Frequency-Dependent Attenuation and Phase Velocity Dispersion of an Acoustic Wave Propagating in the Media with Damages

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Abstract In frame of the self-consistent mathematical model, which includes the dynamics of a material and the state of its defects, the particular qualities of acoustic wave propagation in the material with damages is considered. In this study a constitutive equation of the damaged medium is derived, and the similarity between the models for damaged materials and the medium with memory is confirmed. The dispersion analysis of the model is carried out, and it is shown that the damage of the material gives rise to frequency-dependent attenuation and anomalous dispersion of phase velocity of acoustic wave propagating through that material. This makes it possible to estimate the damage of the material by means of a nondestructive acoustic method.

1 Introduction

Today, the mechanics of a damaged continuum is intensively developed by many authors. The first works in this field were fundamental studies by L. M. Kachanov, which are summarized in his monograph [6], and the detailed investigations and analysis by Yu. N. Rabotnov that are generalized in [11]. The significance of these pioneer works, which presently are recognized as classical, consists in the possibility of using a unified approach for description of the damage of elastic and elastoplastic bodies.

The damage is usually understood as a reduction of an elastic response of the body due to decreasing of the effective area, through which the internal forces are

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transmitting from one part of the body to another. This phenomenon is caused by the appearance and spreading of the scattered field of microdefects (the microcracks in the case of elasticity, the dislocations in the case of plasticity, the micropores in the case of creep, and the surface microcracks in the case of fatigue) [10].

The damage, i.e. the degradation of the mechanical properties of a solid material, cannot be measured directly in the same manner as, for example, velocity, force, or temperature. The damage can be detected indirectly only by analyzing the response of the elastic structure on the various external impacts. According to experimental knowledge, the presence of a damages field inside a solid material can be observed also by changing of physical features of the structure. For example, it may be the decreasing of velocity of ultrasonic signal propagation [17, 5, 16], a decrease in the Young's modulus (the modulus defect) [8], a decrease in material density (loosening) [14], a hardness change [2], a decrease in the stress amplitude under the cyclic testing [9, 12], and an acceleration of the tertiary creep [1].

The purpose of the present study is the modeling of the process of acoustic wave propagation through the damaged material, and estimation of influence of damage on the phase velocity and attenuation of that wave.

2 The self-consistent model for damages description

In accordance with conventional assumptions, the measure of damage under deformation is taken to be a scalar damage parameter $\psi(x,t) > 0$ that characterizes the relative density of microdefects uniformly dispersed in a unit volume. This parameter is zero in the absence of damage and close to unity at the instant of fracture. The process of the damages gain in the structure under study is calculated numerically step by step by solving the kinetic equation of damage at every stage of loading. This procedure is continued until the damage parameter $\psi(x,t)$ reaches an initially prescribed limiting value, which is close to unity.

Generally, in mechanics of deformed solids, the dynamic problems and the problem of defects accumulation are considered separately. In the development of such approach, the usual practice is to postulate the relationship between the velocity of elastic wave and the value of damages by some kind of dependence in advance, and after that, it is assumed that the constant coefficients at this relation can be established on the basis of experimental data.

Usually [13], the phase velocity $V_{ph}(\omega)$ of propagating wave and its attenuation $\alpha(\omega)$ are chosen in the power polynomial form as functions of frequency ω, and as a linear functions of damage ψ as

$$V_{ph}(\omega) = C_0(1 - h_1\omega - h_2\psi\omega^2), \tag{1}$$

$$\alpha(\omega) = (h_3 + h_4\psi)\omega^4, \tag{2}$$

where $C_0 = \sqrt{E/\rho}$ is the velocity of the longitudinal elastic wave propagating in the material in the absence of defects, E is the Young's modulus, ρ is the density of the material, and h_{1-4} are the constant coefficients, which must be determined experimentally.

The evolution of damage is described by the kinetic equation derived in [15] in the form

$$\frac{d\psi}{dt} = f(\sigma, \psi), \tag{3}$$

where σ is the stress due to the external impact.

In most cases, the function f is approximated by a linear function, or, in some cases, by a power polynomial dependence [15].

Although this approach has undoubted advantage such as simplicity, it has also some imperfections, which are typical for any approach that is not based on the physical models of the processes and systems.

The other novel method of materials with damages examination was presented in [3, 4]. In these papers the process of propagation of a longitudinal acoustic wave along a rod is considered. It is also assumed that the rod is subjected to the static or cyclic tests, and during the process of loading the damages may accumulate in the rod's material.

