Piano Hammer-String Interaction

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Abstract

A hammer-string interaction for bass notes is investigated and the problem what can cause the hammer to rebound is clarified. For a linear elastic hammer interacting with a long flexible string the exact solution of the equation describing the hammer motion is derived. It was shown that in some cases no reflected wave is needed to assist the hammer for going away from the string. The numerical simulation carried out for the first ten hammers and strings of *Parlour Grand Piano* shows that the real hysteretic hammer leave the string before the string begins pushing back on the hammer.

1. Introduction

The process of the string excitation by striking with a hammer is under investigation more than a hundred years. There are quite many studies devoted to this problem. We may recollect the well known reviews by Hall [1], Suzuki and Nakamura [2], and Fletcher and Rossing [3].

A central point of many papers was the problem of the contact duration between the hammer and the string, and discussion what can cause the hammer to rebound. The prevailing view about this question is expressed by Hall in [4]. He writes (p. 142) that this is not a gravity of course, because *"gravity will not ordinarily have an appreciable effect before the string begins pushing back on the hammer."*

And further: *"An infinite string has a purely resistive impedance, so it would only slow the hammer to a stop but never reverse its motion. Finite string length gives reactance, so that the string can act like a spring, but the hammer cannot possibly "learn" that the string is finite until there has been time enough for a wave to travel to the near end of the string and back. Only this reflected wave can possibly give the hammer a negative velocity."*

The same understanding of the dynamics of the hammer-string interaction is expressed in [5]: *"When the hammer has less mass than the string, it will most likely be thrown clear of the string by the first reflected pulse."*

In according to this assumption the contact duration t_0 is greater, or at least not less than the minimum time t_α

$$t_0 \geq t_\alpha = 2\alpha L/c\,, \qquad (1)$$

needed for a wave to travel to the near end of the string and back. Here L is the total string length, αL is the distance from the agraffe to the striking point, and c is the speed of waves on the ideal (flexible) string.

In spite of this, it may be established by using the experimental data presented in different articles that the inequality (1) is not hold for the first bass notes. For example in [6] the contact duration for $A1$ note is determined as $t_0 \simeq 3$ ms that is less than $t_\alpha \simeq 4.5$ ms for the same note.

Hall critically reviews his earlier work [4] in [1]: he considers the role of the critical compliance C_0, and shows the possibility *"that the hammer can rebound from the string without the aid of any reflected wave."* The reference to this article we can find in [7], where discussing the problem of the multiple contacts Askenfelt and Jansson wrote (p. 2189) that *"In the bass, where the hammer is relatively light compared to the string, the hammer may lose string contact without the assistance of reflected waves from the agraffe."*

Nevertheless, at present it is widely believed that only the reflected wave is the reason that the hammer leaves the string. Here we shall try to comprehend how this misunderstanding is appeared, and we shall also demonstrate which hammers can go away even from the infinite string.

2. Hammer-string interaction

The displacement $y(x,t)$ of the ideal (flexible) string obeys the simple wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}\,, \qquad (2)$$

where $c = \sqrt{T/\mu}$ is the wave speed in terms of tension and linear mass density of the string.

This equation is satisfied by arbitrary waveforms moving in both directions. At the contact point $x = 0$, we have

$$m\frac{d^2 w}{dt^2} = -F(t)\,, \qquad (3)$$

and

$$F(t) = m\frac{\partial^2 y_1}{\partial t^2} = m\frac{\partial^2 y_2}{\partial t^2} = -T\frac{\partial y_2}{\partial x} + T\frac{\partial y_1}{\partial x}\,, \qquad (4)$$

where m is the mass of the hammer, w is the displacement of the hammer, $F(t)$ is the acting force between the hammer and the string, and y_1 and y_2 are the outgoing waves created by the hammer-string interaction. To these equations we must add the relationship between the hammer and the string displacement, and the initial conditions taking into account the initial hammer velocity V

$$y_1(0,0) = y_2(0,0) = w(0) = 0\,; \quad \left.\frac{dw}{dt}\right|_{t=0} = V\,. \quad (5)$$

Now, let us consider the hammer-string interaction for the different types of hammer.

