Solitons modelled by Boussinesq-type equations

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Abstract

Boussinesq-type equations arise in many areas of fluid and solid mechanics where nonlinearities and dispersion are taken into account. In this paper the analysis of two Boussinesq-type models is presented. One model describes propagation of waves in microstructured solids and another one - waves in biomembranes. The main difference between these equations is the structure of the nonlinearities - in case of the microstructure model these are in terms of displacement gradients and in case of the biomembrane - in terms of displacements. Numerical analysis is carried out and differences in the solutions are discussed. Due to the nonlinear character of biomembranes made of lipids, the smaller solitons in biomembranes may travel faster than higher solitons.

Keywords: Boussinesq-type equation, solitons, microstructure, solids, biomembranes

1. Introduction

Contemporary continuum mechanics has many avenues brilliantly described by G.A.Maugin [1, 2]. Among these avenues is one entitled ‘wave motion’ with one part marked by a label ‘nonlinearity’, another one - by ‘dispersion’. As it is well-known, these physical properties taken into account simultaneously, lead to the concept of solitons and/or solitary waves. Again, G.A.Maugin has described soliton-type waves in many physical situations [3, 4]. The mainstream of studies on solitons is based on evolution equations like the celebrated Korteweg-de Vries (KdV) equation or its modifications. Describing waves in solids, one should pay attention also to Boussinesq-type models [4]. In this paper, based on the legacy of G.A.Maugin, the analysis of certain Boussinesq-type equations is presented for modelling waves in microstructured media.

The Boussinesq-type wave equations have the following characteristics [5]: (i) bi-directionality like in the classical wave equations; (ii) nonlinearity of any order; (iii) terms describing dispersion at least of the fourth order (cf. the second order main terms). The historical background of such models is described by Maugin [4] and Christov et al. [5] but nevertheless some remarks are in order.

First, the dispersion of waves may be caused either by geometrical or physical effects. The geometrical dispersion occurs in waveguides due to the influence of lateral surfaces and depends on the transverse dimensions of wave-guides [6, 7]. The physical dispersion in solids is caused by the existence of the microstructure of the material [3, 8, 9, - see also references therein]. In this case, the scale effects (the scale of a microstructure) are of importance. The governing equations for describing microstructural effects can be derived either from discrete or continuous basis. Starting from the discrete description (atomic structure of materials), the accuracy of governing equations depends on approximations [3, 4] and may lead in some cases to unstable models. The consistent modelling based on continuum theory [10, 11] gives physically realistic results involving both the potential and kinetic energies of a microstructure [8]. In this case, the inertia of a microstructure is explicitly accounted for. Combined with scale effects, the result is described by a hierarchical structure of governing equations [12]. It has been shown that such a modelling guarantees the stability of waves, i.e., the discrepancy that at higher frequencies the velocities are unbounded, is removed.

Second, the character of nonlinearities depends on properties of materials. In solid mechanics, the nonlinearities are usually dependent on the displacement gradient [3, 6, 7, 13]. In biomechanics, it has been recently suggested that the nonlinearities are of the displacement type [14]. This makes an essential difference in modelling of solitary waves when the dispersive and nonlinear effects are balanced [15].

In what follows, the analysis of two Boussinesq-type models is presented. The first case (Section 2) deals with mechanics of microstructured solids with a deformation-type nonlinearity and dispersive effects. This is a typical case of a Boussinesq-type model as stated by Christov et al. [5]. The second case (Section 3) is devoted to the analysis of deformation waves in biomembranes with a displacement-type nonlinearity and dispersive effects. This mathematical model compared with the model proposed by Heimburg and Jackson [14] is improved by taking into account the inertial effects caused by the lipid microstructure of a biomembrane [16]. Finally, in Section 4, the discussion is presented on differences and similarities of models analysed in previous Sections. The main attention is paid to the formation of solitons from arbitrary initial conditions.

