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Weakly Nonlocal Thermoelasticity for Microstructured Solids: Microdeformation and Microtemperature

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Dedicated to Professor G.A.Maugin on His 70th birthday

Abstract Prediction of the thermoelastic behavior of microstructured materials suggests a more general description of thermal processes in addition to the generalized continuum description extending the conventional continuum mechanics for incorporating intrinsic microstructural effects. Double dual internal variables are introduced in order to couple inertial microstructural effects like microdeformation and diffusive microstructural effects like microtemperature. The full coupled system of governing equations provides a complete extension of the classical thermoelasticity theory onto the case of microstructured solids.

Keywords Thermoelasticity · Microstructure · Microdeformation · Microtemperature · Dual internal variables

1 Introduction

Thermoelasticity is a well-known example of coupled phenomena. The classical thermoelastic theory combines the elastic behavior of homogeneous body with heat conduction governed by Fourier's law [1–3, e.g.]. For more realistic microstructured solids, the influence of a microstructure should be taken into account both for the elastic deformation and for the temperature distribution.

The difference between elastic deformation and heat conduction is that the former is a conservative process without dissipation whereas the latter is always a dissipative one. The theoretical description of elasticity in bodies with

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microstructure is known from 1960ties [4, 5]. At the same time, generalizations of the description of heat conduction are aimed in a non-Fourier hyperbolic heat propagation in homogeneous bodies [6–10, e.g.]. Only few papers are devoted to thermoelasticity of microstructured solids [11, 12] excluding formal exploitations of the Grot assumption [13] who introduces the microtemperature as a vectorial quantity.

Among theories of higher order generalized continua, the most developed is the description of micromorphic media [14, e.g.]. This approach is even bridged to atomistic representation of continuum mechanics [15, 16], but it is very complicated to be used in practice due to a large amount of unknown material parameters. Instead, effective media or homogenization technique [17–21, e.g.] is used commonly. Similar methods are applied to heat conduction in microstructured solids [22–24]. Unfortunately, homogenization methods, well elaborated for statics and/or for periodic composites, are not similarly suitable for wave propagation problems [25]. It is worth, therefore, look at alternative approaches.

The introduction of a microdeformation in the consideration [4, 5] means that the corresponding theory is a multi-field one [26]. The size of elements of a microstructure is characterized by a scale which is definitely much less than the macroscale of a specimen. Therefore, the corresponding theory is a multi-scale one by definition. The main difficulty in the multi-field and multi-scale theory is the formulation or derivation of balance equations at the microscale [26, 27]. A constructive solution of such a problem can be found in the internal variables approach [28–30, e.g.]. However, the classical theory of internal variables is concerned with dissipative processes [28, 29].

The uniform treatment of the internal variable theory in case of dissipative and non-dissipative processes is achieved by means of the dual internal variable theory [31]. This theory is successfully applied for the description of dispersion effects in wave propagation in elastic solids [32–34] and of dissipation effects due to microstructure in thermoelastic solids [35, 36]. In the latter case, the dissipation is associated with microtemperature, i.e. fluctuations of temperature due to the difference of thermal characteristics of the macro- and microstructure in a solid [37, 38]. The dual internal variable theory provides a novel approach to generalized mechanics [33, 39]. Compared to the classical approaches based on variational principles [4] or homogenization [5] it has several advantages. Here evolution equations for variables describing the microstructure are direct consequences of the second law without any further assumptions. There is not need to postulate the balance structure [40] neither a variational principle for the internal variables [29]. Moreover, the Euler-Lagrange form of the evolution equations can be derived in the non-dissipative limit [41], therefore, the usually problematic connection of variational principles and dissipative differential equations [42–44] is clear and straightforward with the help of dual internal variables. When compared to homogenization methods (see e.g. [5], or [45]) we want to emphasize that the structure of dissipative terms suggested by our direct thermodynamic derivation and by the homogenization procedure may be completely different [39]. In the previous treatments the influence of microdeformation and that of microtemperature were considered separately. For the complete thermoelastic theory of microstructured solids, we need to introduce dual internal variables both for microdeformation and for microtemperature simultaneously. This results in a double dual internal variable approach.

