Volterra Equations and Applications

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42 SOLUTION OF NONLINEAR INTEGRODIFFERENTIAL EQUATIONS WITH BOUNDARY CONDITIONS USING AN ALGEBRAIC METHOD

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Abstract The constant-profile solutions of an acoustic pulse that propagates in a nonlinear hereditary medium without a form distortion are constructed by means of a direct algebraic method. To satisfy the boundary conditions, for a special type of infinite series describing solution derived the new formulae for inversion of such series are obtained. By using algebraic method, it is shown that the equations considered allow the existence of two types of the solitary waves which velocities depend on the value of its amplitude jump discontinuity across the wave front. These the constant-profile waves propagate with the velocities both less or greater than the velocity of sound in the linear medium.

1 Introduction

The existence of the acoustic pulses moving in nonlinear media without its form distortion is well-known. In nonlinear media with memory and in viscoelastic media the constant-profile wave can propagate also. The wave with exponential profile moving with the same velocity as a speed of sound is described in [1]. The supersonic waves in nonlinear hereditary media are considered in [2]. However, the exact solution obtained in [2] has a significant failure. This solution describes the wave in the implicit form, which makes difficult to construct the wave profile, and what is more, for the some feasible velocities the form of the wave is a complex-valued. This fact can not be explained in the frames of the method used.

The more convenient method to find the solutions of the nonlinear integrodifferential equations is presented in [3,4]. These articles show how to obtain the exact solitary wave solution of the nonlinear integrodifferential evolution and wave equation with a direct algebraic method. This technique is based on the physical concept which says that the constant profile wave may be considered to be a spectrum of harmonic waves traveling with the same velocity. Such waves are the solutions of the underlying linear equation normally traveling with different velocities due to dispersion. The nonlinearity couples these waves by mixing them and locks their velocities and phases so that a solitary wave is formed. By using this direct algebraic method many solutions of the well-known famous nonlinear equations where obtained in [4]. Unfortunately, the initial conditions for such equations where not taken into consideration, and so there is some arbitrary constant in the solutions obtained.

The present paper develops the direct algebraic method described in [3,4] for deriving solitary wave solutions of nonlinear integrodifferential equations with boundary conditions. In addition this method allows to find the range of velocity values of the solitary profiles.

2 Statement of the Problem

The waves in a nonlinear hereditary media may be described by means of the evolution equation

$$U_t(x,t)+cf(U_x)U_x(x,t)+\frac{1}{2}\int_0^t R(t-y)U_y(x,y)dy=0, \tag{1}$$

or by means of the wave equation

$$U_{tt}(x,t)-c^2f(U_x)U_{xx}(x,t)+\int_0^t R(t-y)U_{yy}(x,y)dy=0, \tag{2}$$

with the initial $U(x,0)=0$, and boundary condition $U(0,t)=H(t)\psi(t)$, where $U(x,t)$ is the longitudinal displacement; c is the velocity of sound in linear medium; $R(t)$ is the relaxation kernel function of the hereditary medium; $f(U_x)$ is the function of nonlinearity; $H(t)$ is Heaviside step-function, and $\psi(t)$ denotes an arbitrary continuous function, which has the bounded first-order derivative and satisfies the condition $\psi(0)=0$.

Let us choose the most simple form of the function of nonlinearity

$$f(U_x) = 1 + \alpha U_x(x,t), \tag{3}$$

so for the small deformation $(U_x(x,t) \ll 1)$ we have a linear problem.

Let us seek the stationary solutions for (1), (2) in the form

$$U(x,t) = H(\xi)\psi(\xi), \qquad \xi = \frac{1}{2\tau_0}(t - \frac{x}{v}), \qquad \psi(0) = 0, \tag{4}$$

where ξ is the traveling frame, and v is the anticipated traveling wave velocity. Here τ_0 is the time-dimension parameter, which value would be introduced further.

