



Legacy of Nikolai Alumäe: theory of shells

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Abstract. A short overview on the research of Nikolai Alumäe (1915–1992) in the field of the theory of shells is presented. His brilliant analytical results explaining the stability and transient processes in shells have not lost their importance although obtained in the 1950s and 1960s.

Key words: buckling of shells, vibrations of shells, wave motion in shells.

1. INTRODUCTION



Nikolai Alumäe
(1915–1992)

The progress in naval and aeronautical engineering and technology produced growing interest in the theory of thin elastic shells in the middle of the 20th century. Shells are elements of constructions that are bounded by two curved surfaces the distance between which is small in comparison with their other dimensions. The governing equations of the shell theory are complicated and their analysis needs deep knowledge in the theory of differential equations.

Such an interest in the problems of thin-walled shell constructions is reverberated in many papers and monographs published in the 1930s and 1940s [1–5]. The fundamental ideas of the shell theory were formulated by Reissner [6] and Donnell [2] in the USA; Goldenveiser [7–9], Vlasov [10], and Novozhilov [11] in the

USSR; and by others. The ‘golden age’ in the development of the theory of thin elastic shells, especially in studying their stability and dynamics, was in the 1950s and 1960s. However, as then the computing capacity was not sufficient for solving such complicated problems, attention was directed to simplified theories (in comparison to the 3D model of elasticity) and analytical methods.

Shell theories have been experiencing some renaissance in recent years. The renewed interest in shell theories is on the one hand due to the potentials of smart materials, the challenges of adaptive structures, and the demands of thin-film technologies, and on the other hand, the availability of newly developed mathematical tools, the tremendous increase in computer capacity, and the improvement of commercial software packages. Several textbooks and research papers on shell theory were published at the end of the previous and at the beginning of this century [12,13].

In the 1950s and 1960s the shell theory was studied extensively also in Tallinn by Nikolai Alumäe and his co-workers. Their results were published mostly in Russian and are not widely known to the international community. For example, an overview on the buckling of thin shells by Teng from 1966 [14] does not include any results from the former Soviet Union because of the language barrier. However, in many cases Alumäe was able to solve very complicated problems. The aim of the

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present overview is to bring his excellent ideas and results again into the focus just before his 100th birth anniversary in 2015. A brief review of Alumäe's main results in the shell theory follows. His studies and results may be divided into three main groups: (i) stability of thin elastic shells, (ii) vibrations of shells together with the analysis of their stress states, and (iii) transient processes in shells.

2. STABILITY OF THIN ELASTIC SHELLS

In one of his first papers, Alumäe presented fundamental equations of an approximate nonlinear shell theory for thin elastic shells and conditions that reflected the limiting use of this theory for studying the behaviour of shells in the post-critical state [15]. These equations were based on the assumptions that (i) deformations of the shell are small despite the finite displacements in the post-critical state, (ii) the Kirchhoff–Love hypothesis is valid, and (iii) the governing stress state is of membrane type. The order of neglected terms was analysed qualitatively. The critical stress in the case of a membrane stress state for thin elastic shells may be presented by a similar formula in many cases: of a closed spherical shell under uniformly distributed normal pressure, of a circular cylindrical shell under central axial loading, and of a conical shell of revolution under central axial loading. The initial stress states for an arbitrary shell when similar formulae for critical stress may be obtained were described [16]. A specific form of the generalized Castigliano variation principle for the case of equilibrium of thin elastic shells in the post-critical state was presented [17]. An important result was derivation of a functional that has a stationary value if the normal component of the displacement vector of the middle surface of the shell v and stress function φ satisfy the basic equations. At the same time boundary conditions should be satisfied for the boundary problem of equilibrium of a thin elastic shell in a post-critical state [18]. The proposed method may often be used instead of the Galerkin method in the case of approximate determination of the equilibrium state in the post-critical state, recommended by Vlasov in his monograph [19].

Alumäe used the method of power series of the small parameter for the analysis of the post-critical state of the thin elastic shells [20]. Some examples were analysed in more details: (i) a square plate pressed in one direction, (ii) a circular cylindrical pipe under axial pressing force, and (iii) a circular cylindrical pipe under torsion.

