

Vibration of the String with Nonlinear Contact Condition

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Abstract. We investigate the vibrations of the ideal flexible string, which one end is rigidly clamped, or coupled with a linear damped oscillator, and another one is terminated on the curved contact surface. The vibrating string touches repeatedly this termination, and this, in turn, causes the modulation of fundamental frequency of the string, and the train of high frequency oscillations is generated. The problem is studied both analytically, and numerically. The effect of the contact nonlinearity and of the shape of the contact surface on of the spectral structure of the string vibration is considered. The influence of the impact amplitude on the vibration spectra of struck string is discussed.

Keywords: String vibration, Time-frequency domain, Nonlinear contact condition

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INTRODUCTION

Investigation of the boundary condition of vibrating string is a very important problem in musical acoustics. It is well known that the fundamental frequency of piano string is strictly determined by the type of the string termination. The types of the string support in the piano are different for the bass and treble notes. All the far ends of the piano strings are terminated on the bass and treble bridges, which are the rather complicated resonant systems. The nearest ends of the bass and long treble strings begin from the agraffe that can be considered as an absolutely rigid clamp termination. But the most part of the treble strings starts from the edge of the cast iron frame. These strings turn the rigid edge, and its vibration tone depends on the curvature of this termination. The similar type of the string support we can see on the guitar and some other musical string instruments.

Usually, the changing of tone caused by the curvature of the string support is negligible, but there is a family of Japanese plucked stringed instruments (biwa and shamisen), which sounding strictly determined by the string termination [1, 2]. These lutes are equipped with a mechanism called "sawari" (touch). The sawari is a contact surface of very limited size, located at the nut-side end of the string, to which the string touches repeatedly, producing a unique timbre of the instrumental tone called the sawari tone.

This paper studies the influence of the geometrical nonlinearity of the string termination on the spectrum of its vibrations.

SAWARI MODEL

The nonlinear model of sawari mechanism is considered in [3, 4] and the scheme of this model is shown in Figure 1.

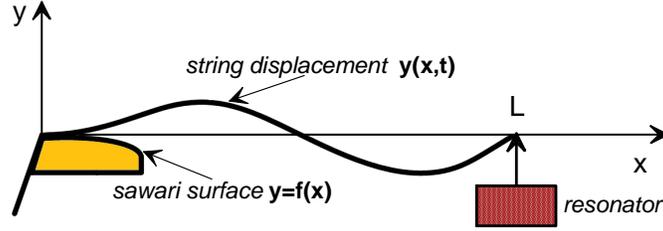


FIGURE 1. Scheme of sawari model

It is assumed that the displacement $y(x,t)$ of the ideal (flexible) string of length L obeys the second-order wave equation. The right-hand end of the string is supported by the bridge, which is considered as a resonator. The left-hand end ($x = 0$) terminates at sawari surface, which is assumed to be rigid enough, and that is defined by $y = f(x)$. When the string pushes this surface from above, the sawari surface deforms, and, as a result, the repelling net force $f = -Ky^*(x,t)$ acts on the string. Here K is the positive constant, which is large enough, and that depends on the material of the sawari (made of tarred bamboo), and the cross section of the silk string.

The nonlinear condition of sawari-string interaction is determined by

$$y^*(x,t) = \begin{cases} y(x,t) - f(x), & \text{if } y(x,t) < f(x) \\ 0, & \text{if } y(x,t) \geq f(x) \end{cases} \quad (1)$$

Hence, the governing equation for the string is given by

$$\mu \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2} - Ky^* \quad (2)$$

where T is the tension and μ is the linear mass density of the string.

The effect of sawari mechanism was studied in [1–4] experimentally and numerically. It was shown that the sawari excites a local disturbance of the string motion, which gets rich spectral components up to very large numbers of the fundamental frequency of the corresponding monochord (without sawari).

It is evident, that the similar mechanism of the contact nonlinearity can also generate the high frequency oscillations of the piano strings. In the following section will be presented another approach to the problem of vibration of the piano string with a nonlinear support.

PIANO STRING

The scheme of position of the treble piano string is shown in Figure 2. The left-hand end of the string wire is fastened to the cast iron frame. Then the string bends around the rigid edge of the frame, runs over the piano bridge, and terminates again on the frame.

The string is assumed to be ideal (flexible). The piano hammer strikes the string at the contact point x_0 , and this generates two simple nondispersive traveling waves $y(t+x/c)$ and $y(t-x/c)$ moving in both directions. At the first moment the amplitude of these waves is always positive.

