

10. DETERMINEERITUS

füüsikaline }
bioloogiline } protsess → matemaatiline mudel → lahendamine
sotsiaalne }

- kas on lahendamine üldse võimalik?
- kas lahend on determineeritud (määratud)
 - üldse
 - mingis piirkonnas

juhuslikud suurused → juhuslik protsess
(algandmed, (jaotusseadus)
parameetrid ...)

Tänapäevane arusaam:

matemaatiline mudel on

- determineeritud:
 - sisaldab kindla hulga liikmed;
 - kõik numbrilised tegurid ja parameetrid on määratud;
 - algandmed on fikseeritud;
- mittelineaarne

↓
Protsess võib mitte olla ennustatav!
mittedetermineeritud
unpredictable

Lahendatavus

Probleem: Olgu

$$u = f(x, y, z)$$

$$v = g(x, y, z)$$

$$w = h(x, y, z)$$

f, g, h – pidevad, diferentseeruvad
vaja leida

$$x = F(u, v, w)$$

$$y = G(u, v, w)$$

$$z = H(u, v, w)$$

Teoreem: Kui u, v, w , on argumentide x, y, z pidevad ja diferentseeruvad funktsioonid ja punktis x_0, y_0, z_0 on nende väärtuseks u_0, v_0, w_0 ning jakobiaan

$$\frac{D(u, v, w)}{D(x, y, z)} = \begin{vmatrix} f'_x & f'_y & f'_z \\ g'_x & g'_y & g'_z \\ h'_x & h'_y & h'_z \end{vmatrix}$$

on nullist erinev ($\neq 0$), siis eksisteerivad ja on üheselt määratud punkti x_0, y_0, z_0 ümbruses niisugused ühesed, pidevad ja diferentseeruvad funktsioonid $F(u, v, w)$, $G(u, v, w)$ ja $H(u, v, w)$, et võrrandsüsteemid

$$\begin{array}{l} u = f(x, y, z) \\ v = g(x, y, z) \\ w = h(x, y, z) \end{array} \quad \text{ja} \quad \begin{array}{l} x = F(u, v, w) \\ y = G(u, v, w) \\ z = H(u, v, w) \end{array}$$

on samaväärsed, St. F, G, H on operaatori f, g, h pöördoperaator

Seega:

Objekti väljundsuurustest kõlbavad pöördülesande lahendamiseks ainult niisugused u , v , ja w , mille puhul jakobiaan

$$\frac{D(u, v, w)}{D(x, y, z)} \neq 0$$

Kui see jakobiaan on ühes punktis 0, siis ei ole pöördülesanne üldse lahenduv või ei ole x, y, z üheselt määratavad u, v , ja w alusel

Vrdl – lineaarvõrrandisüsteemide lahenduvus

NB! Teoreem on lokaalne ja kehtib lineaarsete seoste puhul. Mittelineaarsete f, g ja h puhul võib eksisteerida mitu lahendit.

Vt. I. Petersen, Pöördülesannete olemus

Küb. Inst. Uuringuaruanne Math 45/92.

Kausaalsuse printsiip (põhjus ja tagajärg)

Sündmuste rida on põhjuslikkuse ahel, kas alati?

ajaline järgnevus \neq põhjuslikkuse ahel

Laplace (1749 – 1827):

Kõigi universumi osakeste koordinaatide ja impulsside väärtused antud ajahetkel määravad täiesti üheselt universumi oleku mistahes ajahetkel minevikus või olevikus

Heisenbergi määramatuse printsiip

Kaose paradigma

If *the world*

- were deterministic and
- consisted exclusively of (interacting) particles,

if Newton's Law of Motion, $m \cdot b = K$, were valid without restriction,

if *we knew*

- all laws of nature, in particular all laws of force, and
- all boundary and initial conditions at a definite point in time (i. e., if Newton's laws apply, the positions and velocities of all particles)
- with absolute accuracy, and

if *we could*

- store all these data,
- process them mathematically,
- and solve all relevant equations
- with sufficient speed,

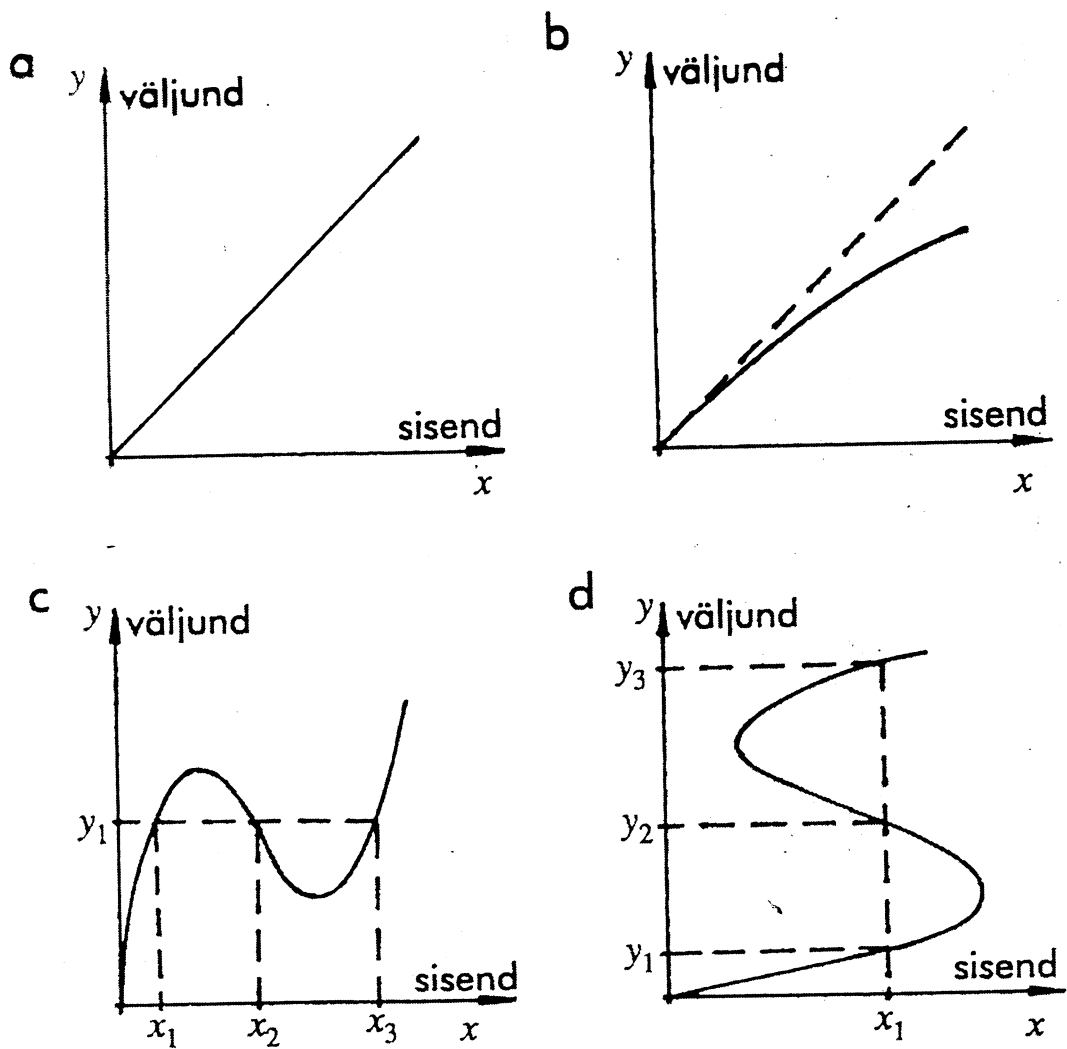
then not only would the course of *the world* be

- unambiguously determined (equal causes have equal effects)
- in every detail,

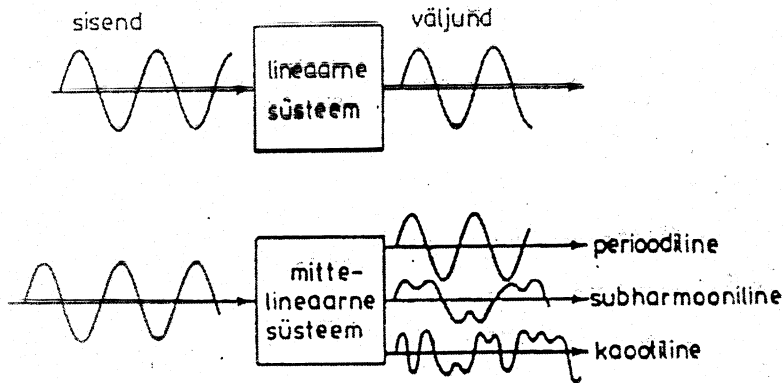
but then could *we* (or at least Laplace's demon, or a gigantic supercomputer) also

- calculate and determine all events
- past and future.

Table 2: Preconditions and consequences of classical determinism.



