

## Piano String Motion and Spectra

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### Abstract

This presentation is focused on the numerical simulation of the hammer-string interaction. The main attention is dedicated to analysis of the special cases of the piano scale design. This study is based on the physical models of hammer and string, and gives the possibility to calculate the motion of strings and hammers, the history of the acting forces, and to simulate the spectra of the string vibrations. The three-parameter hysteretic model, which parameters were obtained experimentally, describes the piano hammer. The string is assumed to be perfectly flexible. Computer simulation of the hammer-string interaction is used for determination of tension of the string terminated on the bass and treble bridges, for optimization of the striking point position, and for investigation of multiple contacts.

### 1. Introduction

The process of string excitation by striking with a piano hammer is a very important problem of the sound formation by a musical instrument. The mathematical modeling of this problem allows predicting the spectrum of the piano string motion, which is very important for piano design. The present paper addresses the problem of how to make such predictions with the hysteretic models of piano hammer and the traveling-wave model of the string.

The displacement  $y(x, t)$  of the ideal (flexible) string obeys the simple wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad (1)$$

where  $c = \sqrt{T/\mu}$  is the wave speed in terms of tension and linear mass density of the string.

Similar to [1], we have the nonlinear system of equations describing the hammer-string interaction

$$\frac{dz}{dt} = -\frac{2T}{cm}g(t) + V, \quad (2)$$

$$\frac{dg}{dt} = \frac{c}{2T}F(t), \quad (3)$$

where  $g(t)$  is the outgoing wave created by the hammer-string interaction,  $F(t)$  is the acting force;  $m$ ,  $z(t)$ , and

$V$  are the hammer mass, the hammer displacement, and the hammer velocity. The hammer felt compression is determined by  $u(t) = z(t) - y(0, t)$ . Function  $y(0, t)$  describes the string deflection at the contact point and is given by

$$y(0, t) = g(t) + 2 \sum_{i=1}^{\infty} g\left(t - \frac{2iL}{c}\right) - \sum_{i=0}^{\infty} g\left[t - \frac{2(i+\alpha)L}{c}\right] - \sum_{i=0}^{\infty} g\left[t - \frac{2(i+\beta)L}{c}\right].$$

It is assumed that the string of length  $L$  extends from  $-\beta L$  to  $\alpha L$  with  $\beta = 1 - \alpha$ . The initial conditions at the moment when the hammer first contacts the string, are taken to be  $g(0) = z(0) = 0$ , and  $dz(0)/dt = V$ .

The string parameters were taken from [2]. In Table 1 there is presented the piano scale for some notes used for numerical simulation. Here  $f$  is the note frequency,  $l = \alpha L$  is the position of the striking point, and  $n$  is the number of strings per note.

*Table 1: Scale of Estonia Parlour Grand Piano.*

$N$	Note	$f(\text{Hz})$	$L(\text{mm})$	$l(\text{mm})$	$T(\text{N})$	$n$
1	$A_0$	27.5	1415	175.8	1307	1
10	$F_1^\sharp$	46.2	1263	156.5	1191	1
11	$G_1$	49.0	1251	154.8	840	2
20	$E_2$	82.4	1083	133.7	777	2
21	$F_2$	87.3	1273	157.2	766	2
25	$A_2$	110.0	1218	150.2	747	2
26	$A_2^\sharp$	116.5	1201	148.1	611	3
49	$A_4$	440.0	381	42.3	628	3

The governing equation connecting the nonlinear force  $F(t)$  exerted by the hammer and the felt compression  $u(t)$  for hysteretic hammer is presented in [3] in the form

$$F(u(t)) = F_0 \left[ u^p + \alpha \frac{d(u^p)}{dt} \right]. \quad (4)$$

The parameters of this model were obtained experimentally in [3], and for numerical simulation these values may be approximated as the functions of the hammer

number  $N$

$$\alpha = 248 + 1.83N - 5.5 \cdot 10^{-2}N^2 + 8.5 \cdot 10^{-4}N^3, \quad (5)$$

$$F_0 = 183 \exp(0.045N); \quad p = 3.7 + 0.015N, \quad (6)$$

$$m = 11.074 - 0.074N + 0.0001N^2. \quad (7)$$

Here the unit for retarded time  $\alpha$  is ( $\mu$ s), the unit for  $F_0$  is (N/mm <sup>$p$</sup> ), and the unit for hammer mass  $m$  is (g).