In according to conservation laws [5], and using the kinematic identity $[U_x] = -v[U_t]$, the value of amplitude jump discontinuity across the wave front $\xi = 0$ is equal to

$$[U_t] = \frac{2v}{\alpha}\left(1 - \frac{v^2}{c^2}\right). \tag{5}$$

Substituting (4) into (1), (2), and using (5), for $\xi > 0$ we have

$$\left[1 - \frac{c}{v} + \frac{\alpha c}{2\tau_0 v^2}\psi'(\xi)\right]\psi'(\xi) + \tau_0\int_0^{\xi} R[2\tau_0(\xi - z)]\psi'(z)dz = 0, \tag{6}$$

$$\left[1 - \frac{c^2}{v^2} + \frac{\alpha c^2}{2\tau_0 v^3}\psi'(\xi)\right]\psi''(\xi) - \frac{8\tau_0{}^2 v^3}{\alpha c^2}\left(1 - \frac{c^2}{v^2}\right)R(2\tau_0\xi) + \tag{7}$$

$$2\tau_0\int_0^{\xi} R[2\tau_0(\xi - z)]\psi''(z)dz = 0.$$

Here $\psi'(\xi) = d\psi(\xi)/d\xi$.

The relaxation kernel function may be chosen in the form

$$R(t) = \frac{\varepsilon}{\tau_0}\exp\left(-\frac{t}{\tau_0}\right), \tag{8}$$

where ε and τ_0 are the hereditary parameters of the medium. In this case (1) and (6) describe the wave propagation in the hereditary medium with E-memory, and (2) and (7) in the standard viscoelastic medium.

By introducing new variables

$$y(\xi) = \frac{\alpha c}{2\tau_0 \varepsilon v^2}\psi'(\xi), \qquad 1 - \frac{c}{v} = a\varepsilon, \tag{9}$$

and

$$z(\xi) = \frac{\alpha c^2}{4\tau_0 \varepsilon v^3}\psi'(\xi), \qquad 1 - \frac{c^2}{v^2} = b\varepsilon, \tag{10}$$

instead of (1) and (2) we have

$$[y(\xi)+a]y(\xi)+\int_0^{\xi} e^{2(\omega-\xi)}y(\omega)d\omega=0, \qquad y(0)=-a, \tag{11}$$

$$[2z(\xi)+b]z^{'}(\xi)-2be^{-2\xi}+2\int_0^{\xi} e^{2(\omega-\xi)}z^{'}(\omega)d\omega=0, \qquad z(0)=-b. \tag{12}$$

After elimination of integrals, we may next integrate (11) once, and (12) twice with respect to ξ, to obtain

$$y(\xi)[2y(\xi)+2a+1]^{\frac{a+1}{a}}=-ae^{-\frac{2a-1}{a}\xi}, \tag{13}$$

$$z(\xi)[z(\xi)+b+1]^{\frac{b+2}{b}}=-be^{-2\xi(\frac{b+1}{b})}. \tag{14}$$

However, the exact solutions (13), (14) have two essential limitations. Firstly, the implicit form of solutions cause difficulties in tracing the wave profiles. Secondly, this solutions are real only for certain wave velocities, and this fact has no explanation.

3 Algebraic Solution

According to the method presented in [3] and [4], let us to construct solutions of problems (11), (12) in the form of series $\sim \sum_{n=0}^{\infty} h_n \exp(-np\xi)$, containing the real exponential solutions of the linear equations. Here p is the constant positive parameter. Instead of (12), we use here

$$[2z(\xi)+b]z^{'}(\xi)+2z(\xi)-4\int_0^{\xi} e^{2(\omega-\xi)}z(\omega)d\omega=0, \quad z(0)=-b, \tag{15}$$

obtained by integration by parts.