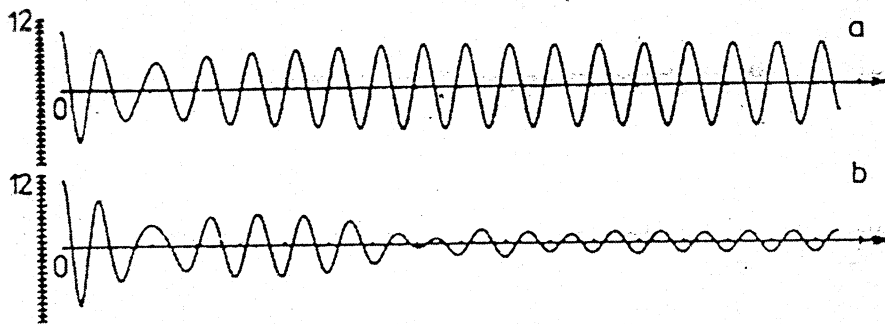
Sisendid ja väljundid lineaarsete ja mittelineaarsete funktsioonide puhul.



Lineaarse ja mittelineaarse süsteemi väljundite erinevus.
Moon, 1987.

Duffingi võrrand

$$\ddot{x} + ax + bx + cx^3 = f \cos \omega t$$



Mittelineaarse sumbuva sundvõnkumise graafik:

- (a) $x_{01} = A, \dot{x}_{01} = 0$;
- (b) $x_{02} = B, \dot{x}_{02} = 0, A < B$; väiksem algamplituud tekitab suurema amplituudiga võnkumise.

Thompson, Stewart, 1986.

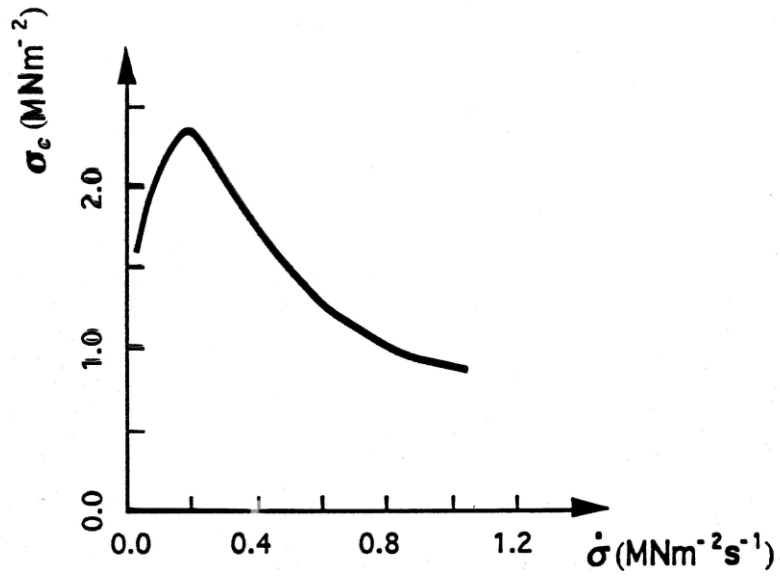


Figure 2. Crushing stress of ice as a function of ice stress rate /11/.

Approximation:

$$\sigma_c = \alpha_j + \beta_j \dot{\sigma} \quad , \quad j=1, 2, 3$$

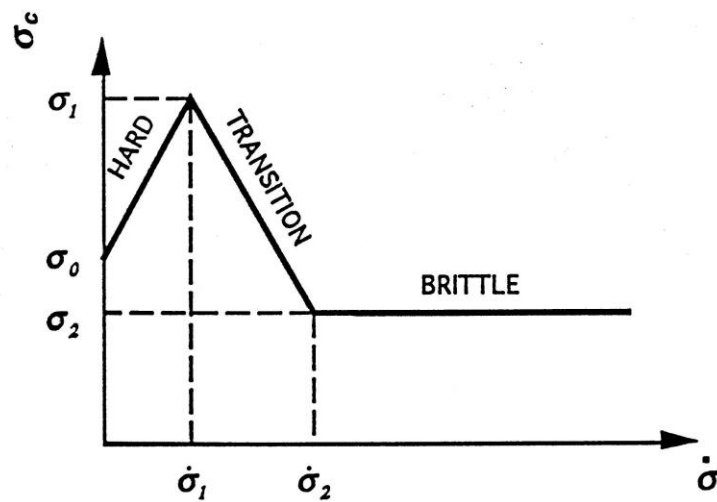


Figure 3. Piece-wise linear approximation of the ice crushing stress.]

ALGIDEED

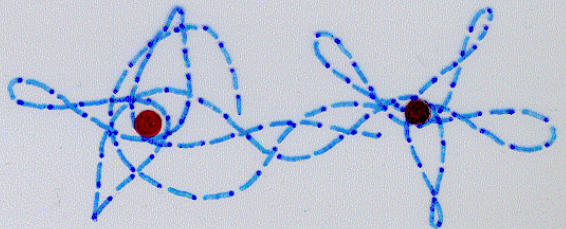
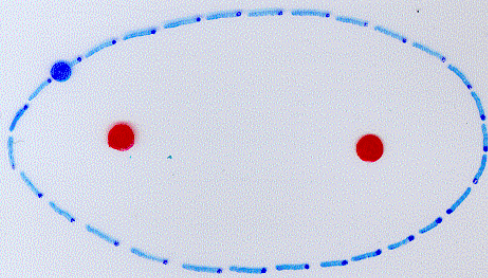
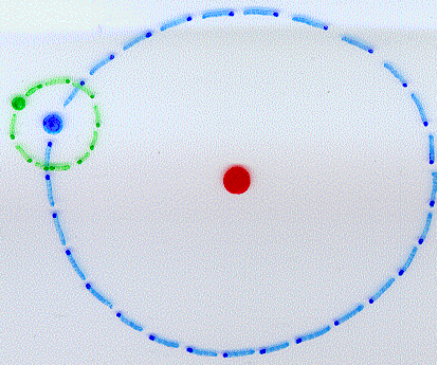
Maxwell 1873

Hadamard 1889

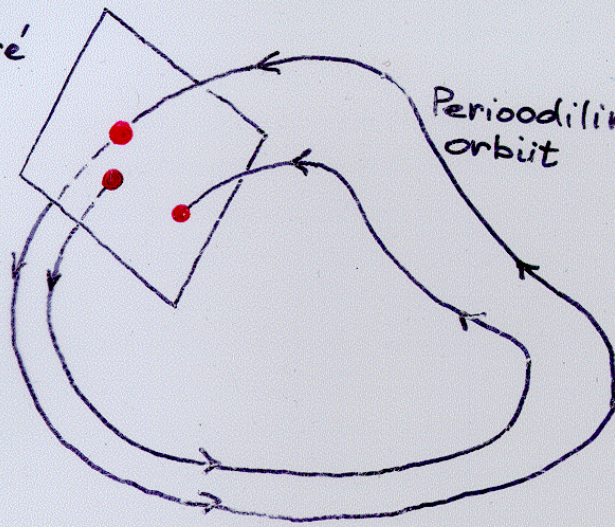
Poincaré 1908

- kolme keha probleem
- “... võib juhtuda, et väikesed erinevused algolukorra kirjeldamisel põhjustavad suuri muutusi vaadeldavas nähtuses.”

Weierstrassi funktsioonid



Poincaré
lõige



Periodiline
orbit

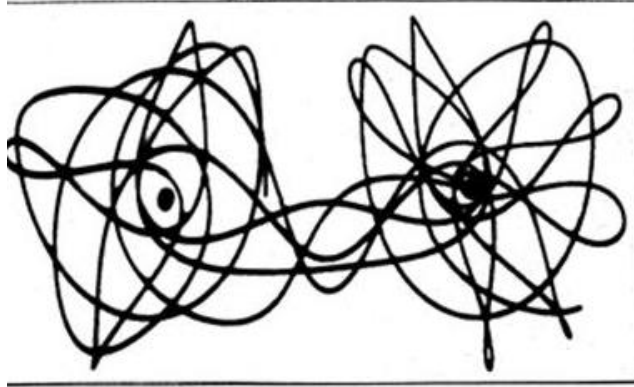


Illustration du mouvement d'un corps attiré par deux masses identiques. Les forces régies par des équations simples, donnent des mouvements d'une extrême complexité.



Une cause très petite, qui nous échappe, détermine un effet considérable que nous ne pouvons pas ne pas voir, et alors nous disons que cet effet est dû au hasard. Si nous connaissions exactement les lois de la nature et la situation de l'univers à l'instant initial, nous pourrions prédire exactement la situation de ce même univers à un instant ultérieur. Mais, lors même que les lois naturelles n'auraient plus de secret pour nous, nous ne pourrions connaître la situation initiale qu'approximativement. Si cela nous permet de prévoir la situation ultérieure avec la même approximation, c'est tout ce qu'il nous faut, nous disons que le phénomène a été prévu, qu'il est régi par des lois; mais il n'en est pas toujours ainsi, il peut arriver que de petites différences dans les conditions initiales en engendrent de très grandes dans les phénomènes finaux; une petite erreur sur les premières produirait une erreur énorme sur les derniers. La prédiction devient impossible et nous avons le phénomène fortuit.