It is naturally of great interest to predict also the spectra of the string motion. The simple method is the calculation of the mode energy spectrum directly from the force history [4]. The general expression for the string mode energy is

$$E_n = \frac{M\omega_n^2}{4}(A_n^2 + B_n^2), \quad (8)$$

where

$$A_n = \frac{2 \sin(\alpha n \pi)}{n \pi c \mu} \int_0^{t_0} F(s) \cos(\omega_n s) ds,$$

$$B_n = -\frac{2 \sin(\alpha n \pi)}{n \pi c \mu} \int_0^{t_0} F(s) \sin(\omega_n s) ds.$$

The mode energy level is determined by  $EL_n = 10 \log(E_n/E_0)$ . Here  $M = \mu L$  is the total string mass;  $\omega_n = n c \pi / L = n \omega_0$  is the string mode angular frequency;  $t_0$  is the contact time;  $E_0 = mV^2/2$  is the initial energy of the hammer.

Presented data and models of string and hammer can be used as a tool for systematical exploring of the process of the hammer-string interaction. For example, in Fig. 1 there are displayed the spectra of the string vibrations simulated for the different hammer velocities.

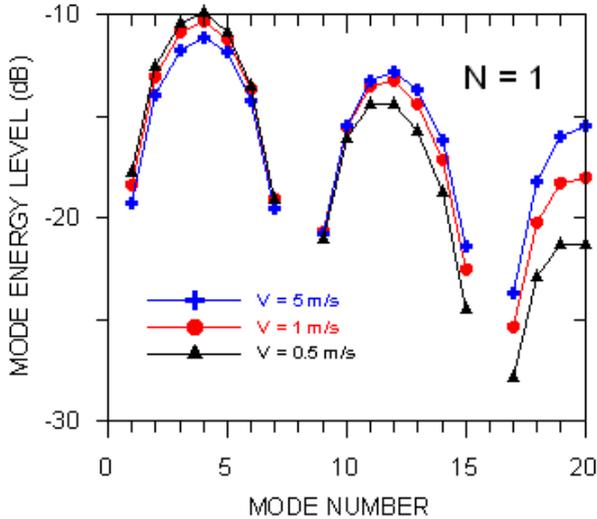


Figure 1: Spectra for  $A_0$  note.

The other characteristics of the process such as the contact duration, the string and the hammer displacement etc. can be found in the similar manner. However, in our opinion, it is very practical to use this tool for the investigation and designing of the piano scale.

## 2. Different number of strings per note

According to the construction of the *Parlour Grand Piano*, the first ten notes ( $A_0 - F_1^\sharp$ ) have only one string per note. The notes from eleven to twenty five ( $G_1 - A_2$ ) have two strings per note, and other notes consist of three strings. Let us denote the tension of strings of the tenth and the eleven note by  $T_{10}$  and  $T_{11}$ , respectively. To obtain the minimum jump of the tension between these notes,  $r_{12}$ , the relative tension per string must equal to the relative tension per note

$$r_{12} = \frac{2(T_{10} - T_{11})}{(T_{10} + T_{11})} = \frac{2(2T_{11} - T_{10})}{(T_{10} + 2T_{11})}. \quad (9)$$

From this equation we have

$$T_{10} = \sqrt{2}T_{11}. \quad (10)$$

Exactly in the same way, for the notes  $A_2$  and  $A_2^\sharp$  we may write

$$r_{23} = \frac{2(T_{25} - T_{26})}{(T_{25} + T_{26})} = \frac{2(3T_{26} - 2T_{25})}{(3T_{26} + 2T_{25})}, \quad (11)$$

and

$$T_{25} = \sqrt{\frac{3}{2}}T_{26}. \quad (12)$$

The relations (10) and (12) give the minimum jump of relative tension in cases when the number of strings in choir varies from one to two and from two to three. Just in accordance with ratios (10) and (12) the jump of the string tension was chosen for this instrument.

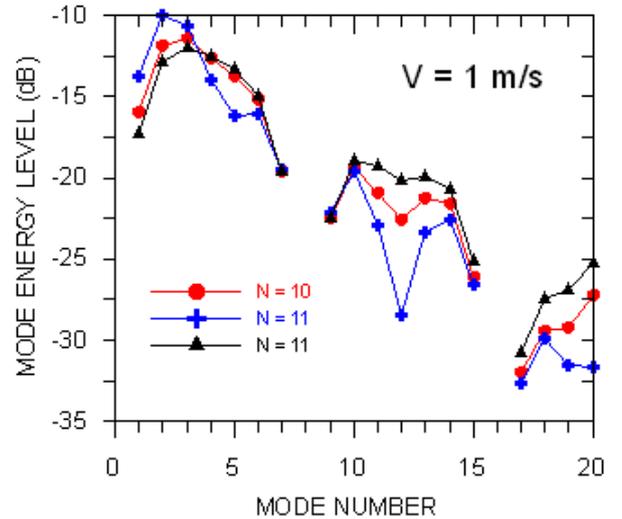


Figure 2: Spectra for  $F_1^\sharp$  and  $G_1$  notes.