This work differs significantly from previous studies. In [3, 4], the authors propose the idea that the problem under study is the self-consistent problem, and therefore, in addition to the damage evolution equation (3), which can be represented in the form

$$\frac{\partial\psi}{\partial t} + \frac{1}{\tau}\psi = \beta_2 E\frac{\partial u}{\partial x}, \tag{4}$$

the additional equation describing the dynamics of the rod given by

$$\frac{\partial^2 u}{\partial t^2} - C_0^2\frac{\partial^2 u}{\partial x^2} + \beta_1\frac{\partial\psi}{\partial x} = 0, \tag{5}$$

must be taken into account.

Here we denote the particle displacement at the rod midline by $u(x,t)$, and the constants τ, β_1 and β_2 characterize the relations between the cyclic process of the rod loading and the speed of the damages accumulation.

Equation (4) may be rewritten in equivalent form as

$$\psi(x,t) = \beta_2 E\int_0^t \frac{\partial u}{\partial x}(x,\xi)e^{(\xi-t)/\tau}d\xi = \beta_2 E\,R(t) * \frac{\partial u}{\partial x}(x,t), \tag{6}$$

where the sign * denotes the convolution sign, and $R(t)$ is the relaxation function given by

$$R(t) = e^{-t/\tau}. \tag{7}$$

Equation (6) describes the process of damages growth as a function of the strain ($\varepsilon = \partial u/\partial x$) history, and one can state that the constant $\tau > 0$ is the relaxation time. Here we assume that the history of the damages appearance starts at $t = 0$.

From Eq. (6), it follows that at the beginning of process, if $t \ll \tau$, there are no defects ($\psi = 0$) in the rod material at all. In the opposite case, if $t \gg \tau$, from Eq. (6) one can obtain the dependence describing the process of damages growth for the case of slow changing of strain in the form

$$\psi = \tau\beta_2 E \frac{\partial u}{\partial x}. \tag{8}$$

Now, using Eq. (6), Eq. (5) can be written as

$$\rho \frac{\partial^2 u}{\partial t^2} = E \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} - \rho\beta_1\beta_2 R* \frac{\partial u}{\partial x}\right). \tag{9}$$

Taking into account the classical equation of motion given by

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x}, \tag{10}$$

we can derive the constitutive equation of the media with damages in the form

$$\sigma = E(1-\rho\beta_1\beta_2 R*)\frac{\partial u}{\partial x} = E\left[\frac{\partial u}{\partial x} - \rho\beta_1\beta_2\int_0^t \frac{\partial u}{\partial x}(x,\xi)\mathrm{e}^{(\xi-t)/\tau}d\xi\right]. \tag{11}$$

Materials described by this equation for which the exerted stress is determined by the history of the deformation are ”materials with memory.”

As indicated by Rabotnov [11], a model of material with memory may be obtained by means of replacing constant elastic parameters of solids by time-dependent operators. So for the case of material with damages, the Young’s modulus is now not a constant, but an operator

$$E_0(t) = E(1-\rho\beta_1\beta_2 R*), \tag{12}$$

and thus the constitutive equation (11) of the media with damages one can rewrite in compact form as

$$\sigma(\varepsilon) = E_0(t)\varepsilon. \tag{13}$$

From Eq. (13), it follows that if $t \ll \tau$, then we obtain the constitutive equation for the fast loading in the form

$$\sigma = E\frac{\partial u}{\partial x} = E_d \frac{\partial u}{\partial x}. \tag{14}$$

Here the constant $E_d = E$ is is the dynamic Young’s modulus.

In the opposite case, if $t \gg \tau$, then we obtain the constitutive equation, which is valid for the slow loading

$$\sigma = E_d(1-\tau\rho\beta_1\beta_2)\frac{\partial u}{\partial x} = \delta E_d \frac{\partial u}{\partial x} = E_s \frac{\partial u}{\partial x}, \tag{15}$$

where the quantity $E_s = \delta E_d$ is the static Young’s modulus of the material, and parameter $\delta = 1-\tau\rho\beta_1\beta_2$ characterizes the material damages.

Due to the evident inequality $E_d > E_s > 0$, parameter $0 < \delta \leq 1$. The value of parameter $\delta = 1$ denotes the absence of damages, and the value of this parameter δ is close to zero at the instant of fracture.