2.1. Rigid hammer

Consider an event of striking of the long flexible string by the absolutely rigid hammer. The solution of this problem was derived many years ago, and it is well known. In this case the string displacement is equal to the hammer displacement

$$y_1(0,t) = y_2(0,t) = w(t)\,, \quad (6)$$

and thus at the contact point $x = 0$, we have

$$\left.\frac{dw}{dt}\right|_{t=0} = \left.\frac{\partial y_1}{\partial t}\right|_{t=0} = \left.\frac{\partial y_2}{\partial t}\right|_{t=0} = V\,. \quad (7)$$

The solution of Eqs. (3, 4) representing the string disturbance at any moment $t > 0$ has the form shown in Fig. 1, where y_1 is the direct wave

$$y_1 = \frac{mcV}{2T}\left\{1 - \exp\left[\frac{2T}{mc^2}(x-ct)\right]\right\},\ (x-ct<0);$$

$$y_1 = 0\,, \qquad (x-ct>0)\,, \quad (8)$$

and y_2 is the return wave

$$y_1 = \frac{mcV}{2T}\left\{1 - \exp\left[-\frac{2T}{mc^2}(x+ct)\right]\right\},\ (x+ct<0);$$

$$y_1 = 0\,, \qquad (x+ct>0)\,. \quad (9)$$

In according to equality (6) the hammer position w at any moment $t > 0$ is defined by

$$w(t) = \frac{mcV}{2T}\left\{1 - \exp\left[-\frac{2Tt}{mc}\right]\right\}, \quad (10)$$

and the hammer velocity V_h at any moment

$$V_h(t) = V\exp\left(-\frac{2Tt}{mc}\right) > 0\,. \quad (11)$$

For this reason, it is obvious that the rigid hammer will never lose the contact with the string until at least one reflected wave returns to $x = 0$. Apparently, due to this well known interpretation of the process of the hammer-string interaction the prevalent misunderstanding exists that only the reflected wave is the cause the hammer to rebound.

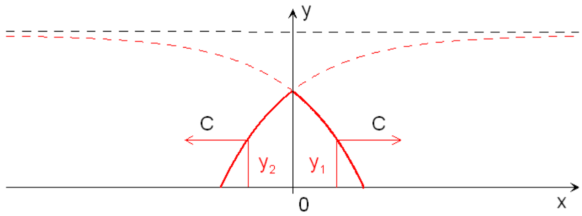

Figure 1: *The waves moving away from the hammer.*

2.2. Elastic hammer

To remove all doubts, consider the process of striking of the long flexible string by an elastic, or deformable hammer. This case is totally different from previous one. First of all, the force F is determined now by the hammer compression u, which is given by

$$u(t) = w(t) - y(0,t)\,. \quad (12)$$

Taking into account that $F(0) = 0$, and using (12) we have the initial conditions at the contact point $x = 0$

$$y(0,0) = w(0) = u(0) = 0\,, \quad (13)$$

and

$$\left.\frac{dy}{dt}\right|_{t=0} = 0\,; \quad (14)$$

$$\left.\frac{dw}{dt}\right|_{t=0} = \left.\frac{du}{dt}\right|_{t=0} = V\,. \quad (15)$$

In case of the rigid hammer the initial string velocity is the same as the hammer speed, but now it is equal to zero. In case of the elastic hammer the initial compression velocity is equal to hammer speed. Thus, in the beginning, the hammer is compressed first, and then the rising of the force compression causes the string acceleration. For the linear elastic hammer described by a function

$$F(u) = F_0\, u\,, \quad (16)$$

we can find the exact solution of the problem of the hammer-string interaction. Using Eqs. (3, 4, 12) we can find that the hammer displacement is determined by equation

$$\frac{d^2w}{dt^2} + 2b\frac{dw}{dt} + 2abw - 2bV = 0\,, \quad (17)$$

where $a = 2T/cm$ and $b = cF_0/4T$. Here F_0 is the hammer stiffness. In [1] the value F_0^{-1} is denoted as C and called the hammer compliance.