In this paper, the double dual internal variables approach is applied for deriving a complete mathematical model of weakly nonlocal thermoelasticity of microstructured solids. The one-dimensional case is chosen in order to present the idea and the governing equations as simple as possible. In Section 2 the concepts of the classical linear thermoelasticity are briefly described in order to create a basis for comparison. Section 3 is devoted to dual internal variables and to their interpretation. The focal point of this paper is Section 4 where the idea of double dual internal variables is used for deriving the governing equations which describe effects of microdeformation and microtemperature simultaneously. Finally, in Section 5 the possible variations of governing equations are given and the importance of the new full system of governing equations is underlined.

2 Governing equations

Governing equations for linear thermoelasticity in the Piola-Kirchhoff and in the material formulation are used in parallel. The balance of linear momentum used as the equation of motion in practical computations is formulated in the Piola-Kirchhoff framework, whereas the material framework provides the transparent introduction of internal variables and their evolution equations. For simplicity, no viscous effects included.

2.1 Piola-Kirchhoff formulation

In the Piola-Kirchhoff formulation, the one-dimensional motion of thermoelastic conductors of heat with no body forces is governed by local balance laws for linear momentum and energy

$$(\rho_0 v)_t - \sigma_x = 0 \tag{1}$$

$$\left(\frac{1}{2}\rho_0 v^2 + E\right)_t - (\sigma v - Q)_x = 0 \tag{2}$$

and by the second law of thermodynamics

$$S_t + \left(\frac{Q}{\theta} + J\right)_x \geq 0 \tag{3}$$

Here ρ_0 is the matter density in the reference configuration, σ is the one-dimensional stress, v is the particle velocity, Q is the heat flux, E is the internal energy per unit volume, S is the entropy per unit volume, θ is temperature, J is the extra entropy flux, subscripts denote derivatives. The extra entropy flux is considered here in the sense of Maugin [46] (see also [47]).

2.2 Material formulation

The canonical energy equation is derived from Eq. (2) by introducing the free energy per unit volume $W := E - S\theta$ and taking into account balance of linear momentum (1)

$$(S\theta)_t + Q_x = h^{int}, \quad h^{int} := \sigma\varepsilon_t - W_t \tag{4}$$

where $\varepsilon = u_x$ is the one-dimensional strain measure, u is the displacement.

Multiplying Eq. (1) by u_x we then check that Eq. (1) yields the following canonical balance of material momentum (cf. [48])

$$\mathcal{P}_t - b_x = f^{int} + f^{inh} \tag{5}$$

where the material momentum $\mathcal{P}$, the material Eshelby stress b, the material inhomogeneity force f^{inh}, and the material internal force f^{int} are defined by [48]

$$\mathcal{P} := -\rho_0 u_t u_x, \quad b := -\left(\frac{1}{2}\rho_0 v^2 - W + \sigma\varepsilon\right) \tag{6}$$

$$f^{inh} := \left(\frac{1}{2}v^2\right)(\rho_0)_x - W_x|_{expl}\,, \quad f^{int} := \sigma u_{xx} - W_x|_{impl} \tag{7}$$

In the case of non-zero extra entropy flux, the second law of thermodynamics gives

$$-\left(W_t + S\theta_t\right) + \sigma\varepsilon_t + (\theta J)_x - \left(\frac{Q}{\theta} + J\right)\theta_x \geq 0 \tag{8}$$

2.3 Classical thermoelasticity in homogeneous solids

In the classical linear thermoelasticity, the free energy depends on strain and temperature [3, e.g.]

$$W(\varepsilon, \theta) = \frac{1}{2}(\lambda + 2\mu)\,u_x^2 - \frac{\rho_0 c_p}{2\theta_0}(\theta - \theta_0)^2 + m\,(\theta - \theta_0)\,u_x \tag{9}$$

where c_p is the heat capacity, the thermoelastic coefficient m is related to the dilatation coefficient a and the Lamé coefficients λ and μ by $m = -a(3\lambda + 2\mu)$, θ_0 is the reference temperature.