C'est la «sensibilité aux conditions initiales» qui engendre des effets «chaotiques» dans des systèmes déterministes régis par des équations différentielles. Poincaré l'avait découvert à ses dépens, c'est le cas de le dire, si l'on se souvient des coûts de réimpression de son mémoire lauréat, qu'il remboursa à Mittag-Leffler.

Les résultats présentés dans ce mémoire seront approfondis et complétés dans les trois volumes des *Méthodes nouvelles de la Mécanique céleste* (1892-1899) et, par la suite, dans les *Leçons de Mécanique céleste* parues en quatre volumes entre 1905 et 1911.

ARVUDE JADA

mõõdetud, arvutatud

$x_0, x_1, \dots, x_{n-1}, x_n, x_{n+1}, \dots$

1, 2, 3, 4, ... $x_{n+1} = x_n + 1$

2, 4, 6, 8, ... $x_{n+1} = x_n + 2$

$$x_{n+1} = f(x_n)$$

$f(x_n)$ - mittelineaarne funktsioon

$$x_{n+1} = \lambda x_n (1 - x_n)$$

λ - konstant

$$x_{n+1} = \lambda_1 x_n (1 - x_n)$$

$$\lambda_1 = 4\lambda$$

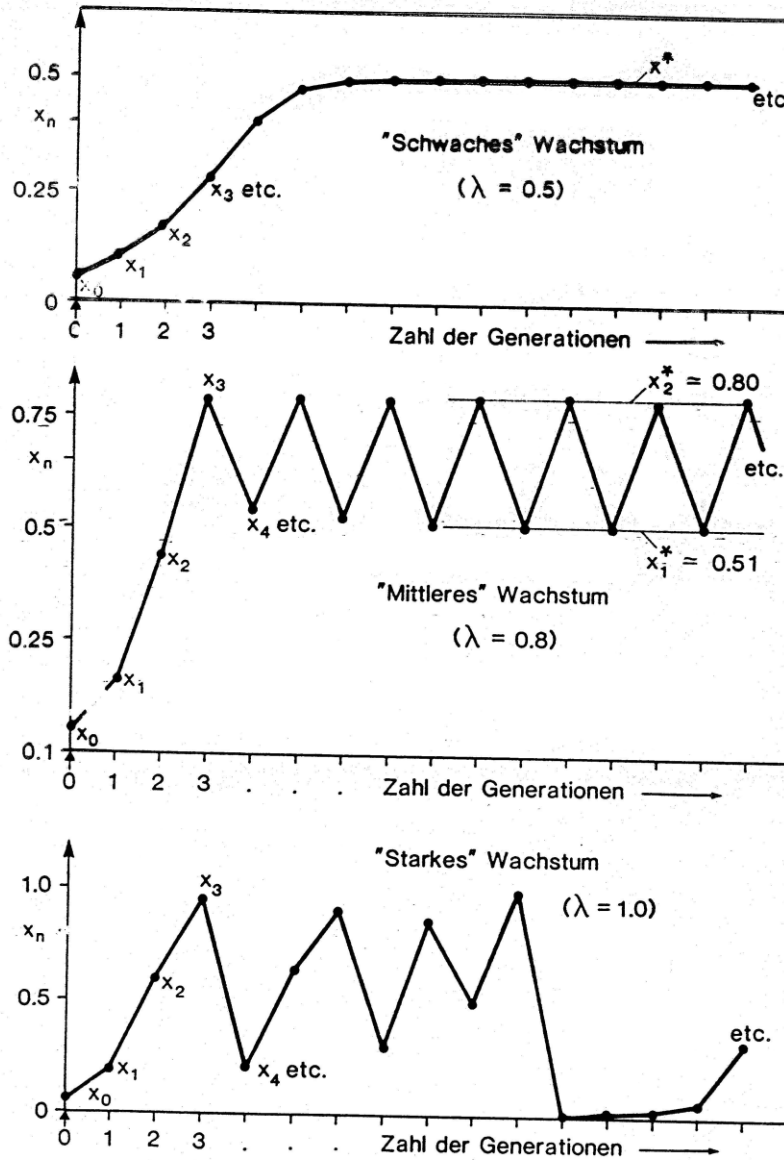


Abbildung 2.1: Das Iterationsschema der Verhulst-Dynamik bei verschiedenen Werten des Wachstumsparameters λ : Schwaches Wachstum ($\lambda = 0.5$) mit einem Attraktor, mittleres Wachstum ($\lambda = 0.8$) mit 2 Attraktoren und starkes Wachstum ($\lambda = 1.0$) mit chaotischem Verhalten der Populationen

$$x_{n+1} = f(x_n)$$

$$f(x_n) = \lambda x_n (1 - x_n)$$

Feigenbaum diagram: periodi kahendumine

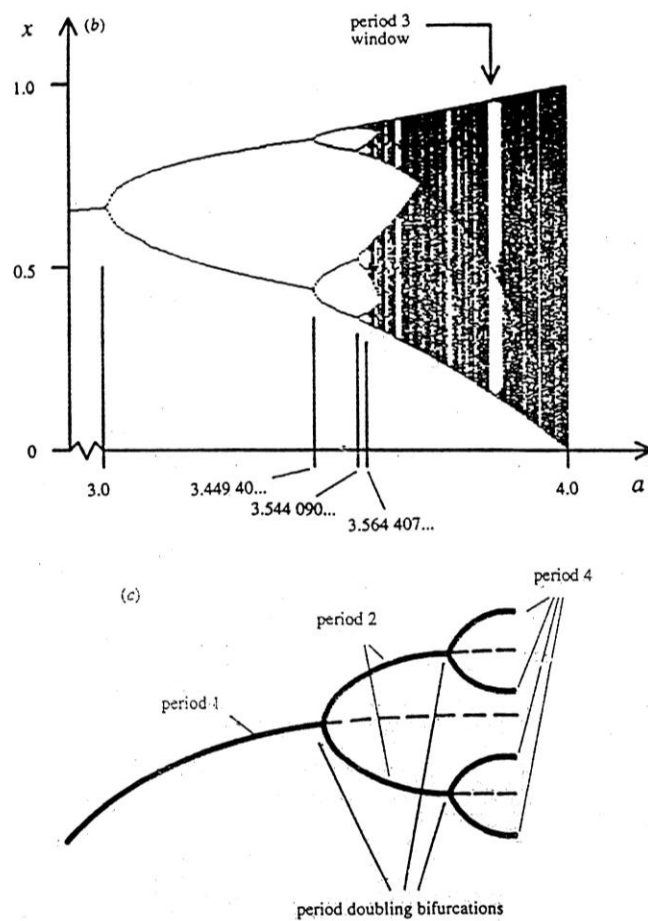
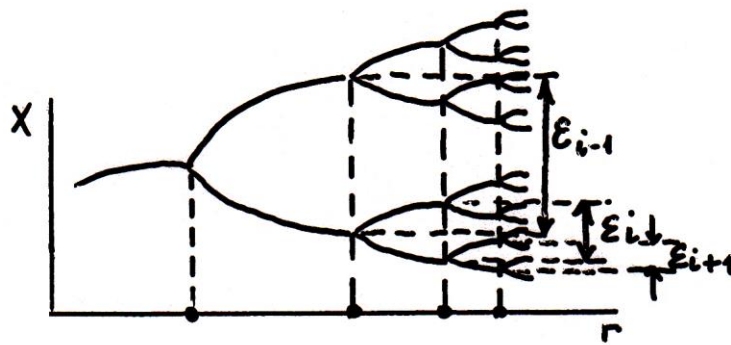
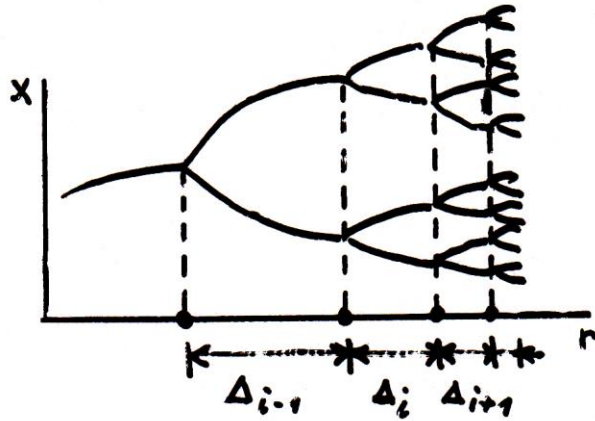


Figure 5.4. The bifurcation diagram for the logistic equation. (a) The bifurcation diagram for the logistic diagram: post-transient solution against control parameter. (b) Zoom into the logistic diagram over the range $3.0 < a < 4.0$. (c) General form of the period doubling bifurcations.