Alumäe developed the ideas for the analysis of flat flexible shells that had been studied earlier with considerable results by Panov [21,22] and Feodosiev [23,24]. Feodosiev used successfully the Papkovich variational method for particular problems. The essence of this method is that the equations of continuity for the

fundamental system of differential equations are integrated exactly, but for the equations of equilibrium the Galerkin method is applied. If the shell is not flat, then, generally speaking, the Papkovich method is difficult to use and therefore one has to integrate also the equations of continuity approximately with the Galerkin method.

Here, however, some additional problems had to be solved. Namely, the Galerkin method assumes that the approximation functions must satisfy all the boundary conditions, including geometrical ones. However, in the case of the approximate integration of the deformation equations of continuity the uniqueness of the displacement field is not determined and the geometrical boundary conditions are not satisfied.

Alumäe exposed the reduction of the problem to the variational problem of the basic system of differential equations of axisymmetric deformations of shells of revolution in the case of finite deformations [25]. Further this variational formula was modified to the form of generalized variational equations of the Galerkin method. The conditions that afford to use the integration of the system of nonlinear differential equations to the variational equations of the Galerkin method in its traditional form were deduced directly from those equations.

In his monograph *General Theory of Shells and Its Applications in Engineering* [19] Vlasov pointed out the difference between shells of the revolution with a positive or a negative Gaussian curvature in the sense of materializing the membrane stress state. The solution of the problem of the membrane stress state in the case of a negative Gaussian curvature was given by Novozhilov [11]. He gave the conditions for the elimination of infinitely small bending of the middle surface of shells and showed that the same conditions cause the membrane stress state in the shell.

Alumäe determined the critical value of the axisymmetric membrane stress state for a long catenoid shell under contour loading. This study was an essential continuation of the problem solved by Novozhilov. Actually, such a problem was first raised by Federhofer [26], but his solution for the loss of stability was limited to the axisymmetric deformation in the case of a membrane stress state. Alumäe showed that his solution leads to higher values of critical loading [27]. This solution demonstrated once more the considerable difference between shells with a positive and a negative Gaussian curvature and pointed out the necessity to create a new version of basic equations and solution algorithms. The reason is that at the centre of the bulking of a catenoid shell the displacements of the middle surface are very small and deformation is similar to the bending of the middle surface. The calculations elicit the loss of accuracy due to small differences of large values in this situation. Alumäe presented a

qualitative analysis of the accuracy of the simplified equations in [27].

The non-axisymmetric state of equilibrium of a cylindrical shell of revolution of medium length under the external hydrostatic pressure and axial loading was studied by Alumäe in [28]. It was assumed that in the axisymmetric stress state forces from axisymmetric loading were smaller than or of the same order as from the external pressure (such a case exists if the loadings are rather small). Fundamental conditions and equations of nonlinear theory for the local loss of stability of the membrane stress state of the shell were presented. Alumäe proposed simplified equations and an equivalent variational formula for the estimation of the critical loading assuming that the character of deformation determined for the critical loading was valid in some initial stage of a post-critical state. The governing equations in terms of the stress function $F(\xi, \theta)$ and the displacement of the mean surface $w(\xi, \theta)$ are nonlinear and should be solved asymptotically. For example, in order to estimate the role of boundary effects, the governing system reads

$$\frac{1}{t} F'''' + w'' + w'' w'' - w' w' = 0,$$

$$\frac{1}{t} F'' - \lambda^2 w'''' + \frac{1}{t} (F'' w'' + F'' w'' - 2F' w') = 0.$$

Here t is the thickness of the shell, λ is the coefficient characterizing material of the shell, and

$$(\dots)' = \partial(\dots)/\partial\xi, \quad (\dots)^{\bullet} = \partial(\dots)/\partial\theta.$$

The derivation of simplified equations was based on the asymptotic properties of the solution of the exact equations (in the sense of the theory of the local loss of stability of the membrane state of a shell). Therefore, the proposed method gives more natural results when the ratio of the thickness of the shell to the radius of the curvature of the middle surface of the shell is small. Alumäe demonstrated that by taking the edge effect of axisymmetric stress state into consideration, the method gives only negligible corrections to the value of the critical load determined by the membrane theory [28].

In the framework of assumptions made, a stress state consists of a membrane state and a mixed stress state. In the case of thin shells the mixed stress state occurs only in the boundary zone or in the zones where the thickness of the shell is changing gradually. A membrane stress state may have local character only when the external pressure acts in the limited zone and the axial loading is relatively small.