We can notice that Eqs. (4) and (5) can be reduced to a single one by eliminating the damage parameter $\psi(x,t)$. In terms of displacement $u(x,t)$, it leads to an equation in the following form

$$\frac{\partial^2 u}{\partial t^2} - \delta C_0^2 \frac{\partial^2 u}{\partial x^2} + \tau \frac{\partial^3 u}{\partial t^3} - \tau C_0^2 \frac{\partial^3 u}{\partial x^2 \partial t} = 0. \quad (16)$$

Dimensionless form of the Eq. (16) is obtained by using the non-dimensional variables that are introduced by relations

$$U = u/\tau C_0, \quad X = \sqrt{\delta} x/\tau C_0, \quad T = \delta t/\tau. \quad (17)$$

Thus Eq. (16) in terms of non-dimensional displacement variable $U(X,T)$ takes the following form [7]

$$\frac{\partial^2 U}{\partial T^2} - \frac{\partial^2 U}{\partial X^2} + \delta \frac{\partial^3 U}{\partial T^3} - \frac{\partial^3 U}{\partial X^2 \partial T} = 0, \quad (18)$$

and describes acoustic wave propagation in the medium with damages.

3 Dispersion relations

The fundamental solution of Eq. (18) has the form of traveling waves

$$U(X,T) = U_0 \, e^{i\kappa X - i\omega T}, \quad (19)$$

where i is the imaginary unit, κ is the wavenumber, ω is the angular frequency, and U_0 is an amplitude.

The dispersion law $\Phi(\kappa, \omega) = 0$ for Eq. (18) is defined by relation

$$i\delta\omega^3 - \omega^2 - i\kappa^2\omega + \kappa^2 = 0. \quad (20)$$

In the case of boundary value problem the general solution of Eq. (18) has the following form

$$U(X,T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Theta(\omega) e^{i\kappa(\omega)X - i\omega T} d\omega, \quad (21)$$

where $\Theta(\omega)$ is the Fourier-transform of the boundary value of disturbance prescribed at $X = 0$

$$\Theta(\omega) = \int_{-\infty}^{\infty} U(0,T) e^{i\omega T} dT. \quad (22)$$

In case of Cauchy problem the general solution of Eq. (18) has the following form

$$U(X,T) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\chi(\kappa)e^{i\kappa X - i\omega(\kappa)T}d\kappa, \tag{23}$$

where $\chi(\kappa)$ is the Fourier-transform of initial disturbance prescribed at $T = 0$

$$\chi(\kappa) = \int_{-\infty}^{\infty}U(X,0)e^{i\kappa X}dX. \tag{24}$$

In general case $\kappa = \kappa(\omega)$ and $\omega = \omega(\kappa)$ are the complex quantities and can be derived from dispersion relation (20). In order to provide the dispersion analysis in context with a boundary value problem we rewrite wavenumber $\kappa(\omega)$ in the form

$$\kappa(\omega) = k(\omega) + i\lambda(\omega), \tag{25}$$

where $k = \mathrm{Re}(\kappa)$ and $\lambda = \mathrm{Im}(\kappa)$. Using this notation, expression (19) can be rewritten as follows

$$U(X,T) = U_0 e^{i(k+i\lambda)X - i\omega T} = e^{-\lambda X}U_0 e^{ikX - i\omega T}. \tag{26}$$

It is clear that for positive values of λ we can observe the exponentially decaying wave that propagates along the positive direction of the space axis. In other words the spectral components $k(\omega) = Re(\kappa)$ decay exponentially as $x,t \to \infty$ for $\lambda(\omega) > 0$. On the other hand, if $\lambda(\omega) < 0$, then the amplitudes of the spectral components grow exponentially as they propagate further along the positive direction of the x-axis. In the latter case the solution of equation (18) becomes unstable for $T \gg 0$.

4 Dispersion analysis

As discussed above, in order to study the wave propagation along the x-axis one needs to solve the dispersion relation (20) against wavenumber κ. This solution takes the form

$$\kappa(\omega) = \frac{\omega\sqrt{1 - i\delta\omega}}{\sqrt{1 - i\omega}}. \tag{27}$$

For real values of k and λ the dispersion relation (20) can be rewritten as follows

$$k^2 + 2ik\lambda - \lambda^2 - ik^2\omega + 2k\lambda\omega + i\lambda^2\omega - \omega^2 + i\delta\omega^3 = 0. \tag{28}$$