$$\delta_i = \frac{\Delta_i}{\Delta_{i+1}} = \frac{r_i - r_{i+1}}{r_{i+1} - r_{i+2}}$$

$$\alpha_i = \frac{\epsilon_i}{\epsilon_{i+1}}$$

$i \rightarrow \infty$

$$\delta_i \rightarrow \delta = 4.6692016091\dots$$

$$\alpha_i \rightarrow \alpha = 2.5029078750\dots$$

Geomeetiline progressioon

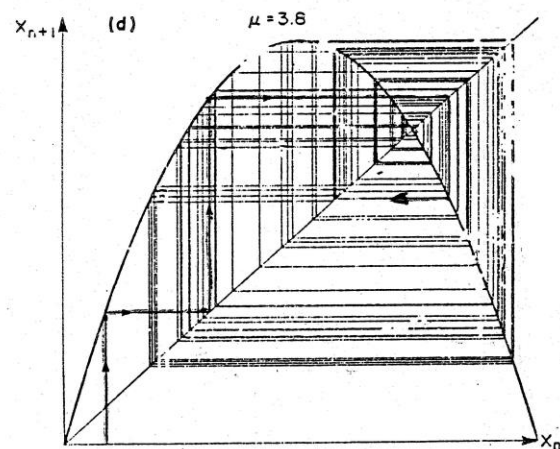
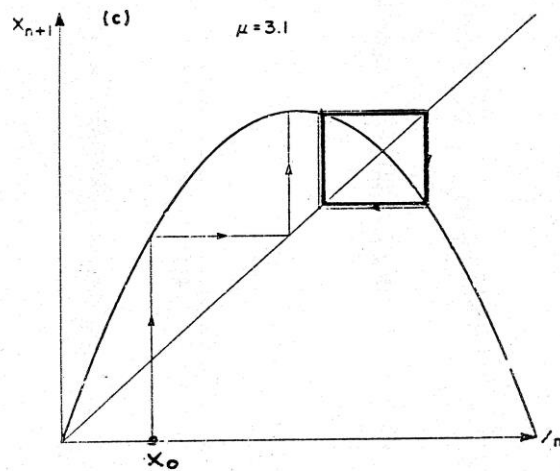
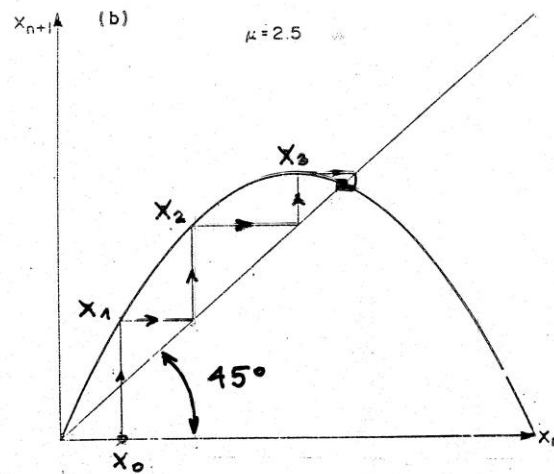
Feigenbaumi arvud