Introducing the definition

$$\delta^2 = 1 - \frac{2a}{b} = 1 - \frac{16\,T^2}{mc^2 F_0}\,, \quad (18)$$

it is easy to derive the different type of solutions of Eq. (17) depending on δ value.

2.2.1. *Case δ^2 = 0*

In the case when

$$\frac{2a}{b} = \frac{16\,T^2}{mc^2 F_0} = 1\,, \tag{19}$$

the solution of Eq. (17) is given by

$$w(t) = \frac{2V}{b}\left(1 - e^{-bt} - \frac{bt}{2}\,e^{-bt}\right), \tag{20}$$

and the hammer velocity V_h at any moment

$$V_h(t) = V(1+bt)\,e^{-bt} > 0\,, \tag{21}$$

and therefore, only the reflected wave can give the hammer a negative velocity.

2.2.2. *Case $\delta^2 > 0$*

In this case we have

$$\frac{2a}{b} = \frac{16\,T^2}{mc^2 F_0} < 1\,, \tag{22}$$

and the solution of Eq. (17) is given by

$$w(t) = \frac{V}{a}\left\{1 - e^{-bt}\left[\cosh(\delta bt) + \delta^{-1}\left(1-\frac{a}{b}\right)\sinh(\delta bt)\right]\right\}. \tag{23}$$

The hammer velocity in this case at any moment

$$V_h(t) = V[\cosh(\delta bt) + \delta^{-1}\sinh(\delta bt)]\,e^{-bt}\,, \tag{24}$$

is also always positive.

2.2.3. *Case $\delta^2 < 0$*

In this case we have

$$\frac{2a}{b} = \frac{16\,T^2}{mc^2 F_0} > 1\,, \tag{25}$$

and introducing the definitions

$$\gamma = \sqrt{\frac{2a}{b} - 1}\,; \qquad \psi = \gamma\, b\, t\,, \tag{26}$$

we can find the solution of Eq. (17) in the form

$$w(t) = \frac{V}{a}\left\{1 - \frac{e^{-bt}}{\gamma}\left[\gamma\cos\psi + \left(1-\frac{a}{b}\right)\sin\psi\right]\right\}. \tag{27}$$

The hammer velocity is given now by

$$V_h(t) = V(\cos\psi + \gamma^{-1}\sin\psi)\,e^{-bt}\,. \tag{28}$$

At the moment $t = t_\star$ determined by equality

$$\psi = \gamma\, b\, t_\star = \frac{\pi}{2} + \arcsin\sqrt{\frac{b}{2a}}\,, \tag{29}$$

the hammer stops shortly and after this moment will continue in motion, but in the opposite direction. The cause of this motion is the compression of the hammer, which acts now as a spring. At this moment $t = t_\star$ the hammer is compressed as much as possible

$$u_\star = V\sqrt{\frac{m}{F_0}}\,\exp(-b\,t_\star)\,. \tag{30}$$

Then the decompression is continued, and at the moment $t_* = \pi/\gamma b$ the hammer is totally decompressed and leaves the string with a constant velocity

$$V_h = -V\,e^{-\pi/\gamma}\,. \tag{31}$$

The string after this moment t_* will remain at rest, and its maximum deflection is determined by

$$y(0, t_*) = \frac{V}{a}(1 + e^{-\pi/\gamma})\,. \tag{32}$$

In this case no reflected wave is needed to help the hammer to rebound.

3. String and hammer parameters

The case which allows the hammer go away from the string without the aid of reflected wave requires

$$\mathcal{K} = \frac{16\,T^2}{mc^2 F_0} > 1\,, \tag{33}$$

which can be replaced by

$$\mathcal{K} = \frac{16\,M_s\,T}{m\,T_h} > 1\,. \tag{34}$$

Here $M_s = \mu L$ is the string mass, and $T_h = F_0 L$ is the so called hammer "tension" - the value, which characterized the hammer stiffness. It is evident, that this case is virtual not only for massive strings, but also for rather smooth hammers. Because the elastic features of the real hammers are essentially nonlinear, we consider the numerical simulation of the hammer-string interaction using a more complicated model of the hammer.

