ACTION OF TRAVELING WAVE ON THE PIANO BRIDGE

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Abstract

We investigate the interaction of the transverse waves in piano string with the string support (bridge), which is considered as an oscillating system consisting of the mass, spring and damper. The part of the string passing the bridge terminates at a point load on a damper. Already in classical papers (J. Rayleigh, J. Larmor, E. Nicolai), it was shown that the elastic waves traveling through various systems transfer with themselves energy and pulse. For this reason the traveling waves exert the pressure on an obstacle. In this paper we show, that the parameters of viscoelastic inertial support of the string can be varying under the action of the traveling wave pressure. This phenomenon, in turn, changes the amplitude-frequency response of the string vibrations, and in particular, the modulation of the natural frequency of the string can be observed.

INTRODUCTION

It seems, L. Euler was the first [Euler, 1746] who come out with a suggestion that the waves bring to bear pressure on any bodies that impeded the free wave propagation. The fact that electromagnetic radiation exerts a pressure upon any surface exposed to it was deduced theoretically by J. C. Maxwell in 1871 [Maxwell, 1991]. The pressure of the electromagnetic radiation upon an immovable obstacle is given by

$$F = \frac{\Phi}{c}, \tag{1}$$

where Φ is the energy flux density, and c is the speed of light.

The results of experimental studies of the sound waves pressure on the acoustic resonators carried out by W. Dvorak in 1876 were explained by John William Strutt (Lord Rayleigh) [Rayleigh, 1945].

After the famous experiments by P. Lebedev [Lebedev, 1901] on detection of pressure of light, J. Rayleigh and J. Larmor independently from each other in 1902 have stated the assumption that any wave motion, of any nature, exerts the pressure on the bodies impeded the waves propagation [Rayleigh, 1902], [Larmor, 1902]. Their studies of the phenomenon of the mechanical pressure of waves have found more complete elaboration in research works by E. Nicolai, who considered in 1912 -1925 years the several problems concerning the interaction of the transverse waves traveling along the string with the movable supports [Nicolai, 1925].

ON THE PRESSURE OF WAVES OF A DIFFERENT NATURE

Let's estimate the value of pressure exerted by waves of a different physical nature on an immovable obstacle. If the obstacle absorbs a wave, the pressure is determined by formula (1), where $c = \omega/k$ is the phase speed, ω is the frequency, k is the wave number, and Φ is the energy flux density of the source in assumption that the whole energy flow is absorbed by an obstacle. The appropriate estimations are presented Table 1.

Table 1. Comparison of different waves.

Wave	Speed c $[m/c]$	Pressure F [N]
Electromagnetic waves in vacuum	$3\cdot 10^{8}$	$3{,}3\cdot 10^{-9}$
Sound waves in steel	$5\cdot 10^{3}$	$2\cdot 10^{-4}$
Sound waves in air	$3\cdot 10^{2}$	$3{,}3\cdot 10^{-3}$
Transverse waves in piano string	$70\div 420$	$(2.4\div 13)^{-3}$
Low frequency gravity waves in fluid	$0.1\div 1$	$1\div 10$

For convenience of comparison the power of sources are taken identical and equal to 1 w. It seems that the action of waves traveling along the string on the string support in some cases must be taken into account.

WAVES IN A STRING WITH VISCOELASTIC INERTIAL SUPPORT

We consider the interaction of the traveling waves in piano string with the string support (bridge), which is presented here as the oscillating system consisting of the mass, spring and damper. The part of the string passing the bridge terminates at a point load on a damper. The scheme of this model is shown in Figure 1.

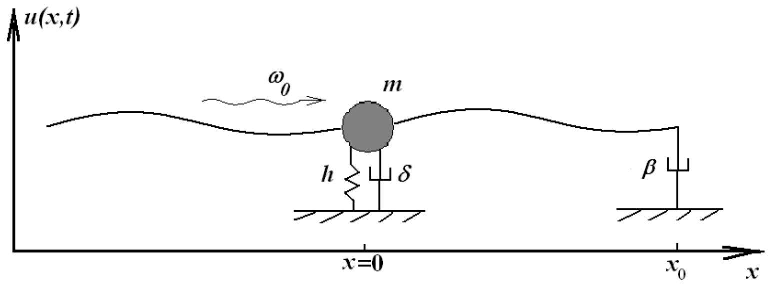

Figure 1: Scheme of model.

The transverse displacement of a stiff string is described by governing equation

$$\rho u_{tt} - N u_{xx} = 0. \qquad (2)$$

The boundary conditions at $x = 0$ can be written as

$$u(-0,t) = u(+0,t) = {}^{0}u(t),$$

$$m^{0}\ddot{u}+h^{0}u+\delta^{0}\dot{u}=\left[-Nu_{x}\right]_{x=0},$$

and at $x=x_0$ we have

$$\beta\,\dot{u}(x_0,t)=-Nu_x(x_0,t).$$

Here ρ, N are the linear mass density, the string tension, and string stiffness; h, m, δ, β are the elastic, the inertial, and the dissipative coefficients of support. The square brackets denote the difference of limit values of this function on the right and on the left sides of the support.