Accordingly, the one-dimensional stress and entropy are defined as

$$\sigma := \frac{\partial W}{\partial u_x} = (\lambda + 2\mu)\,u_x + m\,(\theta - \theta_0) \tag{10}$$

$$S =: -\frac{\partial W}{\partial \theta} = \frac{\rho_0 c_p}{\theta_0}(\theta - \theta_0) - m u_x \tag{11}$$

Dissipation inequality (8) with zero extra entropy flux is reduced to

$$-\left(W_t + S\theta_t\right) + \sigma\varepsilon_t - \frac{Q}{\theta}\theta_x = -\frac{Q}{\theta}\theta_x \geq 0 \tag{12}$$

and the standard choice of the heat flux to satisfy the inequality is the Fourier law of heat conduction

$$Q = -k\theta_x \tag{13}$$

where $k > 0$ is the thermal conductivity of the material.

The internal heat source h^{int} in the classical case is calculated as

$$h^{int} := \sigma\varepsilon_t - W_t = S\theta_t \tag{14}$$

Then energy balance (4) is represented in the form

$$(S\theta)_t + Q_x - h^{int} = S_t\theta - (k\theta_x)_x = \left(\frac{\rho_0 c_p}{\theta_0}\theta_t - m u_{xt}\right)\theta - (k\theta_x)_x = 0 \tag{15}$$

which for small deviations from the reference temperature determines the heat conduction equation

$$\rho_0 c_p \theta_t - (k\theta_x)_x = m\theta_0 u_{xt} \tag{16}$$

The balance of linear momentum is, accordingly,

$$\rho_0 u_{tt} = (\lambda + 2\mu)\, u_{xx} + m\theta_x \tag{17}$$

Governing equations of classical thermoelasticity (16) and (17) are coupled but do not contain any influence of a microstructure.

3 Thermoelasticity in solids with microstructure

As it was mentioned, the influence of a microstructure on the thermoelastic processes is expected to be taken into account by means of an internal variable field [28–30]. We start with the single internal variable theory which will serve as the pattern for further generalizations. Initially, internal variables were introduced as local fields only [28]. We will follow its weakly non-local extension elaborated by Maugin [46].

3.1 Single internal variable theory

It is assumed that an internal variable φ is associated with the influence of a microstructure on the global thermoelastic motion. In the weakly nonlocal theory, the free energy W is specified as a sufficiently regular function of the strain, temperature, the internal variable, and its space gradient [46]

$$W = \overline{W}(u_x, \theta, \varphi, \varphi_x) \tag{18}$$

In a second order weakly nonlocal theory such a dependence is the consequence of the entropy inequality [31, 39, 47]. In addition to equations of state (10), (11), we define partial derivatives of the free energy with respect to the internal variable and its gradient as

$$\sigma = \frac{\partial \overline{W}}{\partial u_x}, \quad S = -\frac{\partial \overline{W}}{\partial \theta}, \quad \tau := -\frac{\partial \overline{W}}{\partial \varphi} \quad \eta := -\frac{\partial \overline{W}}{\partial \varphi_x} \tag{19}$$

The introduction of the internal variable results in a modification of governing equations (16) and (17). To achieve this modification, we start with dissipation inequality (8), which is represented as

$$\begin{aligned}
&-(W_t + S\theta_t) + \sigma\varepsilon_t + (\theta J)_x - \left(\frac{Q}{\theta} + J\right)\theta_x = -\frac{\partial \overline{W}}{\partial u_x} u_{xt} - \\
&-\frac{\partial \overline{W}}{\partial \theta}\theta_t - \frac{\partial \overline{W}}{\partial \varphi}\varphi_t - \frac{\partial \overline{W}}{\partial \varphi_x}\varphi_{xt} - S\theta_t + \sigma\varepsilon_t + (\theta J)_x - \left(\frac{Q}{\theta} + J\right)\theta_x = \\
&= (\tau - \eta_x)\varphi_t + (\eta\varphi_t + \theta J)_x - \left(\frac{Q}{\theta} + J\right)\theta_x \geq 0
\end{aligned} \tag{20}$$