By numerical simulation of these strings vibrations, we can estimate the validity of our choice. In Fig. 2 there are displayed the calculated spectra for neighbor notes  $F_1^\sharp$  and  $G_1$  shown by circles and crosses. The curve marked by triangles is the simulated spectrum of  $G_1$  note excited

by a hammer having a half of a mass (5.15 g) of usual hammer  $N=11$  ( $m = 10.3$  g). This spectrum is more similar to the spectrum of  $F_1^\sharp$  note and this fact indicates a good matching of the string tension. The same procedure was used for determination of the string tension for  $A_2$  and  $A_2^\sharp$  notes as well.

### 3. Bridges over

The first twenty strings of the piano terminate on a bass bridge, and the others on a treble bridge. Therefore, there is a significant jump of the string length between the neighbor notes  $E_2$  and  $F_2$ . The simulation of the vibration of these strings gives the possibility to match suitable tension of the strings and to choose their masses (diameters) to obtain the minimum difference of the excited spectra.

The force-time dependencies for these notes simulated for the initial hammer velocity  $V = 2.3$  m/s are presented in Fig. 3. The curve marked with triangles is simulated for non-hysteretic hammer ( $\alpha=0$ ).

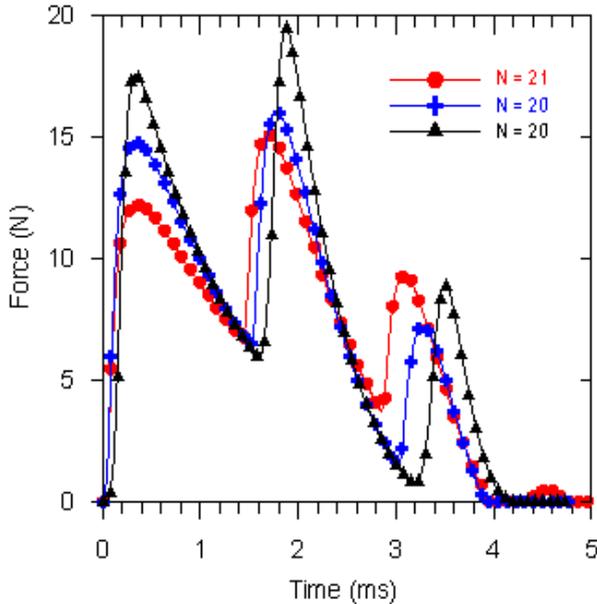


Figure 3: Force histories computed for  $E_2$  and  $F_2$  notes.

The course of the curves in Fig. 3 clear demonstrates the arrival of reflected waves. For the long string  $N = 20$  we can see also the third reflected pulse that is the repeated strike indeed, because it is quite separate from the previous. It is also evident that the unloading of the hammer felt caused by the felt elasticity begins before the moment of the first reflected wave arrival. This time is equal to 1.41 ms for the string  $N = 21$ . The force history curve for non-hysteretic hammer is more "sharp" than the curves describing the behavior of the hysteretic hammer.

The difference between hammers is demonstrated also in Fig. 4, where the corresponding spectra of the

string vibration are presented.

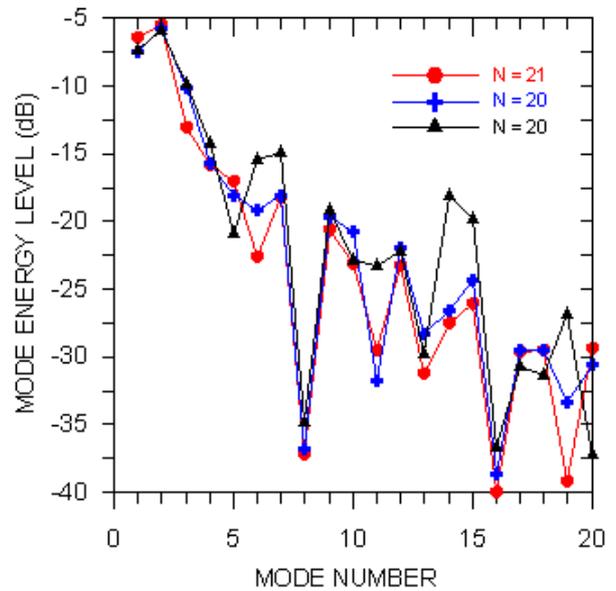


Figure 4: Spectra for  $E_2$  and  $F_2$  notes.