A non-axisymmetric form of equilibrium of the shell possible at certain values of loading parameter was described concerning the existing axisymmetric form

that may exist at these loading parameters. In the studies to establish non-axisymmetric forms of equilibrium of the shells of revolution the axisymmetric component of the membrane state is taken as a leading factor, but the initial state and edge effects are not taken into consideration. Such an approach is not valid in the case of very flat shells.

Alumäe showed that the determination of critical loading for long and medium-length cylindrical shells under torsion is equivalent to the integration of the fourth order equation that satisfies two boundary conditions on contours of the middle surface [29].

Alumäe was certainly not the only one who studied such complicated problems. For example, in 1945 Goldenveiser showed that there is a simple analogy in the linear shell theory between static and geometric relations [8]. Later Goldenveiser introduced a non-symmetric metric tensor of deformation in order to guarantee that components of tensors of deformation are the energetic components of deformation and the analogy between static and geometric relations is retained [30]. Another version showing that basic relations of linear shell theory have analogy between static and geometric relations and with energetic components was presented by Novozhilov [11] and Lurye [31]. Alumäe proposed a version of basic relations of nonlinear shell theory that is an analogue to Novozhilov's version of the linear shell theory: formal symmetrical tensors of tangent forces and moments and the symmetrical deformation tensor were introduced in such a way that quite a simple analogy between static and geometric relations existed and the components of the deformation tensor were also energetic components [32].

Alumäe presented a method for determining the critical pressure for an elastic thin shell of revolution with a boundary of a hyperboloid of one sheet, symmetrical to the thinnest part of the surface, with exclusive flat bottoms rigid in plane but flexible in bending out of its plane [33]. The case of uniformly distributed external pressure was studied and algorithms for calculations were proposed.

Alumäe showed that the problem of determination of the critical loading of a thin elastic conical shell of revolution under all-round external pressure may be reduced to the determination of the smallest eigenvalue of the system of ordinary differential equations under given boundary conditions. The problem may be simplified and the asymptotical integration method was used for obtaining a solution in the case of a very thin shell [34]. This solution allowed analysis of the accuracy of the use of a simplified system. As an example, a shell closed in the apex was studied.

A shell on a helical surface surrounded by asymptotic contour lines was also studied. The critical loading and the shell stress state after the loss of stability 'in small' were established. A simplified governing equation was

proposed and then integrated by the asymptotic method [35].

3. VIBRATIONS OF SHELLS

Alumäe studied small stationary elastic axisymmetric vibrations of a truncated conical shell of revolution in [36], and noted that in some frequency range the equations of membrane vibrations have a branching point in the interval of integration. The membrane theory gives infinite amplitudes for displacements at this branching point. In the interval of determination of the independent variable (a coordinate point on the middle surface) there are no transition points in any frequencies of vibrations if the membrane vibrations have linear damping. In spite of this, there are regions near these points in the complex plane of arguments in some definite range of frequencies, and consequently the stress state changes rapidly. It is quite natural to suppose that in some neighbourhood of the branching point the membrane equations do not describe movements correctly and one has to use the general theory, which in this problem involves the determination of edge effects. At once a question arises: What result can be expected when correcting the membrane solution using a general theory? It appears that the question is not formulated correctly assuming that the general theory smooths out singularities of the membrane theory. This assumption is not correct because the membrane solution is not unique in the complex plane of arguments.

The study of the small stationary elastic axisymmetric vibrations of a truncated conical shell of revolution leads to the integration of the ordinary linear differential equation of the sixth order in terms of displacement Y :

$$-\varepsilon^4 \left\{ A_6(x) \frac{d^6 Y}{dx^6} + \dots + A_2(x) \frac{d^2 Y}{dx^2} \right\} + B_2(x) \frac{d^2 Y}{dx^2} + B_1(x) \frac{dY}{dx} + B_0(x) Y = f(x).$$

Here ε is a small parameter and coefficients $A_i(x)$ and $B_j(x)$ depend on the parameters of the shell and $f(x)$ is related to loading.

