Two Nonlinear Hysteretic Models of Piano Hammer

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Abstract. The introduction to the theory of piano hammer is provided based on the hysteretic (hereditary) models of piano hammer and upon the large number of experimental data. It is experimentally proved that a standard piano hammer possesses history–dependent properties, meaning, that it is made of a material with memory. It has been shown that dynamical behavior of the piano hammer can be described by two different mathematical hysteretic models. The both models demonstrated here makes predictions in good agreement with experimental data for various types of piano hammers and for a broad range of hammer velocity.

INTRODUCTION

The first work describing an experimental research of a piano hammer *in situ* as an independent object was the remarkable experiment provided by Yanagisawa and Nakamura [1]. In this paper for the first time the main dynamical and very important features of piano hammers were demonstrated: nonlinearity of the force-compression characteristics of the hammer, the strong dependence of the slope of the loading curve on the hammer velocity, and the significant influence of hysteresis, i.e. the loading and unloading of the hammer felt are not alike. It was shown that the hammer felt is still deformed even after the acting force is removed.

Another experimental investigation [2, 3], using a similar method was carried out almost 20 years later. A special high precision device [2] has been developed for this purpose in order to provide the dynamical testing of piano hammers. The experimental arrangement makes it possible to obtain the force and compression histories of the hammer-string interaction, and investigate the dynamic force-compression characteristics of the various piano hammers [3].

A new nonlinear hysteretic model of the piano hammer that is in a good agreement with experimental data obtained was developed and described in [4]. This model is based on an assumption that the hammer felt made of wool is a microstructural material possessing history-dependent properties. Such a physical substance is called the material with memory. Particularly, it has been shown that the physical assumption about the history-dependent properties of the hammer felt are confirmed by the experiments.

HAMMER MODEL I

According to the hammer models considered earlier, the loading and unloading of the hammer are reversible. Usual model of the hammer relates the force exerted by hammer F and the hammer felt compression u in the form of the power law

$$F = F_0\, u^p\,, \tag{1}$$

where F_0 is the hammer stiffness and p is the compliance nonlinearity exponent. Thus the features of the hammer are determined by these two parameters which may be easily measured in static experiments. However, dynamic features of piano hammers are significantly more complicated. As it was mentioned above, it is necessary to take into consideration both the hysteresis of the force–compression characteristics and their dependence on the hammer speed. Such theoretical model of the piano hammer was derived in [4] in the form

$$F(u(t)) = F_0 \left[u^p(t) - \frac{\varepsilon}{\tau_0} \int_0^t u^p(\xi)\, exp\left(\frac{\xi - t}{\tau_0}\right) d\xi \right]. \tag{2}$$

Here $F(u)$ is the force exerted by a hammer and u is the hammer compression. The instantaneous hammer stiffness F_0 and compliance nonlinearity exponent p are the elastic parameters of a hammer, and constants ε and τ_0 are the hereditary parameters. According to this model, a real piano hammer possesses history-dependent properties, or in other words, is made of a material with memory. This analytical model takes into account all the important features of the hammer-string interaction, and it is the base of our theoretical studies.

NUMERICAL SIMULATION OF EXPERIMENTS

The piano hammer parameters may be obtained by numerical simulation of the dynamic experiments. This procedure was presented in [2, 3] and it is based on the mathematical model of the experiment with the piano hammer testing device. The impact of the hammer can be described by the equation of motion

$$m\frac{d^2u}{dt^2} + F(u) = 0, \tag{3}$$

with the initial conditions

$$u(0) = 0,\ \frac{du}{dt}(0) = V_h. \tag{4}$$

Here m and V_h are the hammer mass and the velocity respectively, and $F(u)$ is defined by Eq. (2).

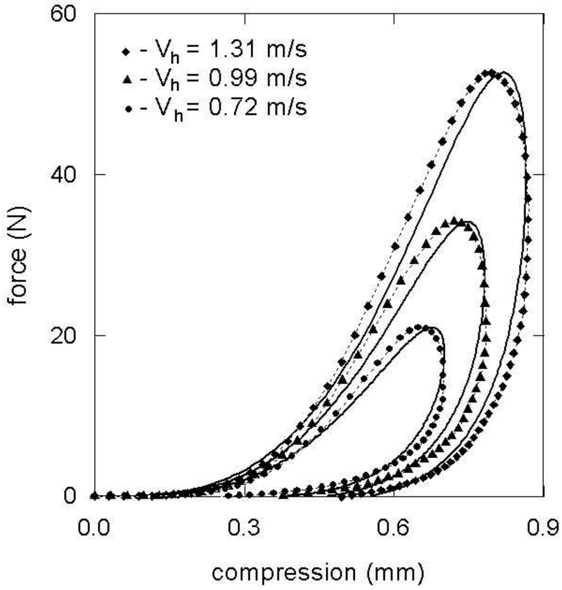

FIGURE 1. The measured (various signs) and simulated (solid lines) force-compression characteristics of the piano hammer for the various hammer velocity.