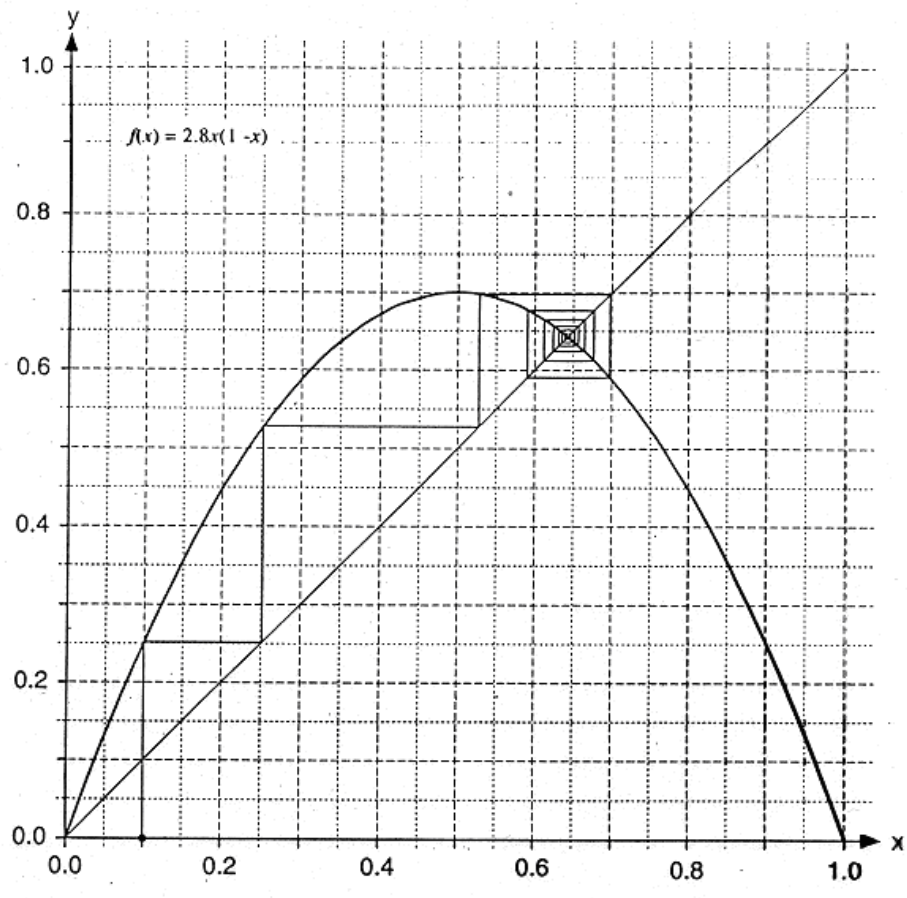
Universaalsus

$$x_{n+1} = \lambda_1 x_n (1 - x_n)$$

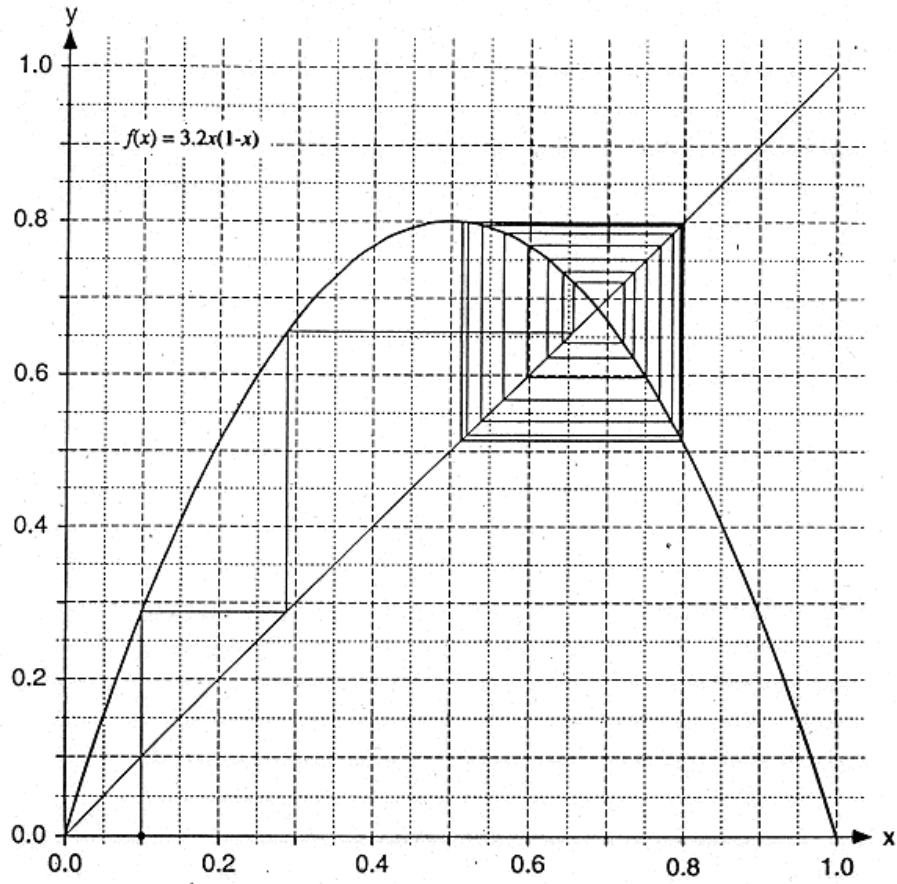
logistic map

logistiline kujutis

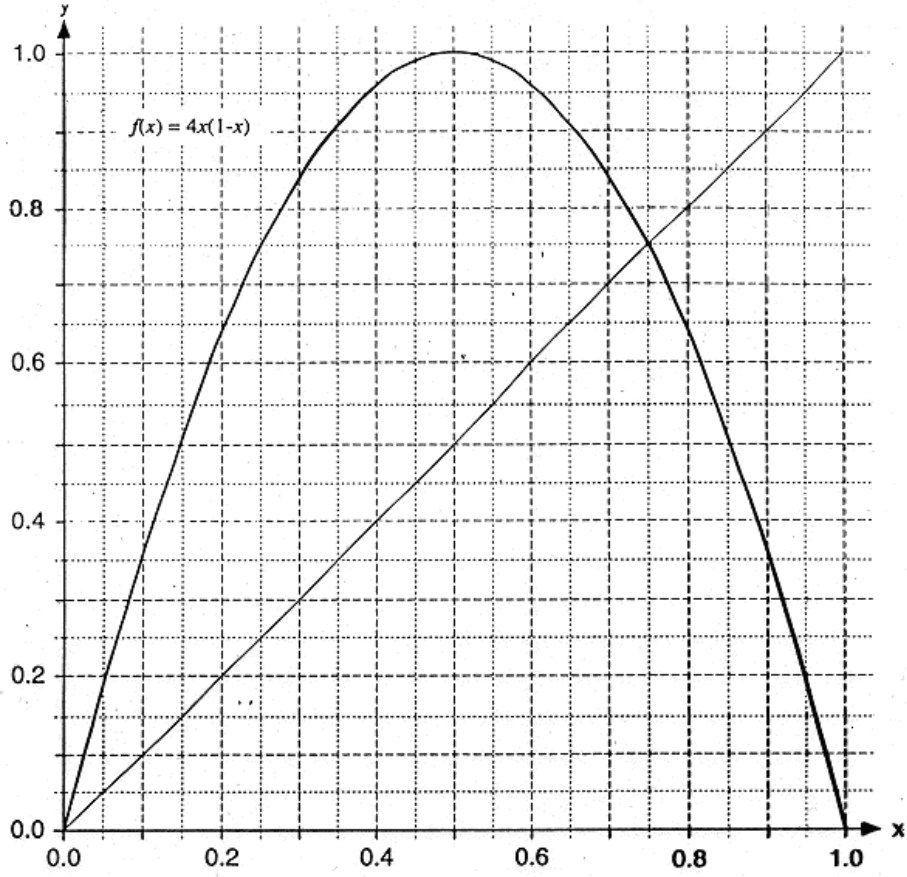




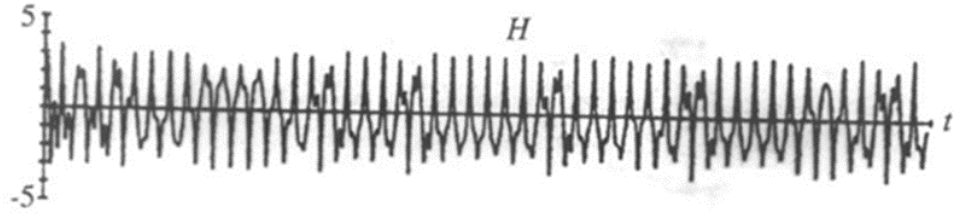
- $x_0 = 0.10$
- $x_1 = \underline{\hspace{2cm}}$
- $x_2 = \underline{\hspace{2cm}}$
- $x_3 = \underline{\hspace{2cm}}$
- $x_4 = \underline{\hspace{2cm}}$
- $x_5 = \underline{\hspace{2cm}}$
- $x_6 = \underline{\hspace{2cm}}$
- $x_7 = \underline{\hspace{2cm}}$



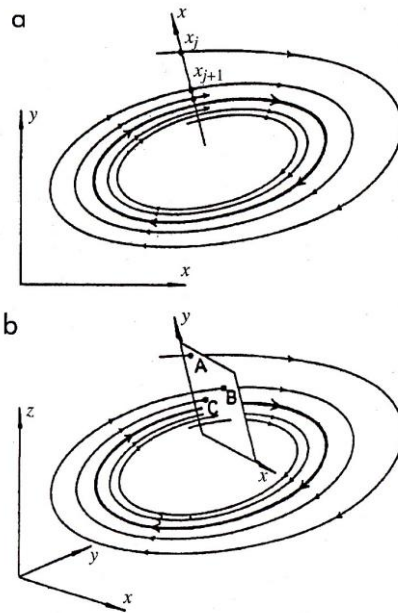
- | | |
|---------|-------|
| $x_0 =$ | 0.10 |
| $x_1 =$ | _____ |
| $x_2 =$ | _____ |
| $x_3 =$ | _____ |
| $x_4 =$ | _____ |
| $x_5 =$ | _____ |
| $x_6 =$ | _____ |
| $x_7 =$ | _____ |



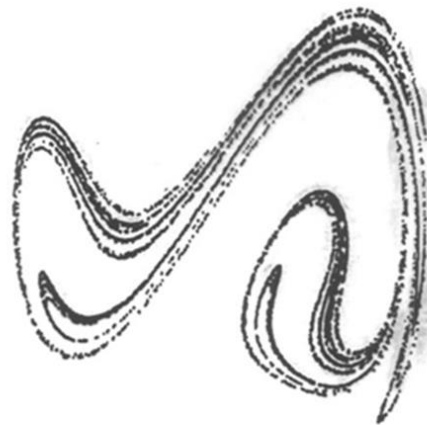
- $x_0 =$ _____
- $x_1 =$ _____
- $x_2 =$ _____
- $x_3 =$ _____
- $x_4 =$ _____
- $x_5 =$ _____
- $x_6 =$ _____
- $x_7 =$ _____



Aja-tee diagramm Ueda atraktorile. Thomson, Stewart, 1986.



a) ühedimensionaalne Poincaré lõige; b) kahedimensionaalne Poincaré lõige



Ueda atraktori Poincaré lõige. Moon, 1987.

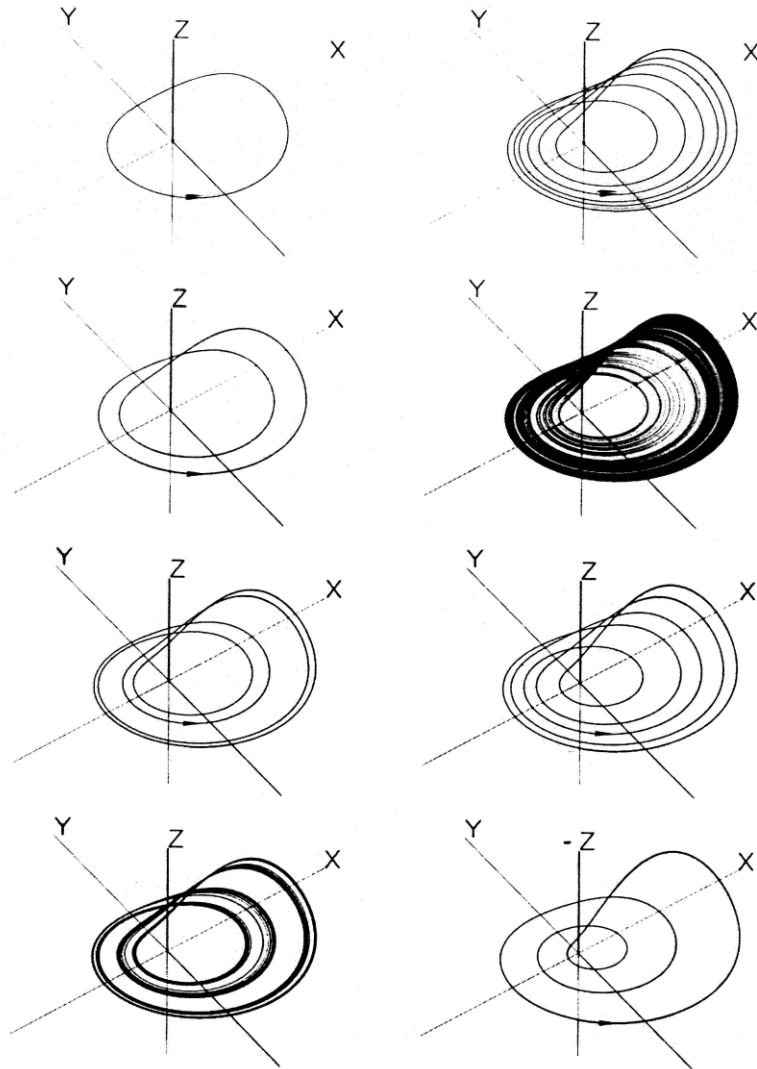


Figure 12.4 Final trajectories of equations (12.1) for different values of the parameter a . Left row, top to bottom: limit cycle, $a = 0.3$; period 2 limit cycle, $a = 0.35$; period 4, $a = 0.375$; four-band chaotic attractor, $a = 0.386$. Right row, top to bottom: period 6, $a = 0.3909$; single-band chaos, $a = 0.398$; period 5, $a = 0.4$; period 3, $a = 0.411$. In all cases $b = 2$, $c = 4$