### 4. Striking point position

The fact that string vibration spectrum is very sensitive to the position of the fractional striking point  $\alpha$  is well known. Now we have the possibility to compare the spectra of the string vibrations calculated for the different values of  $\alpha$ .

In Fig. 5 are presented the spectra calculated for  $A_4$  note.

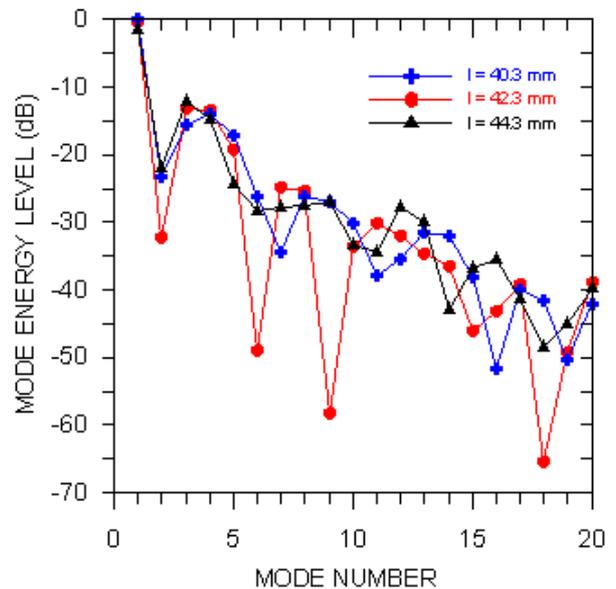


Figure 5: Spectra for  $A_4$  note.

The values of  $\alpha$  are 0.1058, 0.1110, and 0.1163. The corresponding values of the  $l = \alpha L$  are displayed also in Fig. 5. It is obvious that 5% shift of the striking point position is quite a big shift. The analysis of the presented results demonstrates that the value  $l = 42.3$  mm shown in Table 1 is not the best quantity. The more uniform spectrum may be obtained by using the value  $l = 44.3$  mm.

## 5. Hammer felt compression

The dynamical behavior of the hammer felt during the hammer-string interaction is naturally of great interest. It seems that the better way of analysis of the hammer felt compression is the presentation of the data in the form of the force and compression histories measured or simulated for this study. The example of such data is presented in Fig. 3.

The data in the form of the force-compression characteristics of the hammer felt are useful in the case when hammer strikes the immovable object. In case when hammer interacts with vibrating string, due to the complexity of this process we will obtain a very knotty picture. The examples of the hammer-string interaction in the form of the force-compression characteristics calculated for hammers  $N = 10$  and  $N = 21$  are presented in Fig. 6. The curve marked by triangles corresponds to the non-hysteretic model of the hammer.

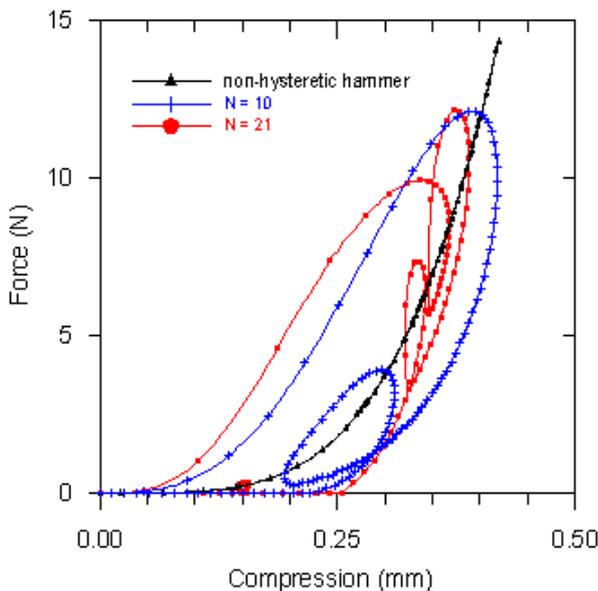


Figure 6: Force-compression characteristics for various hammers.

These are the samples of a very nice, but absolutely useless presentation of the obtained results.

## 6. Conclusions

We have presented the simple and effective method for the analysis of the piano scale. The numerical simulation of the hammer-string interaction demonstrated here makes predictions in a good agreement with experiments and it is really useful for piano designing.

## 7. Acknowledgments

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