2. Solitons in microstructured solids

Boussinesq-type equations do not only exist in case of water waves but can also arise in solid materials when the inherent microstructure is taken into account. Starting from lattice theory, the Boussinesq-type equation is derived by replacing the

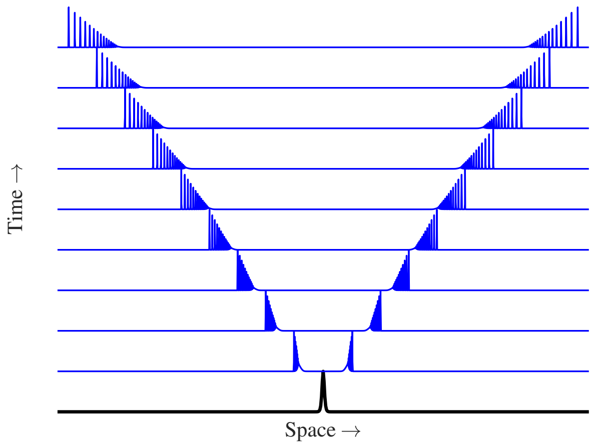

Figure 1: Formation of trains of solitons from pulse-type initial condition for Eq. (3). Right- and left-going structures are plotted at every $\Delta T = 1750$. Here $b = 0.7188$, $\mu = 1.1394$, $\delta = 0.09$, $\beta = 56$, $\nu = 9.3867$, $\lambda = 1.1470$. Width and amplitude of the pulse-type ($U_0 \mathrm{sech}^2(B_0 X)$) initial condition are $B_0 = 0.01$ and $U_0 = 2$ respectively [18].

discrete degree of freedom of the underlying crystal structure with continuous variable by using the Taylor expansion [3, 17].

In the theory of microstructured continua [8, 10] the macro- and microcontinuum are separated and the balance laws are formulated separately for the macro- and micro-scale. In terms of macrodisplacement u and microdeformation φ, the simplest free energy W function is a quadratic function

$$W = \alpha u_x^2 + A\varphi u_x + \frac{1}{2}B\varphi^2 + \frac{1}{2}C\varphi_x^2 + \frac{1}{6}Nu_x^3 + \frac{1}{6}M\varphi_x^3, \quad (1)$$

where α, A, B, C, N and M denote material parameters. Here and further the indices x, t (later X, T) denote differentiation with respect to these variables. The balance laws are derived from the Euler-Lagrange equations:

$$\rho u_{tt} = \sigma_x, \quad I\varphi_{tt} = \eta_x - \tau, \quad (2)$$

where ρ is the density, I is the microinertia, $\sigma = \partial W/\partial u_x$ is the Cauchy stress, $\eta = \partial W/\partial \varphi_x$ is the microstress and $\tau = \partial W/\varphi$ is the interactive force.

Introducing dimensionless variables $X = x/L_0$, $T = c_0 t/L_0$, $U = u/U_0$ where $c_0^2 = \alpha/\rho$ and U_0 and L_0 are certain constants (e.g. amplitude and wavelength of the initial excitation), along with geometrical parameters $\delta = (l_0/L_0)^2$ and $\varepsilon = U_0/L_0$, where l_0 is the characteristic scale of the microstructure and making use of the slaving principle (see [8] for details) a Boussinesq-type equation in terms of deformation ($V = U_X$) is obtained [18]

$$V_{TT} - bV_{XX} - \frac{\mu}{2}(V^2)_{XX} = \delta\left(\beta V_{TT} - \nu V_{XX} + \frac{\lambda\sqrt{\delta}}{2}(V_X)_X^2\right)_{XX}, \quad (3)$$

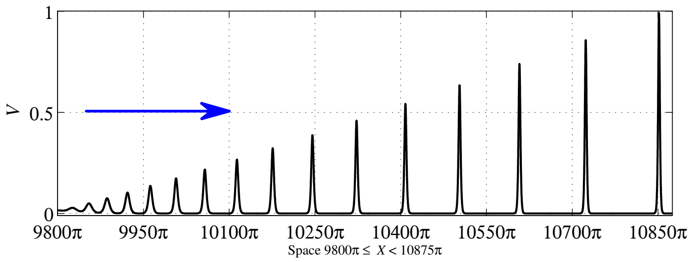

Figure 2: A soliton train solution of Eq. (3) in terms of deformation at $T = 16000$ propagating to the right. Here $b = 0.7188$, $\mu = 1.1394$, $\delta = 0.09$, $\beta = 56$, $\nu = 9.3867$, $\lambda = 1.1470$. Width and amplitude of the pulse-type ($U_0 \mathrm{sech}^2(B_0 X)$) initial condition are $B_0 = 0.01$ and $U_0 = 2$ respectively [18].

where $b = 1 - A^2/(\alpha B)$, $\mu = NU_0/(\alpha L_0)$, $\beta = IA^2/(\rho l_0^2 B^2)$, $\nu = CA^2/(\alpha B^2 l_0^2)$ and $\lambda = A^3 M U_0/(\alpha B^3 l_0^3 L_0)$ are constants. In case of $\lambda = 0$, Eq. (3) possesses a closed solution [7].