Following the scheme originally developed by Maugin [49] for materials with diffusive dissipative processes described by means of internal variables of state, we select the extra entropy flux J in order to eliminate the divergence term in Eq. (20)

$$J = -\theta^{-1}\eta\varphi_t \tag{21}$$

In this case, the internal material force and heat source each split in thermal and intrinsic terms according to

$$f^{int} = f^{th} + \widetilde{f}^{intr}, \quad h^{int} = h^{th} + \widetilde{h}^{intr} \tag{22}$$

where the introduced thermal source terms involve only temperature

$$f^{th} := S\theta_x, \quad h^{th} := S\theta_t \tag{23}$$

whereas intrinsic source terms are determined by the internal variable

$$\widetilde{f}^{intr} := (\tau - \eta_x)\varphi_x, \quad \widetilde{h}^{intr} := (\tau - \eta_x)\varphi_t \tag{24}$$

so that we have the following consistent canonical equations of momentum and energy:

$$\mathcal{P}_t - \widetilde{b}_x = f^{th} + \widetilde{f}^{intr}, \quad (S\theta)_t + \widetilde{Q}_x = h^{th} + \widetilde{h}^{intr} \tag{25}$$

with dissipation inequality

$$\varPhi = (\tau - \eta_x)\varphi_t - \frac{\widetilde{Q}}{\theta}\theta_x \geq 0 \tag{26}$$

where we have introduced the modified heat flux [46]

$$\widetilde{Q} = Q - \eta\varphi_t \tag{27}$$

and the modified Eshelby stress tensor

$$\widetilde{b} = -(\rho_0 v^2/2 - W + \sigma u_x - \eta\varphi_x) \tag{28}$$

which includes all effects presenting gradients since the material gradient of φ plays a role parallel to that of the deformation gradient u_x. The latter demonstrates the main advantage of the use of the material formulation of thermomechanics: the conservation of the form of canonical balance laws and a clear separation of thermal and intrinsic dissipation.

If intrinsic dissipation is still independent of temperature gradient, then we are forced to modify the Fourier law

$$Q - \eta \varphi_t = -k\theta_x \tag{29}$$

The standard choice to provide the non-negativity of the intrinsic part of dissipation inequality (26)

$$\Phi = (\tau - \eta_x)\varphi_t \geq 0 \tag{30}$$

leads to the evolution equation for the internal variable in the form

$$\varphi_t = d(\tau - \eta_x), \quad d \geq 0 \tag{31}$$

The obtained reaction-diffusion-like equation can be found under different names in numerous applications. For example, for the quadratic free energy dependence

$$\overline{W} = \frac{1}{2}\left(\lambda + 2\mu\right) u_x^2 - \frac{\rho_0 c_p}{2\theta_0}\left(\theta - \theta_0\right)^2 + m\left(\theta - \theta_0\right) u_x + A\varphi u_x + \frac{1}{2}B\varphi^2 + \frac{1}{2}C\varphi_x^2 \tag{32}$$

stress components $(19)_{3,4}$ are determined as follows:

$$\sigma = \frac{\partial \overline{W}}{\partial u_x} = \left(\lambda + 2\mu\right) u_x + m\left(\theta - \theta_0\right) + A\varphi, \quad \eta = -\frac{\partial \overline{W}}{\partial \varphi_x} = -C\varphi_x \tag{33}$$

and τ coincides with the interactive internal force

$$\tau = -\frac{\partial \overline{W}}{\partial \varphi} = -Au_x - B\varphi \tag{34}$$

Consequently, the balance of linear momentum is rewritten as

$$\rho_0 u_{tt} = \left(\lambda + 2\mu\right) u_{xx} + m\theta_x + A\varphi_x \tag{35}$$

and evolution equation for the internal variable (31) is the Ginzburg-Landau (or the Allen-Cahn) equation (cf. Cross and Hohenberg [50])

$$\frac{1}{d}\varphi_t = C\varphi_{xx} - Au_x - B\varphi \tag{36}$$

Correspondingly, energy balance equation $(25)_2$ for small deviations from the reference temperature is represented in the form

$$\rho_0 c_p \theta_t - (k\theta_x)_x = m\theta_0 u_{xt} + (C\varphi_{xx} - Au_x - B\varphi)\varphi_t = m\theta_0 u_{xt} + \frac{1}{d}\varphi_t^2 \tag{37}$$

The influence of the microstructure in the single internal variable theory manifests itself in a dissipative manner. Such an internal variable cannot be associated with the microdeformation, as it is represented in theories of generalized continua. It is shown, however, that the microdeformation can be treated as an internal variable in the dual internal variables approach [33, 39].