If parameter $\varepsilon = 0$, then one has an equation of membrane vibrations with the branching point $x = 0$. In the neighbourhood of this point the character of the stress state of the shell has peculiarities that are not in conformity with the main assumptions of the membrane theory of the shells. Alumäe pointed out that in such a case more effective methods for the analysis of shells should be developed. Note that sixth-order differential equations with variable coefficients were practically not studied at that time. For the investigation of these

problems Alumäe derived a simplified equation in the form

$$-\varepsilon^4 \{ y^{(6)} + d_1 y^{(5)} + d_0 y^{(4)} \} + zu^{(2)} + (2 + d_1 z) y^{(1)} + (c_0 + d_0 z) y = 0$$

as a ‘model equation’ in the neighbourhood of the branching point. Here y and z are used in the same sense as Y and x in the basic equation and

$$y^{(i)}(z) = \partial^i y(z) / \partial z^i.$$

The model equation afforded to present the solution in the form of contour integrals from which one should obtain asymptotic presentations for integrals with a large index of variability using the saddle point method. The integrals of the model equation were used for formal construction of integrals of the basic equation using known methods of asymptotic integration.

Alumäe concluded that on the basis of detailed mathematical analysis it is possible to forecast basic features of solutions that behave somewhat unusually for integrals of the shell theory. Some of these integrals have quite a complicated character. On one hand, the integral of the membrane theory occurs in the formula for boundary effects and on the other hand, the oscillating part of the solution occurs in the formula for the membrane state.

4. TRANSIENT PROCESSES IN SHELLS

In the case of the slowly applied axisymmetric load the stress state of quite a wide class of shells of revolution may be separated into the membrane state and the boundary effects [37–39]. The non-stationary wave processes in the shells under fast loading were practically not studied at that time.

To clarify the basic phenomena of vibrations during the starting stage of motion Alumäe studied the problem of non-stationary vibrations of half-infinite circular cylindrical shells under sinusoidal boundary loading [40]. As a mathematical model for describing axisymmetric motions of a circular cylindrical shell the following simple system of linear differential equations of hyperbolic type was used

$$\begin{aligned} \frac{\partial^2 u}{\partial \alpha^2} - \frac{\partial^2 u}{\partial \tau^2} - v \frac{\partial w}{\partial \alpha} &= 0, \\ v \frac{\partial u}{\partial \alpha} + k^2 \frac{\partial^2 w}{\partial \alpha^2} - \frac{\partial^2 w}{\partial \tau^2} + \frac{k^2}{\varepsilon} \frac{\partial v}{\partial \alpha} &= 0, \\ -\frac{k^2}{\varepsilon} \frac{\partial w}{\partial \alpha} + \frac{\partial^2 v}{\partial \alpha^2} - \frac{\partial^2 v}{\partial \tau^2} - \frac{k^2}{\varepsilon} v &= 0. \end{aligned}$$

Here u , v , and w are dimensionless deformations, α is a dimensionless coordinate, and τ is dimensionless time, which are defined in [40].

These Timoshenko-type shell equations had been proposed already earlier by Herrmann and Mirsky [41]. However, until the 1960s they were used only for the determination of phase and group velocities of the distribution of elastic disturbances.

It was interesting to estimate the limits of the application of simplified equations for vibrations in the case of fast processes in the shells. The calculations on the basis of the equations of hyperbolic type are considerably more complicated than the calculations on the basis of theories based on the hypothesis of Kirchhoff–Love. Alumäe presented a solution of this system of equations in the form of contour integrals using the Laplace transform technique [40]. That particular study elucidated the role of these integrals in the solution.

The main conclusion presented by Alumäe was the following: for the solution of such a problem the separation of a general stress state into a membrane state and boundary effects is possible (i) if a period of the change of loading is longer or commensurable with the time during which an elastic wave passes the distance equal to the radius of the middle surface, and (ii) if the aim is only determination of maximal displacements and stresses. These elementary states may be determined by simplified computational relations using the equilibrium theory with added inertial terms. This situation often allows separate solution of mixed (with initial and boundary conditions) problems of axisymmetric vibrations of the shell when first the membrane problem is solved and then the boundary effects are taken into account.

Practically there is a need to use hyperbolic equations only in case the characteristic period of loading is very short. In the limiting case (in the sense of shell theory) when the period is commensurable with the time when an elastic wave passes a distance equal to the thickness of the shell, there is no two-dimensional theory that describes correctly the vibrations of the shell at the beginning of wave motion.