Due to the existence of nonlinearities and dispersion in Eq. (3), the possible balance between the effects caused by them may occur resulting in solitons. This can be demonstrated by solving Eq. (3) numerically under localised initial and periodic boundary conditions making use of the pseudospectral method (see [18] for details). The solution for Eq. (3) is depicted in Fig. 1 where it can be seen that the initial pulse $U_0 \mathrm{sech}^2(B_0 X)$, where U_0 and B_0 are the initial amplitude and width of the pulse, splits into two counter-propagating solitary trains. In Fig. 2 only the right-going solitary train is shown.

Due to the nonlinearities in microstructure ($\lambda \neq 0$) the solitary wave solution of Eq. (3) is asymmetric [19] and this property can be used for solving inverse problems of nondestructive evaluation of material properties [20].

3. Solitons in biomembranes

Boussinesq-type equations can also be derived for describing the deformation waves in lipid bilayers. It has been demonstrated by the experiments [21, 22] that a mechanical wave propagates along a nerve axon together with the action potential. A mathematical model describing such a wave was proposed by Heimburg and Jackson [14] and later improved by Engelbrecht et al. [16].

The starting point for a model is a wave equation in terms of longitudinal density change ($\Delta\rho_A = u$)

$$u_{tt} = (c_e^2 u_x)_x \quad (4)$$

and an assumption that effective velocity in biomembrane (c_e) is dependent of the density change as

$$c_e^2 = c_0^2 + pu + qu^2, \quad (5)$$

where c_0 is the velocity if low amplitude sound and p, q are coefficients determined from experiments. In addition an *ad hoc* fourth order term explaining elasticity of the microstructure in the biomembrane was added [14]. In order to account for the microinertia of the biomembrane, the fourth order mixed

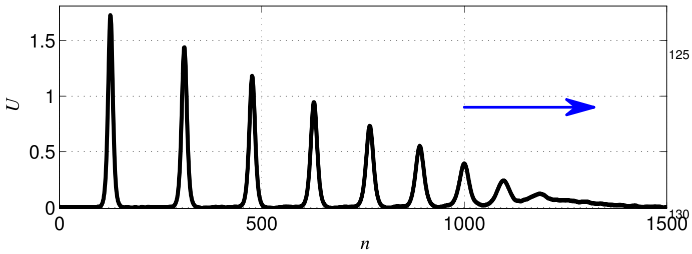

Figure 3: A soliton train solution of Eq. (6) at $T = 98001$ propagating to the right. Here $P = -0.2186$, $Q = 0.004230$, $H_1 = 72.14$, $H_2 = 1$.

derivative term has to be added [16]. The governing equation in the dimensionless form is then

$$U_{TT} = (1 + PU + QU^2)U_{XX} + (P + 2QU)U_X^2 - H_1 U_{XXXX} + H_2 U_{XXTT}, \quad (6)$$

where $X = x/l$, $T = c_0 t/l$, $U = u/\rho_A$ and $P = p\rho_A/c_0^2$, $Q = q\rho_A^2/c_0^2$, $H_1 = h_1/(c_0^2 l^2)$, $H_2 = h_2/l^2$. Here l is a certain length, for example, the fibre diameter. We also note that while mathematically there are no restrictions for the parameters P and Q, for the case of biomembrane the conditions $P < 0$ and $Q > 0$ are satisfied [14, 23]. Equation (6) has a closed solution [23]:

$$u(\xi) = \frac{6(c^2 - 1)}{P + \sqrt{P^2 + 6Q(c^2 - 1)} \cosh\left[\xi \sqrt{(1 - c^2)/(H_1 - H_2 c^2)}\right]}, \quad (7)$$

where $\xi = X - cT$ and c is the velocity of the solitary wave. Note that in this case the solitary wave solution is symmetric contrary to the case of similar wave in microstructured material modelled by Eq. (3) [19].