To obtain a solution of (15), we substitute the series expansion

$$z(\xi)=-b+A_0\left[1+\sum_{n=1}^{\infty} h_n \exp(-np\xi)\right], \tag{16}$$

into it, and apply Cauchy's rule for the product appearing in the quadratic nonlinearity. By simple algebraic methods it is easy to show that (16) satisfies (15), only if

$$b=A_0\left(1-2\sum_{n=1}^{\infty}\frac{h_n}{np-2}\right), \tag{17}$$

$$h_n=\frac{p(p-1)}{(2-p)A_0}d_nq^n, \qquad d_n=\frac{\Gamma(np-1)}{\Gamma(n)\Gamma(np-n-1)},$$

and coefficient A_0 is the root of quadratic equation $(p-2)A_0^2+(p-4)A_0-2=0$. Here $\Gamma(x)$ is gamma-function, and $q=q(p)$ is the root of an equation $F(q)=0$, where

$$F(q)=1-\frac{p(p-1)}{A_0(p-2)}\sum_{n=1}^{\infty} d_n q^n. \tag{18}$$

Using asymptotic of gamma-function, it is easy to show that series in (18) uniformly converges for $p>1$ if $|q|<q_0$, where $q_0=(p-1)^{p-1}p^{-p}$.

If $q=q_0$, series in (18) converges, because of under $n\to\infty$

$$d_n q^n \sim \left[2\pi p^3(p-1)\right]^{-1/2} n^{-3/2}.$$

Due to $d_1=(p-1)^{-1}$, series in (18) converges also for $p=1$. Thus series in (18) converges for any $p\geq 1$, if $|q|\leq q_0$.

4 Series Characteristics

The special kind of coefficients d_n gives us the possibility to invert the series in (18). To find the dependence $q=q(p)$, consider the series

$$\sum_{n=1}^{\infty} d_n q^n = S_1. \tag{19}$$

Using the formula of the series inversion [6], we obtain

$$q(p)=s\left[1-(p-1)s+\frac{1}{2}(p-1)(p-2)s^2-\frac{1}{6}(p-1)(p-2)(p-3)s^3+\right.$$

$$\left.\frac{1}{24}(p-1)(p-2)(p-3(p-4)s^4-\ldots\right]=s\left[\sum_{n=0}^{\infty}\frac{(-s)^n}{n!}\frac{\Gamma(p)}{\Gamma(p-n)}\right]=s(1-s)^{p-1},$$

where $s=(p-1)S_1$. By using this formula, we have

$$q(p)=(p-1)S_1[1-(p-1)S_1]^{p-1}. \tag{20}$$

Now, using (19) we may find the sums of the series

$$\sum_{n=1}^{\infty}\frac{nd_n q^n}{np-2}=S_2, \qquad \sum_{n=1}^{\infty}\frac{d_n q^n}{np-2}=S_3. \tag{21}$$

Dividing the first by (19), we may write

$$\frac{S_2}{S_1}=\frac{\sum_{n=1}^{\infty}\frac{nd_nq^n}{np-2}}{\sum_{n=1}^{\infty}d_nq^n}=\sum_{n=0}^{\infty}c_nq^n.$$

The coefficients c_n are defined by relation

$$\frac{nd_n}{np-2} = \sum_{m=0}^{n} d_m c_{n-m} = d_n c_0 + \sum_{m=1}^{n-1} d_m c_{n-m}\,,$$

from which:

$$c_0 = \frac{1}{p-2}\,, \qquad c_n = -\left(\frac{p-1}{p-2}\right) d_n, \quad n \geq 1\,.$$

Thus, we have

$$\frac{S_2}{S_1} = \frac{1}{p-2} - \frac{p-1}{p-2} S_1,$$

and by using the obvious relation $S_1 = pS_2 - 2S_3$, we find

$$S_2 = \frac{S_1}{p-2}\left[1-(p-1)S_1\right], \qquad S_3 = \frac{S_1}{2(p-2)}\left[2-p(p-1)S_1\right]. \tag{22}$$

Using (20), from (18) for $A_0 = A_{01} = 2/(p-2)$, we have

$$q = q_1(p) = \frac{2}{p-2}\left(\frac{p-2}{p}\right)^p, \tag{23}$$

and for $A_0 = A_{02} = -1$, we have

$$q = q_2(p) = \frac{2-p}{2(p-1)}\left[\frac{2(p-1)}{p}\right]^p. \tag{24}$$

Taking into account the region of convergence $|q| \leq q_0$ of the series (18), we may find that relation (23) is valid for any $p > 2$, and relation (24) is valid for $1 \leq p < p_0, (p \neq 2)$. Here p_0 is the root of an equation $p_0 = 2(1+2^{-p_0})$, and $p_0 \approx 2.3833$.