The question whether a membrane theory can determine tangential characteristics of shell deformation in a transient process was also of Alumäe's interest. As a special case, he studied membrane stresses in a closed circular cylindrical shell due to sinusoidally distributed membrane edge forces that are suddenly applied in time and maintain a constant value [42]. The Timoshenko-type linear shell theory and the procedure of the Laplace transform were used. For establishing the early behaviour the inverse integral was evaluated by a rational approximation, and for finding stresses for a longer time the saddle-point method was used. An analysis indicated that transient membrane stresses in a thin shell at early times may be obtained with the aid of the dynamic membrane theory of shells; for longer transient times the semi-membrane dynamic theory of shells including circumferential moments and shear should be used.

Qualitative analysis of stress states in shells was carried out by Goldenveiser [30] for quite a general case when the distortion line of the stress state tangent to the characteristics of the system of the differential equations is singular, and the perturbations evolving near the tangent point do not localize but propagate along the asymptotic lines. The static problem of the shell theory solved by Goldenveiser does not contain enough examples of that phenomenon because the general moment theory is of elliptic type and only its degeneration, the membrane theory, is described by equations that have real characteristics in the case of negative or zero curvature. In the Timoshenko-type theory, used quite often for solving dynamic problems, governing equations are of hyperbolic type and the effects connected with tangent intersections of lines of perturbation and characteristics must emerge quite clearly. As an example, Alumäe studied a one-dimensional problem of the behaviour of a spherical segment of the shell of revolution under a plane pressure wave [43]. He assumed that (i) the front of a pressure wave is moving with a constant speed in the direction of the axis of the shell; (ii) the pressure behind the wave front remains constant (interactions of the wave and the shell are not considered); and (iii) the value of the pressure is such that only small deformations are initiated. In such a case the lines of perturbation of a stress state are caused by the pressure front moving with a changing speed along the shell and the perturbation lines in the coordinate-time plane are tangent to the characteristics of the system of differential equations in two points. Physically this means that the speed of the pressure wave at these points on the shell surface is equal to the speed of the compressional wave or to the speed of the shear wave, respectively. Asymptotical analysis showed that the solution has discontinuities at the fronts of the propagating waves. Quite a strong discontinuity is generated in the normal component of acceleration. Study of the characteristics of discontinuities may be useful for composing computing algorithms using simple numerical methods for smooth functions [43].

5. FINAL REMARKS

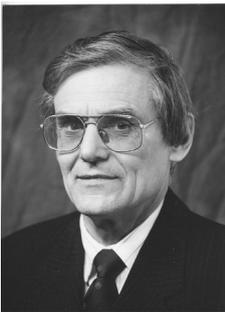
The mathematical models derived and studied by Nikolai Alumäe were mostly related to high order partial differential equations (PDEs) or their systems, both linear and nonlinear, which were based on clear physical considerations. His ingenious ideas for the analysis of such complicated PDEs were at the front of research of his time. His results have not lost their importance because they included explanation of many specific phenomena such as the rates of changing the variables, the existence of possible branching points, the possible existence of discontinuities or the infinite

values of some variables, just to name a few. Such effects should be understood also today although powerful computational methods are used in solving many problems. For example, the buckling of thin metal shells got much attention recently because shell buckling is an important element in the design of deep-space vehicles [44]. NASA experiments have shown that studies on Shell Buckling Knockdown Factor (SBKF) enable significant weight savings of space vehicles. This shows clearly that Alumäe's ideas were ahead of his time.

It is clear even from this brief overview that Alumäe was a brilliant researcher and his papers were written in a transparent way combining physics with mathematical analysis. Here we also tried to explain not only his results but described the background and studies of his colleagues from Moscow and Leningrad (now St. Petersburg). As a leader of the Estonian research community in mechanics, Alumäe created the Estonian National Committee for Mechanics, a member of the International Union of Theoretical and Applied Mechanics (IUTAM). Alumäe was also the founder of the Institute of Cybernetics (1960) – an interdisciplinary research centre for computer science and mechanics.

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Nikolai Alumäe pärand: koorikute teooria

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On esitatud lühike ülevaade Nikolai Alumäe (1915–1992) uuringute tulemustest koorikute teooria valdkonnas. Tema suurepärase analüütilised tulemused koorikute stabiilsuse ja võnkeprotsesside selgitamisel ei ole oma aktuaalsust kaotanud, kuigi need on pärit 20. sajandi 1950. ning 1960. aastatest. Ülevaade tähistab N. Alumäe 100. sünniaasta-päeva.