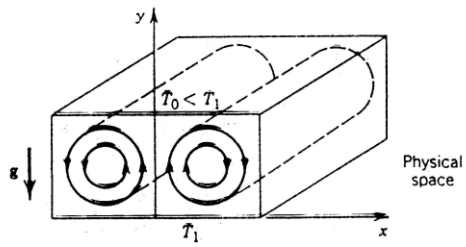
$$\dot{x} = -y - z$$

$$\dot{y} = x + ay$$

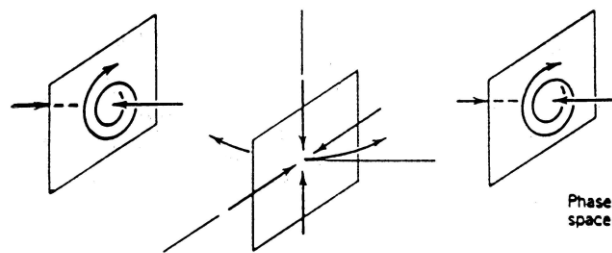
$$\dot{z} = b + z(x - c)$$

Rössleri atraktor:
periodi kahendumine

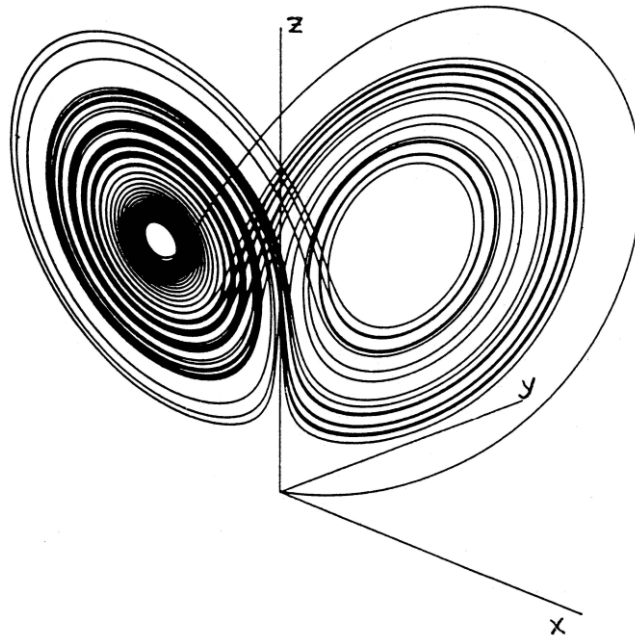
Lorenzi atraktor

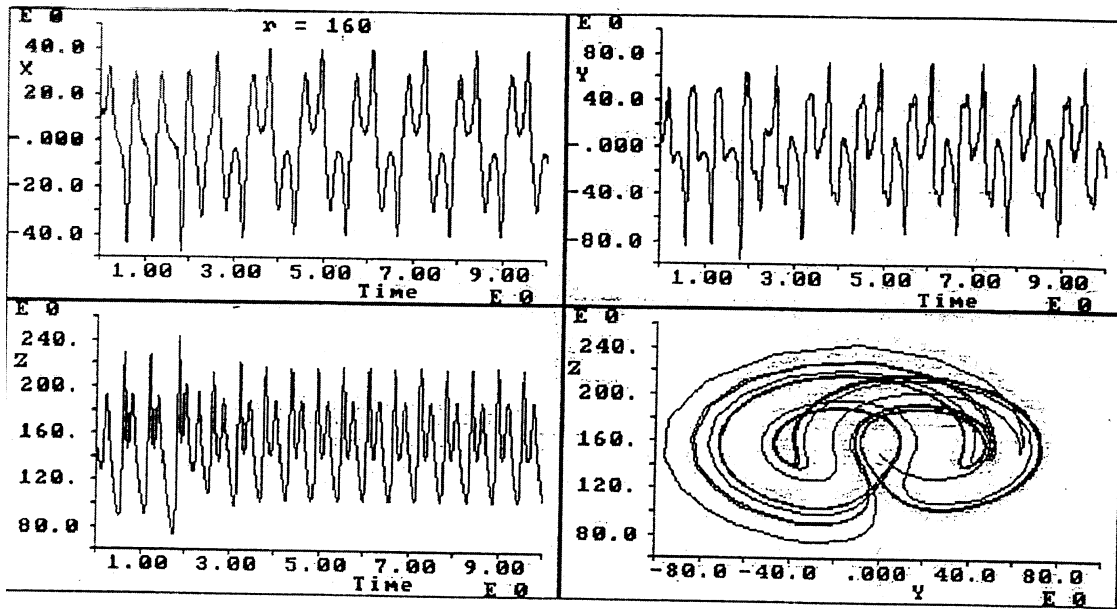


(a)



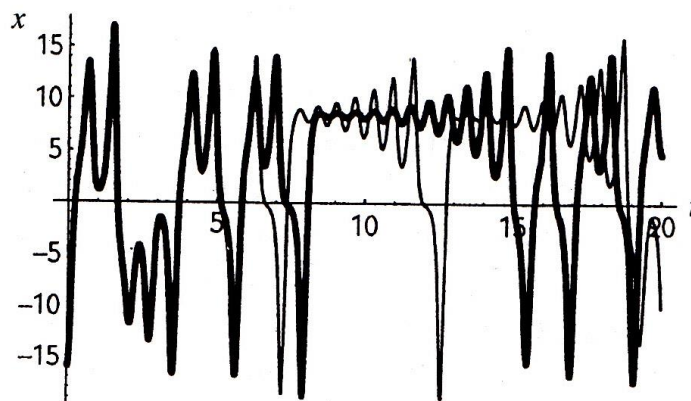
Navier - Stokes eqs. \Rightarrow ^(b)
$$\begin{cases} \dot{x} = \underline{\sigma}(y - x) \\ \dot{y} = \underline{\rho}x - y - \underline{x}z \\ \dot{z} = -\underline{\beta}z + \underline{xy} \end{cases}$$



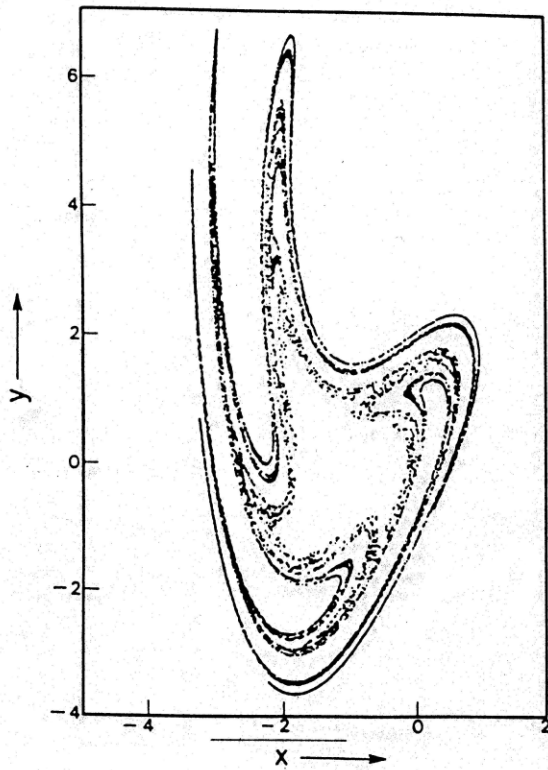
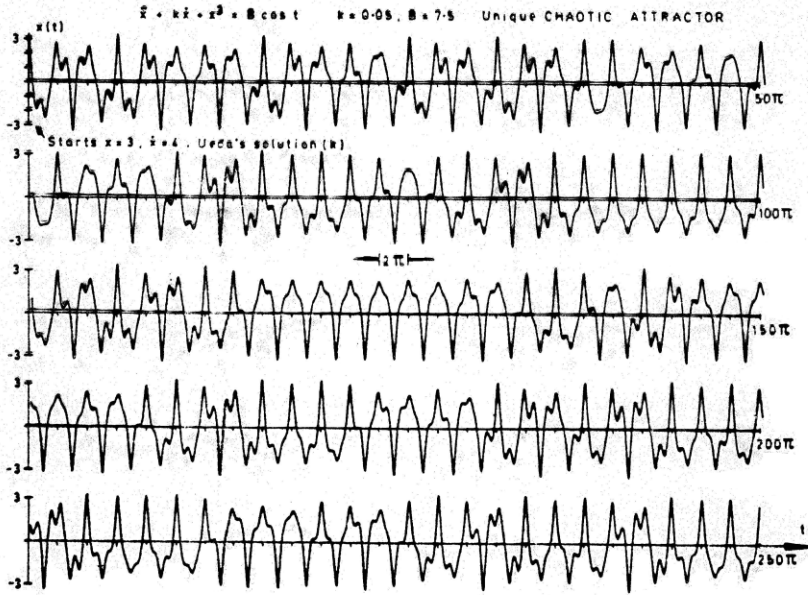
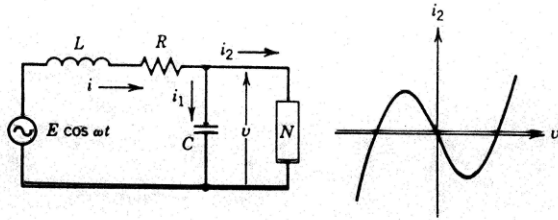


Solutions of the Lorenz equations with $r = 160$. After an initial transient that lasts until about $t = 3$, the solutions are periodic (but not sinusoidal).