Like in Section 2, we shall solve the governing equation (6) at an arbitrary initial input. Solution for Eq. (6) can be seen in Fig. 3 where like in Fig. 2 only soliton train propagating to the right is depicted.

4. Discussion

One major difference between the solutions shown in Figs 2 and 3 is that the higher amplitude solitons travel faster in case of the solution of Eq. (3) while in the case of a solution of Eq. (6) with $P < 0$ in Fig. 3 the faster solitons are with smaller amplitudes. Another interesting fact is that the solitons with negative amplitude are even faster under such a parameter combination as used in Fig. 3 [23]. It should be noted that systems where smaller amplitude solitons travel faster are possible even in the case of the classical Boussinesq equation under some parameter combinations [24]. However, such a realistic parameter combination which would result for a smaller amplitude solitons to have greater velocity than the higher amplitude ones is not known for Eq. (3) currently.

What is similar for the solutions of both Eqs. (3) and (6) is that these solutions are strictly speaking not solitons but solitary waves as far as the interactions are not fully elastic [18, 23]. If the radiation from the interactions is small, such trains are often still called solitons. In both systems if the parameters of the system are suitable for solitonic solutions an arbitrary input will be decomposed normally into a train of solitons and a 'tail' of lower amplitude osccilations. However, it should be emphasised that the wave structure (for example, the number of emerging solitons in the train) and even the existence of a solitonic solution can be sensitive on some parameters of the initial condition [18, 23]. For example, for Eq. (6) it is possible that the amplitude of the initial condition (positive or negative) can change between the solution types of an solitary wave train and the oscillatory structure [23].

In general it can be noted that as both Eqs. (3) and (6) are of the Boussinesq type like stated by Christov et al. [5], the solutions can qualitatively be remarkably similar. However, the structure and type of the nonlinear and dispersive terms can add different nuances to the behaviour of the solutions, as demonstrated, for example, by a solution for Eq. (6) where smaller amplitude solitons can travel faster than the higher amplitude ones. Note also that the soliton of Eq. (3) are in terms of deformation and those of Eq. (6) are in terms of displacement.

The dispersion analysis of linear versions of Eqs (3) and (6) shows that when the nonlinear terms are neglected then the behaviour of Eqs (3) and (6) is identical with the exception of coefficients used in the equations. The linear version of Eq. (6) can be rewritten as [25]

$$U_{TT} = U_{XX} + H_2(U_{TT} - \gamma^2 U_{XX})_{XX}, \quad (8)$$

where $\gamma = H_1/H_2$ is the dimensionless bounding velocity related to the front of the soliton train. This form clearly shows that when elastic (the term $H_1 U_{XXXX}$) and inertial (the term $H_2 U_{TTXX}$) properties of the underlying structure are considered then the dispersive effects behave as an additional higher order wave operator. In case of microstructure (Eq. (3)) the bounding velocity is given by ν/β which are the parameters related to the properties of the microstructure.

In many cases the physical situation is best modelled by including nonlinearities into the governing equations [13]. This is reflected above in Eqs (3) and (6) derived following certain physical assumptions. However, in this context it would be of interest to discuss other possible nonlinearities in such systems. In case of microstructured solids, the modelling of martensitic-austenitic alloys leads to quadratic and quartic nonlinearities in governing equations [26]. As a result, solitonic structures may emerge. In case of seismic shear waves, the nonlinear body force capture the effect of attenuation for small amplitudes and amplification for higher amplitudes due to the releasing of embedded energy [13, 27].

In biomembranes, the assumption of effective velocity (Eq. (5)) includes a polynomial up to the quadratic order which describes displacements in a biomembrane close to the melting transition [14]. If such a dependence is described with a polynomial of a higher order then theoretically it would open more possibilities. The pseudopotential [23] which governs the solitonic solutions of the governing equation is then also of higher order. Consequently it may lead to more coexisting solution types including oscillatory ones. However, the displacements of biomembranes are limited to the possible phase change [14]. If displacements are larger then the temperature effects must also be taken into account [28]. It means that the present model (Eq. (6)) of waves with solitonic solutions is not valid anymore and must be changed.

In conclusion, one could say that while the Boussinesq paradigm, as outlined in [5] can give a good idea what to expect from the solutions, the material properties reflected in the finer structure of the equation under consideration are capable of influencing the behaviour of the solutions.

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