5 Solutions Expansion

Finding the dependence $q(p)$, and using (22), from (17) we may obtain function $b = b(p)$. For corresponding values A_0 and $q(p)$, we have

$$b = b_1(p) = \frac{2}{p-2}\,, \qquad b = b_2(p) = \frac{2(1-p)}{p-2}\,. \tag{25}$$

All the relations obtained allow us not only to construct (16) but, taking into account (10), to determine the dependence of the wave velocity on the value of the amplitude jump discontinuity across the wave front.

Thus, in the form of the exponential series two the solutions of (15) are constructed

$$z(\xi) = z_1(\xi) = \frac{p(1-p)}{p-2} \sum_{n=1}^{\infty} d_n [q_1(p)]^n \exp(-np\xi), \quad p > 2\,, \tag{26}$$

$$z(\xi) = z_2(\xi) = \frac{p}{p-2}\left\{1-(p-1)\sum_{n=1}^{\infty} d_n[q_2(p)]^n \exp(-np\xi)\right\}, \quad 1 \le p < p_0\,, \tag{27}$$

that however, have the different values of the amplitude jump discontinuity across the front

$$z_1(0) = z_{10} = -2/(p-2), \quad z_2(0) = z_{20} = 2(p-1)/(p-2). \tag{28}$$

These two solutions obtained are equivalent, and they represent the same general solution (14). Taking into account (20), from (26) we have

$$\left(\frac{2-p}{2}\right) z_1(\xi)\left[\frac{p}{p-2}+z_1(\xi)\right]^{p-1} = \exp(-p\xi), \tag{29}$$

and from (27)

$$\left[z_2(\xi)-\frac{p}{p-2}\right]\left[\frac{(p-2)z_2(\xi)}{2(p-1)}\right]^{p-1} = \exp(-p\xi). \tag{30}$$

Using (28), it is easy to reduce both the solutions (29) and (30) to the form

$$z(\xi)[z(\xi)-z_0+1]^{1-2/z_0} = z_0 \exp[-2\xi(1-z_0{}^{-1})], \quad z_0 = -b. \tag{31}$$

This general solution allows the existence of two types of the power series expansions in the forms (29) and (30) for the various regions of the parameter p, and thus for the various velocities of the solitary wave. So, taking into account (10) and (25), for the solution (26) we have

$$\frac{v^2}{c^2} = \left(1-\frac{2\varepsilon}{p-2}\right)^{-1}.$$

Since, $v^2/c^2 > 0$, in this case $p > p_1 = 2(1+\varepsilon)$, and $v > c$. For the solution (27) we have

$$\frac{v^2}{c^2} = \left[1+\frac{2\varepsilon(p-1)}{p-2}\right]^{-1},$$

and in the region $1 \le p < p_2 = 2(1+\varepsilon)(1+2\varepsilon)^{-1}$, $v \ge c$ too. In the region $2 < p < p_0$ the velocity of the solitary wave is smaller than the velocity of sound in linear medium, because in this region $0 < v^2/c^2 < (1+2\varepsilon)^{-1}$. The intervals $2 < p < p_1$ for (26), and $p_2 < p < 2$ for (27) are forbidden intervals, because in these regions $(v/c)^2 < 0$, i.e. the velocity of the wave can not be a real value.

Equation (11) is also solved by using the same procedure. The solution is obtained in the form of the series with the same coefficients d_n. Thus, we have $y(\xi) = \frac{1}{2}z_2(\xi)$ (see (27)). But in this case $a = (1-p)/(p-2)$. It is easy to show, that this equation (11) allows the existence of the ”fast” and ”slow” solitary waves too.

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