Initially unknown, the values of the hammer parameters were obtained by means of numerical simulation of the model. The force–compression characteristics $F(u)$ was numerically calculated from Eq. (3) by assuming initial values of the parameters. The model was run repeatedly, each time with different parameter values, until the prediction from the model gave a good agreement with the experimental data.

In Fig. 1 are presented the force –compression characteristics of piano hammer measured for the various initial hammer velocities and the simulated force–compression curves. All the calculated curves shown in Fig. 1 are obtained by using one certain combination of hammer parameter values: F_0 = 8800 N/mm^p; p = 3.95; ε = 0.992; τ_0 = 2.0 μs. Only the value of the hammer velocity was varied. It is interesting, that the value of the relaxation time τ_0 obtained is much less than the time of the hammer–string interaction which is equal to 1.6 ms for a hammer speed 1.31 m/s and 2.5 ms for a hammer speed 0.72 m/s. Nevertheless, the simulation of the experimental data gives the good result, and it seems only the certain and unique set of the initial hammer parameters fix the definite force–compression curve.

However not all is so simple. Unexpectedly, during the numerical simulation of the model (2) it was found that the same (very similar) force–compression curves presented in Fig. 1 can be obtained for the hammer parameters: F_0 = 3520 N/mm^p; p = 3.95; ε = 0.98; τ_0 = 5.0 μs, for example. This fact explanation leads to the new hysteretic model of the piano hammer.

HAMMER MODEL II

The equation (3) with the function (2) may be written also in the form

$$m\frac{d^2u}{dt^2}+m\tau_0\frac{d^3u}{dt^3}+F_0\left[(1-\varepsilon)u^p+\tau_0\frac{d(u^p)}{dt}\right]=0\,. \qquad (5)$$

The analysis of this equation shows that the first term is much greater than the second one. This fact corresponds to the non equality $F(t) \gg \tau_0\, dF/dt$, which is valid for the chosen values of τ_0 (rather small), and for any reasonable value of the piano hammer velocity – up to 10 m/s. If it is so, the second term may be neglected, and introducing the new parameters $Q_0 = F_0(1-\varepsilon)$ and $\alpha = \tau_0/(1-\varepsilon)$, we have

$$m\frac{d^2u}{dt^2} + Q_0\left[u^p + \alpha\frac{d(u^p)}{dt}\right] = 0\,. \tag{6}$$

Thus, according to Eq. 3 we can determine the new piano hammer model in the form

$$Q(u(t)) = Q_0\left[u^p + \alpha\frac{d(u^p)}{dt}\right]\,, \tag{7}$$

where $Q(u)$ is the force exerted by a hammer, Q_0 is the static hammer stiffness, and α is the new time dimension parameter. This hysteretic model permits a description of the hammer felt compression that is consistent with experiments also. Thus, the simulated curves (exactly the same) shown in Fig. 1 may be single valued obtained by using the hammer parameters: Q_0 = 70.4 N/mm^p; p = 3.95; α = 0.25 ms, and for almost the same hammer velocities.

CONCLUSIONS

The both models of the piano hammer makes predictions in good agreement with experimental data for various types of hammers and for a broad range of hammer velocities. Furthermore, the dependence of the slope of the force-compression characteristics of the hammer for very slow or static compression is the same for both models: $F(u) = F_0(1-\varepsilon)\ u^p$ and $Q(u) = Q_0\ u^p$, because $F_0(1-\varepsilon) = Q_0$.

However, for very fast loading these two models are quite different. The first model gives $F(u) = F_0\ u^p$ and the second $Q(u) = p\alpha V_h Q_0\ u^{p-1}$. Because the force $Q(u)$ exerted by hammer is proportional to the hammer velocity and its value is unlimited, it seems the first model is more physical and reasonable by nature. To decide this problem and to choose the correct model the new additional experiments with a very fast hammer loading must be provided.

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