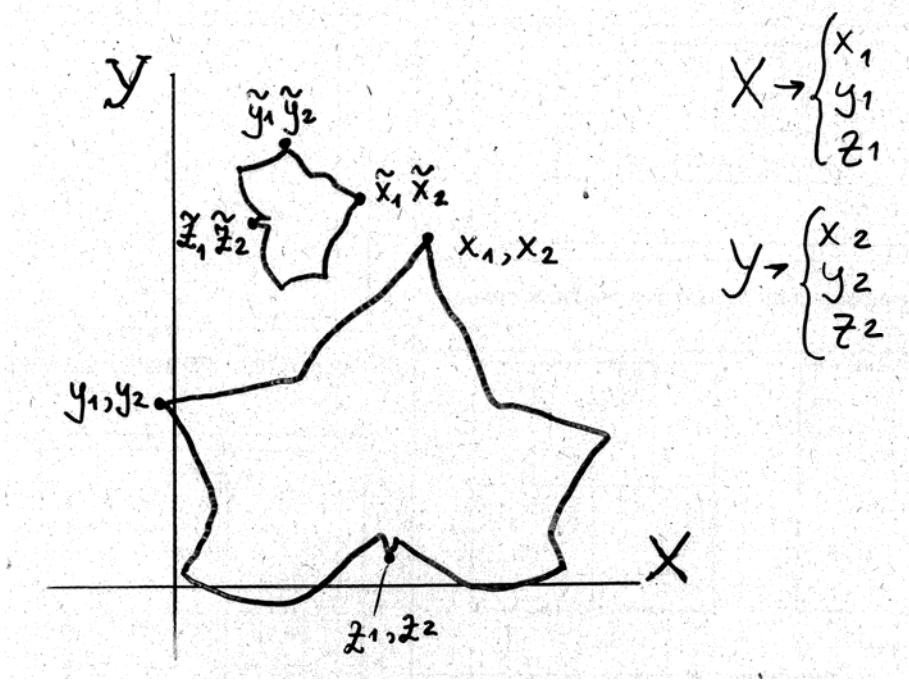
$$5(0.382)^D = 1$$

$$D = \frac{\ln 1/5}{\ln 0.382} \cong 1.6725$$

H.O. Peitgen, H. Jürgens, D. Saupe.
Fractals for the Classroom. Part One
Springer, Berlin et al. 1992

M. Barnsley. Fractals Everywhere.
Academic Press, Boston et al., 1988.

Pöördülesanne



$$w = Ax + t \quad , \quad A - ? \quad t - ?$$

$$\begin{cases} x_1 a + x_2 b + e = \tilde{x}_1 \\ y_1 a + y_2 b + e = \tilde{y}_1 \\ z_1 a + z_2 b + e = \tilde{z}_1 \end{cases} \rightarrow a, b, e$$

$$\begin{cases} x_1 c + x_2 d + f = \tilde{x}_2 \\ y_1 c + y_2 d + f = \tilde{y}_2 \\ z_1 c + z_2 d + f = \tilde{z}_2 \end{cases} \rightarrow c, d, f$$

$$A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}, \quad t = \begin{vmatrix} e \\ f \end{vmatrix}$$