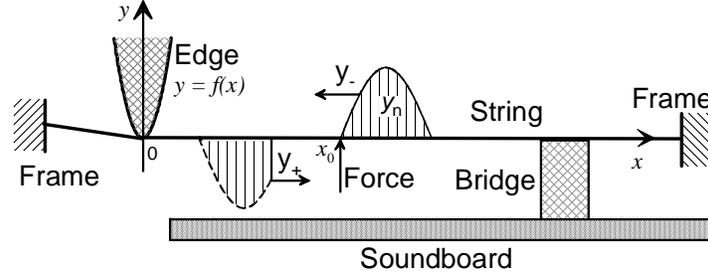


FIGURE 2. Scheme of piano string model

Let's consider the wave $y_- = y(t+x/c)$ moving to the edge termination given by $f(x)$. This wave reflects back from the edge, so that at any moment t on the rigid surface we have $y_- + y_+ = 0$. It means, that the reflected wave $y_+ = -y_-$, but the phases of these waves are different. The phase shift and the form of the reflected wave depend on the amplitude of the incident wave.

In musical acoustics all functions of time $y(t)$ are limited to the band from 0 to $\omega_{max} = 3 \cdot 10^5 \text{ s}^{-1}$, approximately. Therefore, such function is completely determined by giving its ordinates at a series of discrete points [5]

$$y_-(t) = \sum_{n=-\infty}^{\infty} y(t_n) \frac{\sin \omega_{max}(t - t_n)}{\omega_{max}(t - t_n)} \quad (3)$$

where $t_n = n\pi/\omega_{max}$. Because each ordinate $y(t_n)$ reflects back at the moment $t = t_n^*$ when $y_n = y(t_n) = f(x)$, the reflected wave can be represented in the form

$$y_+(t) = - \sum_{n=-\infty}^{\infty} y(t_n) \frac{\sin \omega_{max}(t - t_n + t_n^*)}{\omega_{max}(t - t_n + t_n^*)} \quad (4)$$

where $t_n^* = \frac{1}{c} f^{-1}(y_n)$. Here $f^{-1}(y_n)$ denotes the inverse function of $f(x)$.

The spectrum of reflected wave $\mathcal{Y}_+(\omega)$ is related through the spectrum of incident wave $\mathcal{Y}_-(\omega)$ by equation

$$\mathcal{Y}_+(\omega) = \mathcal{Y}_-(\omega) \mathcal{D}(\omega) \quad (5)$$

where $\mathcal{D}(\omega)$ is defined by

$$\mathcal{D}(\omega) = \sum_{n=-\infty}^{\infty} \exp\left[-\frac{i\omega}{c} f^{-1}(y_n)\right] \quad (6)$$

Equations (5, 6) show that even if the left-hand end of the piano string has a rigid termination, which only changes the sign of reflected waves, the spectral structure of traveling waves had changed significantly. The new trains of high frequency oscillations that do not exist initially grow up eventually, and its distribution depends on the amplitude of the initial wave excited by the piano hammer and also on the frame edge curvature.

RESULTS FOR THE PIANO STRING

The simplest use of the presented model is for plotting displacement–time curves and power spectra for particular wave parameters. It is of most interest to plot families of curves showing how the reflected wave spectrum changes as the amplitude of incident wave is varied. Figure 3 shows the spectra of the incident y_- and reflected y_+ waves.

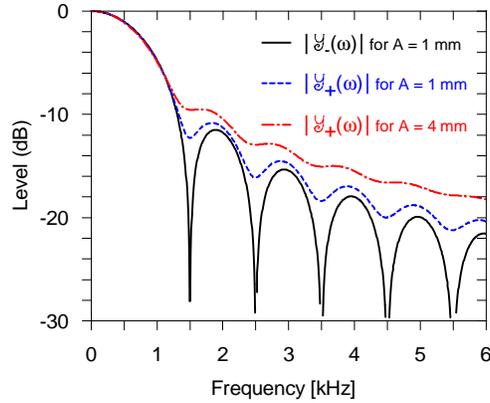


FIGURE 3. Normalized power spectra for various amplitude A

The incident wave pulse has been chosen in the form $y_- = A \sin \pi t / t_0$ with $t_0 = 1$ ms. The frequency band is limited to $\omega_{max} = 3 \cdot 10^5 \text{ s}^{-1}$. The curvature radius of the edge is equal to $R = 5$ mm, thus the edge form function is determined by $y = ax^2$, where $a = 0.1 \text{ mm}^{-1}$. It is evident that the power spectrum of reflected wave grows up significantly and essentially reshapes with increasing of the incident wave amplitude.

CONCLUSIONS

It has been shown that presented model of piano string with nonlinear support provides a good practical way to make predictions about the vibration spectra of struck string. One respect in which this model is still idealized is its assumption of very simple string boundary condition at the piano bridge. One could perhaps use a more sophisticated representation of finite bridge impedance to generalize the simple delay-line treatment of the traveling waves used here.

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