Let's choose the value of dissipation factor β to obtain the ideal matching damping structure, at which the waves reflected from the point $x=x_0$ are absent. For this purpose the value of β must be equal to the string impedance $\beta=\sqrt{N\rho}$.

Let the source located on the left side from the object radiates a simple-harmonic wave $u_f(x,t)=A_0\exp[i(\omega_0 t-k_0x)]$, which due to the interaction with the object creates the reflected wave $u_r(x,t)=A_1\exp[i(\omega_1 t-k_1x)]$ and the transmitted wave $u_p(x,t)=A_2\exp[i(\omega_2 t-k_2x)]$, where A_j are the complex amplitudes (j=1,2).

Substituting the solution in the form

$$u(x,t)=\begin{cases}u_f(x,t)+u_r(x,t), & x<0\\ u_p(x,t), & x>0\end{cases}, \tag{3}$$

into Eq. (2) and using the boundary conditions, the following frequencies ω_j, wave number k_j, and amplitudes A_j results

$$\omega_1=\omega_2=\omega_0,\; k_{1,2}=\mp\,\omega_0/c,$$

$$A_1=A_0\sqrt{\left(\omega_0^2\delta^2+\left(h-\omega_0^2m\right)^2\right)\Big/\left(\omega_0^2(2z+\delta)^2+\left(h-\omega_0^2m\right)^2\right)}\exp(i\varphi),$$

$$A_2=2z\omega_0A_0\Big/\left(\omega_0^2(2z+\delta)^2+\left(h-\omega_0^2m\right)^2\right)\exp(i\psi),$$

where

$$\tan\frac{\varphi}{2}=\left\{\sqrt{\left(\omega_0^2\delta^2+\left(h-\omega_0^2m\right)^2\right)\left(\omega_0^2(2z+\delta)^2+\left(h-\omega_0^2m\right)^2\right)}+\right.$$
$$\left.+\,\omega_0^2\delta(2z+\delta)+\left(h-\omega_0^2m\right)^2\right\}\Big/2\omega_0z\left(h-\omega_0^2m\right), \tag{4}$$

$$\tan\frac{\psi}{2}=\left\{\left(\omega_0^2(2z+\delta)^2+\left(h-\omega_0^2m\right)^2\right)-\omega_0(2z+\delta)\right\}\Big/\left(h-\omega_0^2m\right), \tag{5}$$

and $z=\sqrt{N\rho}$ is the string impedance.

In case of the mass of the bridge or its stiffness tends to infinity, we have only the reflected wave $u_r(x,t)$, because $A_2 = 0$.

As the string support possesses elastic and inertial properties, it is the power-consuming object too. On a bridge the energy of an incident wave transforms into the potential and kinetic energy of the support, and then it is transferred back to the string. A time delay to provide this process can be determined in according to formula [Vesnitskij, 2001]

$$\tau_{gr} = \left| d\varphi / d\omega_0 \right|,$$

where φ is phase of the wave. Using this formula, and expressions (4) and (5) we find that the time delay for the reflected wave is

$$\tau_{gr1} = 2z\left(h + \omega_0^2 m\right)\left(-\omega_0^2 (2z+\delta)\delta + \left(h - \omega_0^2 m\right)^2\right)\left\{\left(\omega_0^2 \delta^2 + \right.\right.$$
$$\left.\left. + \left(h - \omega_0^2 m\right)^2\right)\left(\omega_0^2 (2z+\delta)^2 + \left(h - \omega_0^2 m\right)^2\right)\right\}^{-1},$$

and the time delay for transmitted wave is

$$\tau_{gr2} = (2z+\delta)\left(h + \omega_0^2 m\right)\left(\omega_0^2 (2z+\delta)^2 + \left(h - \omega_0^2 m\right)^2\right)^{-1}.$$

If the dissipative losses can be neglected, the time delay for the reflected wave is the same as for the transmitted wave.

The traveling wave acts on a bridge with the pressure that can be determined in according to formula [Vesnitskij, 2001]

$$F_{pr} = -\left[\left(\rho\, u_t^2 + N u_x^2\right)/2\right]_{x=0}.$$

The average value of the pressure exerted by the incident wave for the period $2\pi/\omega_0$ is given by

$$< F_{pr} >= \rho \omega_0^2 A_0^2 \left\{ 1 - \frac{2z\omega_0^2 (2z+\delta)}{\omega_0^2 (2z+\delta)^2 + \left(h - m\omega_0^2\right)^2} \right\}.$$

In Figure 2 are presented the dependencies of dimensionless constant component of an average pressure $< F_{pr} > / \rho\, \omega_0^2 A_0^2$ as a function of dimensionless frequency ω_0/Ω_0. Curve (1) is calculated for the case of absence of the dissipative losses, and curve (2) is calculated for the case of presence of dissipative losses. Here $\Omega_0 = \sqrt{h/m}$.