In order to study real and imaginary parts separately, the system of equations in the form

$$\begin{cases} k^2 - \lambda^2 + 2k\lambda\omega - \omega^2 = 0 \\ 2k\lambda - \omega(k^2 - \lambda^2) + \delta\omega^3 = 0 \end{cases} \tag{29}$$

is solved and analyzed. Solutions with respect to k and λ are

$$k(\omega) = LM\left(\sqrt{1+M^2}-1\right)^{-1/2}, \tag{30}$$

$$\lambda(\omega) = L\left(\sqrt{1+M^2}-1\right)^{1/2}, \tag{31}$$

where

$$L = \omega\sqrt{\frac{1+\delta\omega^2}{2(1+\omega^2)}}, \quad M = \frac{(1-\delta)\omega}{1+\delta\omega^2}. \tag{32}$$

The frequency dependencies $k(\omega) = \mathrm{Re}(\kappa)$ and $\lambda(\omega) = \mathrm{Im}(\kappa)$ of dispersion relation (20) are displayed in Fig. 1 for the various values of the material parameter δ. Parameter δ can have values on the interval $\delta = [0,1]$.

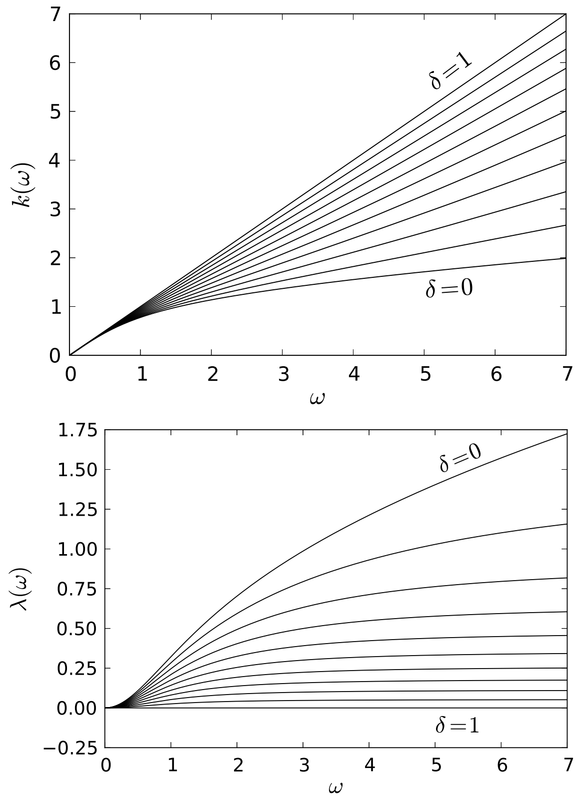

Fig. 1 Dispersion relations $k(\omega)$ and $\lambda(\omega)$ for various values of parameter δ in range [0.0, 1.0] with step 0.1

If $\delta = 1$, then from (30) and (31) one can find

$$k(\omega) = \omega, \quad \lambda(\omega) = 0. \tag{33}$$

These relations correspond to the ideal elastic material without damages, and in which the wave propagates without attenuation.

In case of $\omega \to \infty$ it is easy to see that $k(\omega) \to \omega\sqrt{\delta}$ and that

$$\lim_{\omega\to\infty} \lambda(\omega) = \frac{1-\delta}{2\sqrt{\delta}}. \tag{34}$$

For large frequencies, the exponential decay constant λ depends only on the parameter δ.

The phase velocity is defined as $v_{ph}(\omega) = \omega/k$, and it takes the following general form

$$v_{ph} = \frac{\sqrt{2(1+\omega^2)(N-\delta\omega^2-1)}}{(1-\delta)\omega}, \tag{35}$$

where

$$N = \sqrt{(1+\omega^2)(1+\delta^2\omega^2)}. \tag{36}$$

The frequency dependence $v_{ph}(\omega)$ for various values of parameter δ is shown in Fig. 2.

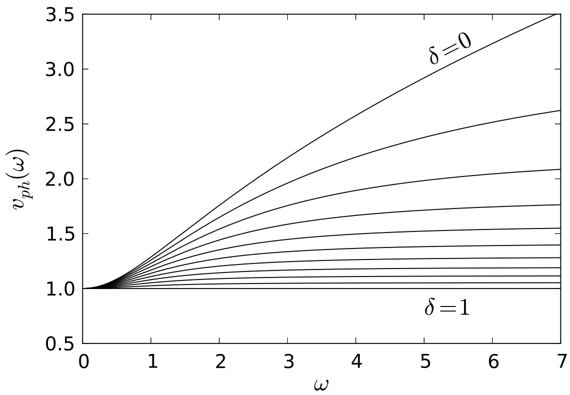

Fig. 2 Phase velocity as a function of frequency for various values of the parameter δ in range [0.0, 1.0] with step 0.1