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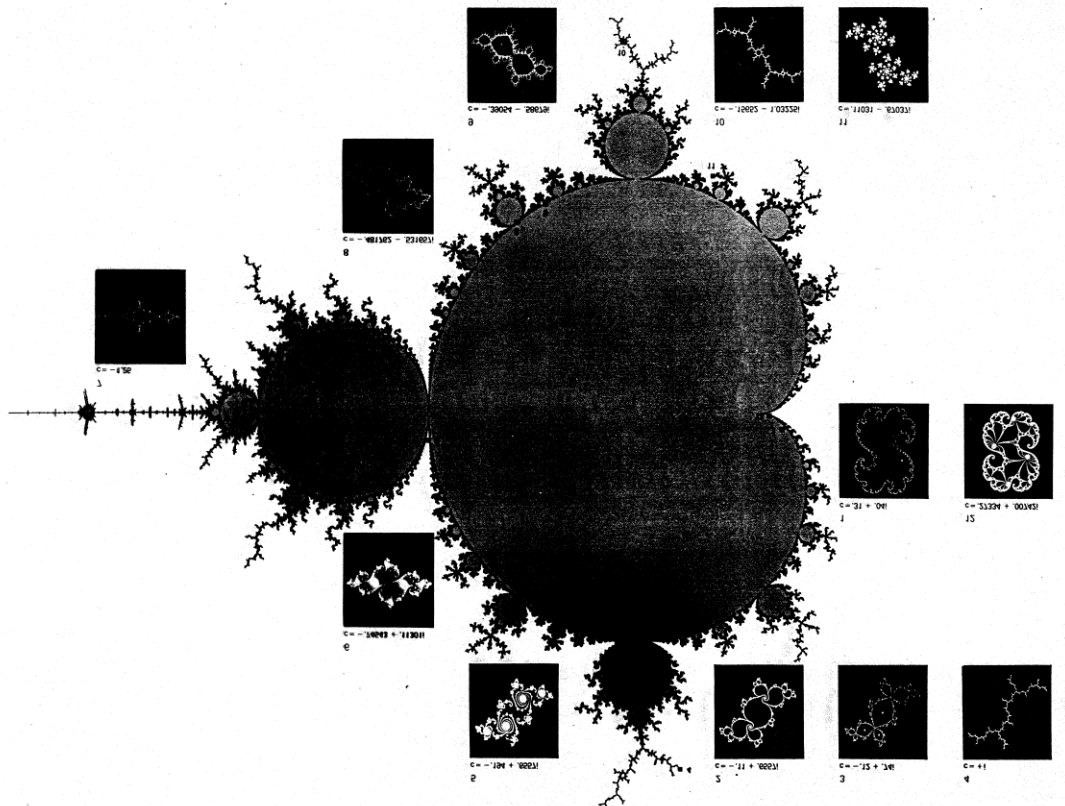


Joonis 6.6. Lorenzi atraktori x -komponent erinevatel algtingimustel. Kaplan, Glass, 1995.



Faasiruumil on fraktaalne iseloom

Mandelbroti fraktal



$$z_{n+1} = z_n^2 + C$$

C , mille puhul iteratsioon koondub, on Mandelbroti hulk
 fikseeritud C puhul moodustavad algväärtused, mille puhul
 iteratsioon koondub, Julia hulgad

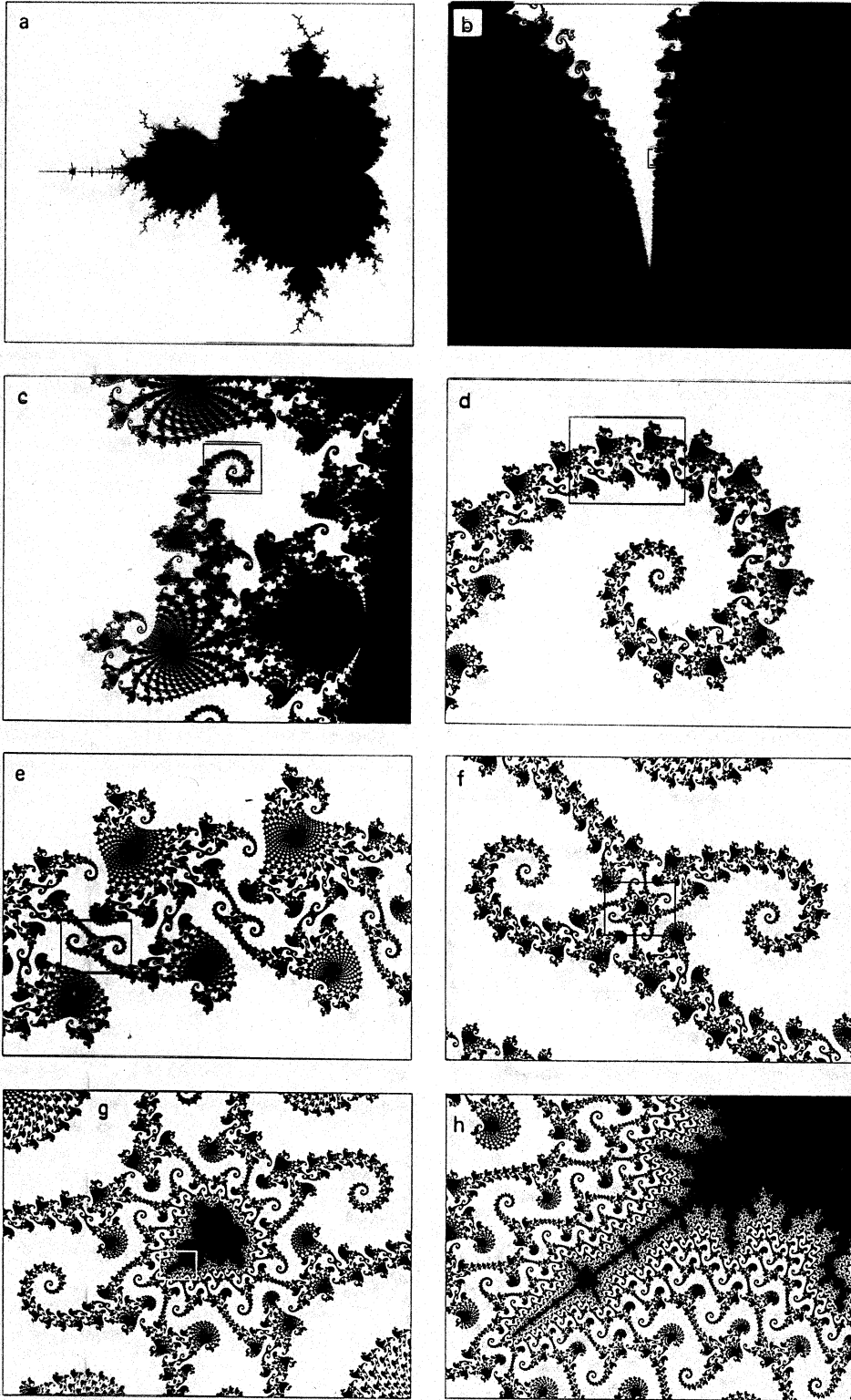


Fig. 4.21: 8 blow ups at the boundary of the Mandelbrot set obtained by DEM/M

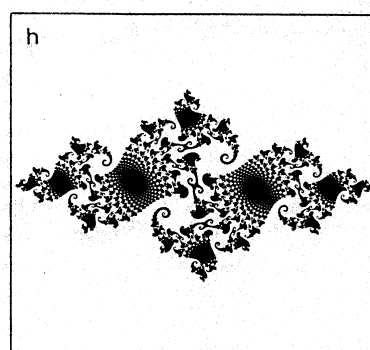
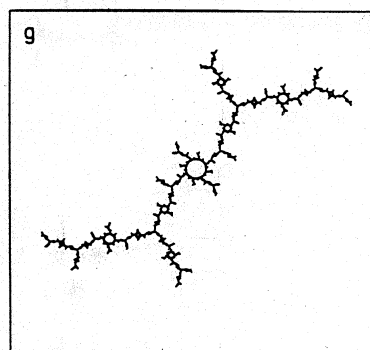
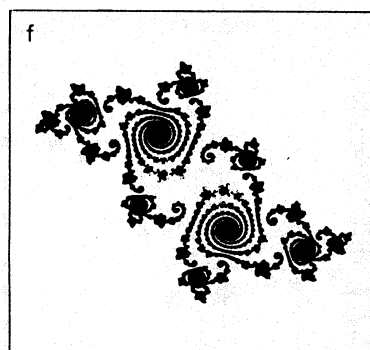
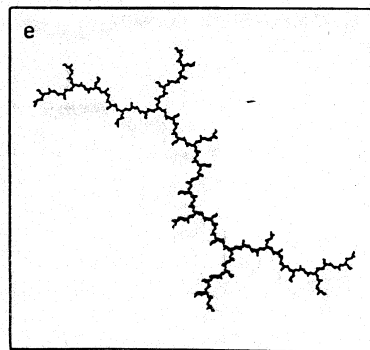
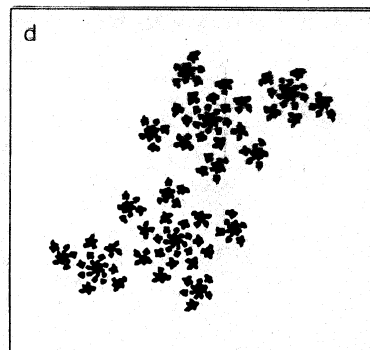
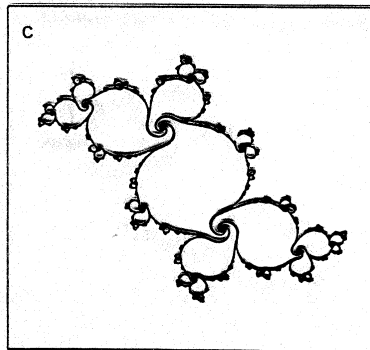
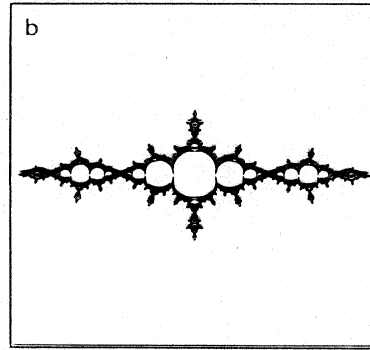
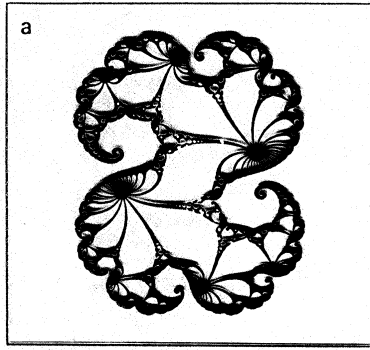