If the frequency of an incident wave is equal to the main frequency of the bridge vibrations, the pressure exerted by the wave is minimal, but the transverse displacement of the bridge achieves the maximum value.

Generally speaking, the bridge parameters can change under the action of the wave pressure, for example, the elasticity of the support can increase. So, if the fastening spring of length ℓ_0 is become longer for a value $\Delta\ell$, it will result in change of elastic forces in transverse direction.

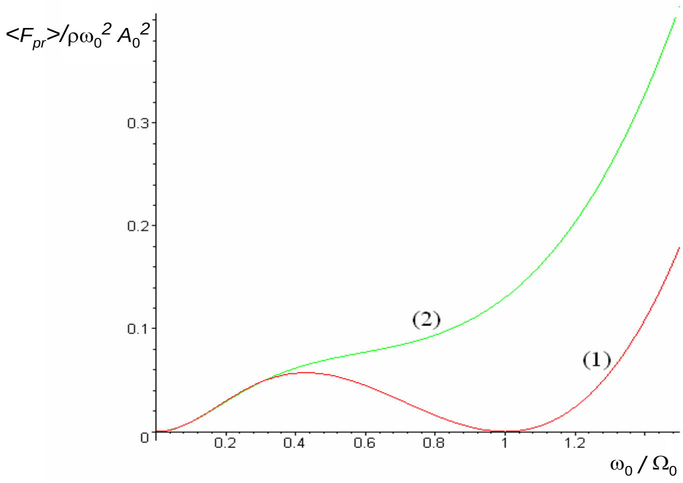

Figure 2: Pressure exerted by the incident wave as a function of frequency. Curve (1) - without of dissipative losses, and curve (2) – with dissipative losses.

Therefore, taking into account the pressure of waves, the coefficient of elasticity of the support becomes the following

$$h_0 = h\left(1 + \frac{F_{pr}}{F_{pr} + 2h\,\ell_0}\right). \qquad (6)$$

If the speed of longitudinal waves is much greater than the speed of transverse waves in a string, it is enough to consider in expression (6) only the constant component of the wave pressure. This will lead only to minor alterations of reflection and transmission coefficients shown in Fugure3.

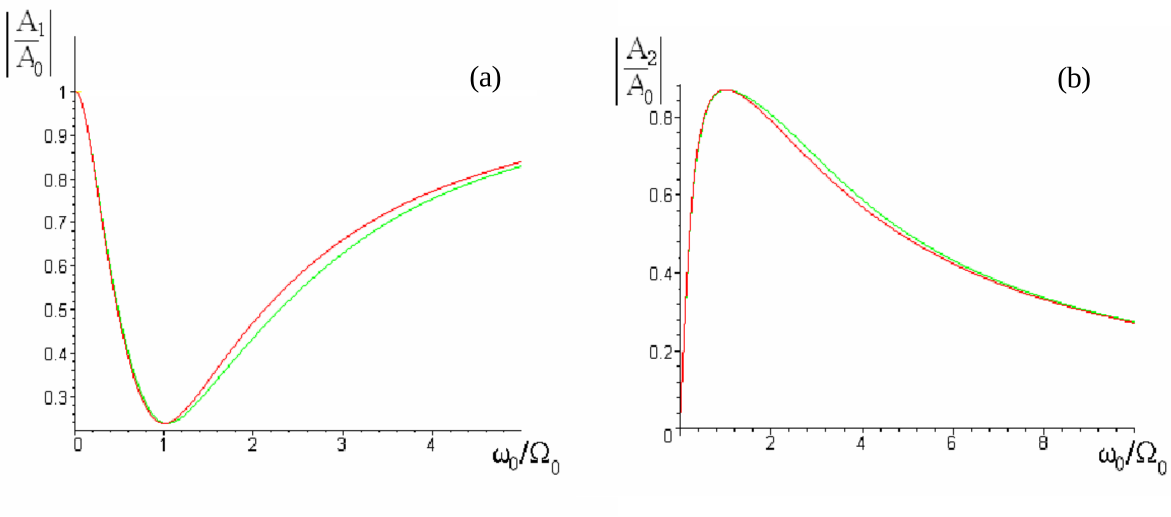

Figure 3: Amplitudes of reflection (a) and transmission (b) coefficients vs. frequency for a case of the speed of longitudinal waves is much greater than the speed of transverse waves.

If the speeds of longitudinal and transverse waves differ no more than on two orders (and it is true for piano strings), it is necessary to consider also the variable component of the wave pressure. In this case it is required to solve a problem with periodically changing coefficient, which frequency of change is equal to the double frequency of an incident wave. It provides modulation of frequencies of the transmitted and the reflected waves.

SUMMARY

It was shown, that the transverse waves traveling along the string transfer with themselves energy and pulse. For this reason the traveling wave exerts the additional pressure on the piano bridge. It was shown, that the parameters of viscoelastic inertial support of the string can be varying under the action of the traveling wave pressure. This phenomenon, in turn, changes the amplitude-frequency response of the string, and, in particular, it is the cause of modulation of the natural frequency of the string vibrations.

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