The hysteretic hammer model presented in [8], which is very similar to nonlinear Voigt model and that is consistent also with experiments is used here in the form

$$F(u(t)) = F_0\left[u^p + \alpha\frac{d(u^p)}{dt}\right]. \tag{35}$$

The parameters of this model were obtained experimentally in [8], and for numerical calculations these values may be approximated as the functions of the hammer number N in the form

$$\alpha = 248 + 1.83N - 5.5\cdot 10^{-2}N^2\,, \tag{36}$$

$$F_0 = 183\exp(0.045N)\,; \quad p = 3.7 + 0.015N\,. \tag{37}$$

Here the unit for retarded time α is (μs), and the unit for F_0 is (N/mm^p). The string parameters were taken from Table 1. *Scale of Estonia Parlour Grand Piano* [9].

4. Numerical simulation of piano strings

The description of a numerical method for the case of hysteretic hammer was presented in [10]. The process of the hammer-string interaction was computed for the first ten hammers and strings of *Parlour Grand*.

First of all, in Fig. 2 is displayed t_α dependence shown by black line and crosses. The contact durations

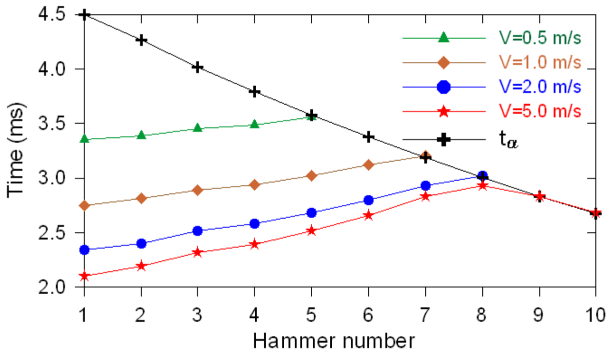

Figure 2: *Contact times as functions of hammer number.*

computed for the different initial hammer velocities are shown by colored lines and symbols. Due to the fact that the contact duration is decreased with increasing the hammer velocity, all the points calculated for the hammer velocity 5 m/s for all ten hammers are located under the black line. It means that all the hammers considered have time to decompress fully and leave the string without the assistance of reflected wave. For the ninth and tenth hammers the end of entirely decompression coincides with the reflected wave appearance.

Nevertheless, the result even of the tenth string simulation shown in Fig. 3 demonstrates that the hammer has a negative velocity, or in other words moves away from the string, before the moment t_α. Black circles mark these parts of the force histories. The moment of the reflected wave arrival and the beginning of the second contact are observed in Fig. 3 rather well.

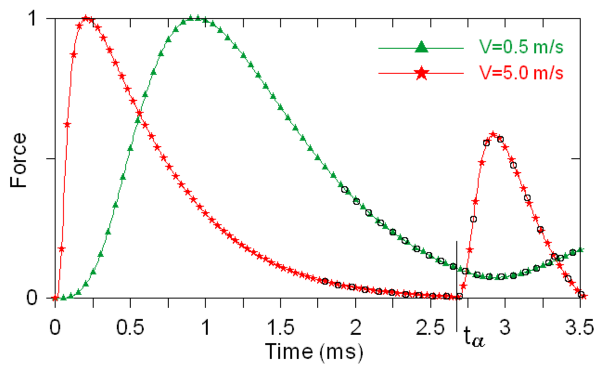

Figure 3: *Normalized force histories computed for hammer N=10.*

5. Conclusions

We have shown that the real piano hammer can leave at least the lowest ten strings of the piano without the aid of any reflected wave. At that, with the increasing of the initial hammer velocity the contact duration is decreased due to the hammer compression and decompression come to an end more rapidly. On the other hand, the hammer compresses hardly with the increasing of its stiffness. It means, that in frame of the nonlinear hysteretic hammer model the contact duration increases with increasing of parameter p. The numerical simulations confirm this effect completely.

6. References

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