In case of $\delta = 1$, the phase velocity becomes $v_{ph}(\omega) = 1$ (*cf.* relationship (33)). For large frequencies, the phase velocity has a limit

$$\lim_{\omega\to\infty} v_{ph}(\omega) = \frac{1}{\sqrt{\delta}}. \tag{37}$$

The group velocity, which is defined as $v_{gr}(\omega) = d\omega/dk = (dk/d\omega)^{-1}$ takes in this case the following general form

$$v_{gr} = \frac{2(1+\omega^2)^2 \sqrt{2(1+\delta^2\omega^2)}\left(N-\delta\omega^2-1\right)^{3/2}}{\omega(1-\delta)[(1+3\delta^2)\omega^4-(2N+2\delta N-3\delta^2-5)\omega^2-4(N-1)]}, \qquad (38)$$

where N is defined by relation (36). The frequency dependence $v_{gr}(\omega)$ for various values of the parameter δ is presented in Fig. 3.

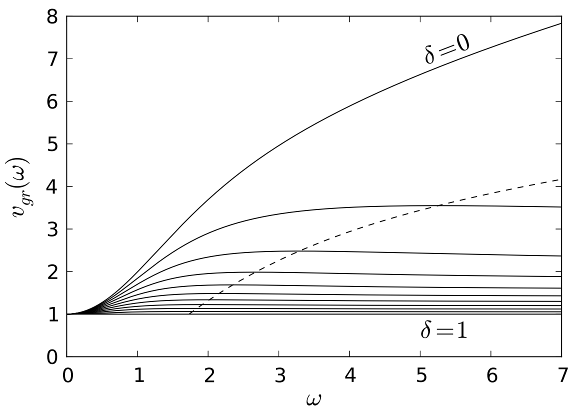

Fig. 3 Group velocity as a function of frequency for various values of the parameter δ in range [0.0, 1.0] with step 0.1. Maximum of v_{gr} for $\delta < 1$ is shown by *dashed line*

In case of $\delta = 1$, the group velocity $v_{gr}(\omega) = 1$ (*cf.* relationship (33)). For large frequencies the group velocity has the same limit as the phase velocity did

$$\lim_{\omega\to\infty} v_{gr}(\omega) = \frac{1}{\sqrt{\delta}}. \qquad (39)$$

The essential difference between the behavior of phase and group velocities is that the phase velocity is a monotonic function of frequency, while the group velocity has a maximum. The maximum of different values of δ are located on the dashed line shown in Fig. 3.

Comparison of phase and group velocities for a single value of δ is presented in Fig. 4. In the material with damages the group velocity is always greater than the phase velocity for any frequency. This fact means that the material with damages is a medium with anomalous dispersion. This is true for any value of parameter $\delta < 1$. In case of $\delta = 1$, then $v_{gr} = v_{ph} = 1$, and we have the non-dispersive case.

5 Conclusions

We have presented results for simulation of acoustic wave propagation in the medium with damages. Based on the self-consistent model for damages description,

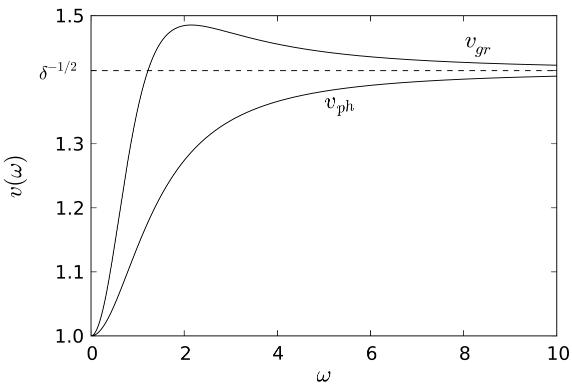

Fig. 4 Comparison of group and phase velocities for single value of the parameter $\delta = 0.5$. The *dashed line* shows the limit for the large frequencies

we have been able to demonstrate the similarity between the models for damaged materials and the medium with memory.

We have derived the constitutive equation of the material with damages, and examine the influence of the parameters of damage on the process of wave propagation in that medium.

The dispersion analysis of the model have been carried out, and the effect of the material damage on attenuation and phase velocity of propagating acoustic wave have also been estimated. It has been shown that the damage causes the anomalous dispersion and the frequency-dependent attenuation of the wave propagating through that material.

The results obtained may be of some use for developing of a technique for nondestructive acoustic detection of damages in solids and structural elements.

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