Fig. 4.3: 8 Julia sets

Julia hulga

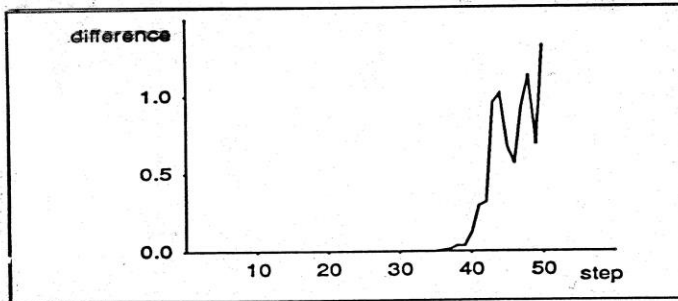
Arvutustäpsus mittelineaarses süsteemis

$$\text{Näide: } p_{n+1} = p_n + rp_n(1 - p_n)$$

evaluations	CASIO	HP
1	0.0397	0.0397
2	0.15407173	0.15407173
3	0.5450726260	0.545072626044
4	1.288978001	1.28897800119
5	0.1715191421	0.171519142100
10	0.7229143012	0.722914301711
15	1.270261775	1.27026173116
20	0.5965292447	0.596528770927
25	1.315587846	1.31558435183
30	0.3742092321	0.374847695060
35	0.9233215064	0.908845072341
40	0.0021143643	0.143971503996
45	1.219763115	1.23060086551
50	0.0036616295	0.225758993390

CASIO fx-7000G
Versus HP 28S for
 $p + rp(1 - p)$

Table 1.31 : Two different calculators at the same job do not produce the same results.



Differences in the Race

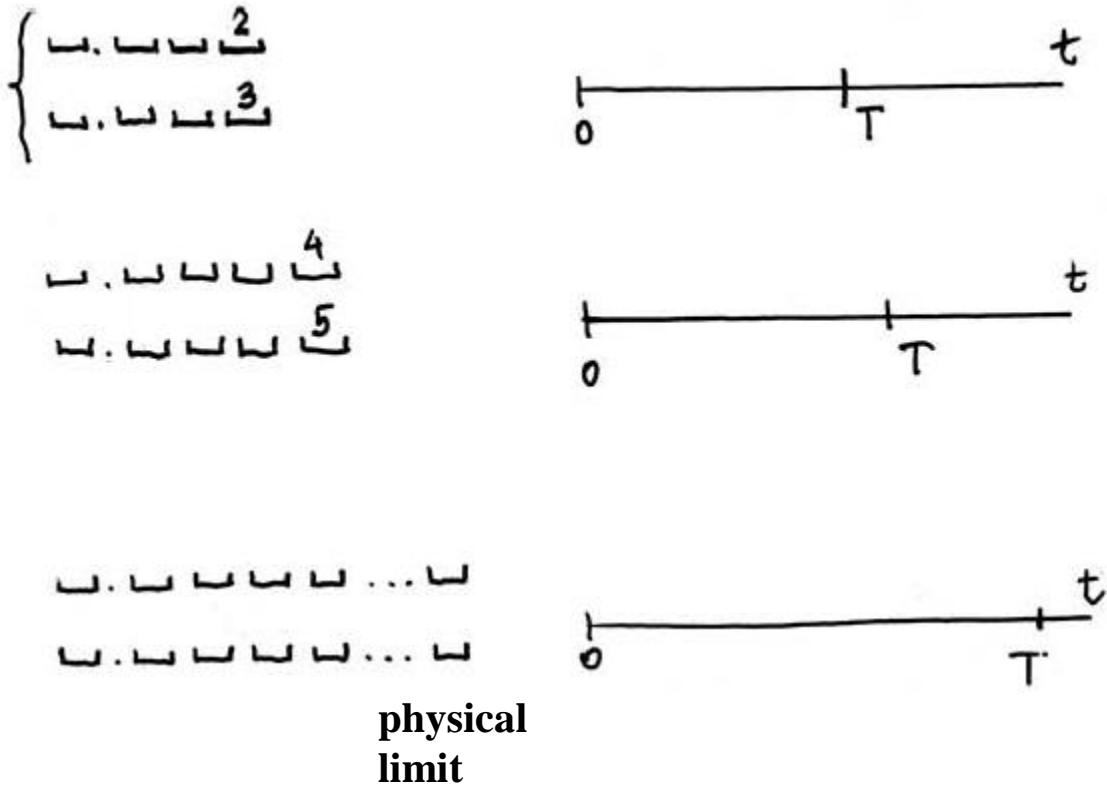
Figure 1.32 : Plot of the difference between the computed iteration values of HP and Casio.

evaluations	$p + rp(1 - p)$	$(1 + r)p - rp^2$
1	0.0397	0.0397
2	0.15407173	0.15407173
3	0.5450726260	0.5450726260
4	1.288978001	1.288978001
5	0.1715191421	0.1715191421
10	0.7229143012	0.7229143012
11	1.323841944	1.323841944
12	0.03769529734	0.03769529724
13	0.146518383	0.1465183826
14	0.5216706225	0.5216706212
15	1.270261775	1.270261774
20	0.5965292447	0.5965293261
25	1.315587846	1.315588447
30	0.3742092321	0.3741338572
35	0.9233215064	0.9257966719
40	0.0021143643	0.0144387553
45	1.219763115	0.0497855318

$p + rp(1 - p)$ Versus
 $(1 + r)p - rp^2$

Predictability horizon J. Lighthill, 1986

Time after which solutions with initial conditions that are “nearest neighbours” to the accuracy of specification being used,
become remote from one another differ beyond
recognition



Ennustatavuse horisont:

aeg millest alates naaberalgandmetest arvutatud protsessi
parameetrid erinevad teineteisest kaootiliselt

KAOSE IDENTIFITSEERIMINE

- 1) kas protsessis on mittelineaarsus
- 2) kas algtingimused ja protsessi parameetrid on determineeritud
- 3) milline on protsessi ajaline kulg
- 4) milline on faasiportree
- 5) milline on Poincaré lõige (kujutis)
- 6) milline on Fourier' spekter
- 7) mis juhtub süsteemi parameetrite muutumisel

Tunnused:

- 1) tundlikkus algtingimuste suhtes
- 2) laiaribaline Fourier' spekter kus ülekaalus on madalad sagedused
- 3) positiivne Ljapunovi eksponent (eksponendid)
- 4) atraktori fraktaalne struktuur faasiruumis
- 5) korrapärase liikumiste bifurkatsioonid
- 6) mitteregulaarsete intervallide esinemine (viitab juhumuutuvale kaosele)
- 7) Kolmogorovi entroopia nullist erinev, kuid mitte lõpmatu (vrdl. Ljapunovi eksponent)

KAOSE SÜMBOLID

Feigenbaumi diagramm

Mandelbroti fraktal

Lorenzi liblikas

Barnsley sõnajalg

ja veel:

Sierpinski vaip (kolmnurk)

Julia hulgad

Cantori hulk

Kochi lumehelbeke

Smale'i hobuseraud

.....

ETTEVAATUST

Determineeritud füüsikaliste süsteemide käitumine pole üheselt ülekantav sotsiaalsete (nt. ühiskonna või indiviidi käitumine) protsesside kirjeldamiseks.

**Summa on suurem kui tema üksikosad kokku liidetuna
Aristoteles**

Kokkuvõte

MUDEL

determineeritud

lineaarne

determineeritud

mittelineaarne

Täpsemalt:

Probleem:

PROTSESS

determineeritud

ennustatav

lahend võib puududa

võib olla

kaootiline

mitteennustatav

mitmene lahend

kursus

“Mittelineaarne dünaamika ja kaos”

kuidas muuta mudeli parameetreid nii,
et lahend poleks kaootiline
(controlling chaos)