3.2 Dual internal variables

Now we suppose that the free energy depends on two internal variables φ, ψ and their space derivatives $W = \overline{W}(u_x, \theta, \varphi, \varphi_x, \psi, \psi_x)$. Corresponding constitutive equations are similar to those which were previously presented

$$\begin{gathered}\sigma := \frac{\partial \overline{W}}{\partial u_x}, \quad S = -\frac{\partial \overline{W}}{\partial \theta}, \quad \tau := -\frac{\partial \overline{W}}{\partial \varphi}, \quad \eta := -\frac{\partial \overline{W}}{\partial \varphi_x}, \\ \xi := -\frac{\partial \overline{W}}{\partial \psi}, \quad \zeta := -\frac{\partial \overline{W}}{\partial \psi_x}\end{gathered} \tag{38}$$

Taking into account constitutive relations (38), we can represent the source term in Eq. (4) as follows:

$$\begin{aligned} h^{int} &= \sigma \varepsilon_t - W_t = S\theta_t + \tau\varphi_t + \eta\varphi_{xt} + \xi\psi_t + \zeta\psi_{xt} = \\ &= S\theta_t + (\tau - \eta_x)\varphi_t + (\eta\varphi_t)_x + (\xi - \zeta_x)\psi_t + (\zeta\psi_t)_x \end{aligned} \tag{39}$$

Defining thermal and intrinsic parts of the internal heat source,

$$h^{th} := S\theta_t, \quad \widetilde{h}^{intr} := (\tau - \eta_x)\varphi_t + (\xi - \zeta_x)\psi_t \tag{40}$$

we arrive at the canonical energy balance in the form like $(25)_2$

$$(S\theta)_t + \widetilde{Q}_x = h^{th} + \widetilde{h}^{intr} \tag{41}$$

but with the appropriate modification of heat flux

$$\widetilde{Q} = Q - \eta\varphi_t - \zeta\psi_t \tag{42}$$

Similarly, calculating the internal force

$$\begin{aligned} f^{int} &= \sigma u_{xx} - W_x|_{impl} = S\theta_x + \tau\varphi_x + \eta\varphi_{xx} + \xi\psi_x + \zeta\psi_{xx} = \\ &= S\theta_x + (\tau - \eta_x)\varphi_x + (\eta\varphi_x)_x + (\xi - \zeta_x)\psi_x + (\zeta\psi_x)_x \end{aligned} \tag{43}$$

and defining its thermal and intrinsic parts

$$f^{th} := S\theta_x, \quad \widetilde{f}^{intr} := (\tau - \eta_x)\varphi_x + (\xi - \zeta_x)\psi_x \tag{44}$$

we can represent the canonical equation of momentum like $(25)_1$

$$\mathcal{P}_t - \widetilde{b}_x = f^{th} + \widetilde{f}^{intr} \tag{45}$$

again with the appropriate modification of the Eshelby stress

$$\widetilde{b} = -\left(\frac{1}{2}\rho_0 v^2 - W + \sigma\varepsilon - \eta\varphi_x - \zeta\psi_x\right) \tag{46}$$

Comparing modified heat flux (42) with the expression in parenthesis in the last term of dissipation inequality (8), we conclude that the extra entropy flux should be chosen as

$$J = -\theta^{-1}\eta\varphi_t - \theta^{-1}\zeta\psi_t \tag{47}$$

The latter means that dissipation inequality (8) reduces to

$$\varPhi = (\tau - \eta_x)\varphi_t + (\xi - \zeta_x)\psi_t - \left(\frac{Q - \eta\varphi_t - \zeta\psi_t}{\theta}\right)\theta_x \geq 0 \tag{48}$$

Again, under the independence of the intrinsic dissipation of temperature gradient, we are forced to modify the Fourier law accordingly

$$Q - \eta\varphi_t - \zeta\psi_t = -k\theta_x \tag{49}$$

to satisfy the thermal part of the dissipation inequality, as it was shown by means of the Liu procedure in [39].

The remaining intrinsic part of dissipation inequality (48) is nothing else but a linear combination of products of thermodynamic fluxes and forces. It is straightforward to point out the simplest solution of the intrinsic part of the dissipation inequality assuming linear relationships between the thermodynamic fluxes and their multipliers, the thermodynamic forces [51]

$$\varphi_t = R_{11}(\tau - \eta_x) + R_{12}(\xi - \zeta_x) \tag{50}$$

$$\psi_t = R_{21}(\tau - \eta_x) + R_{22}(\xi - \zeta_x) \tag{51}$$

Coefficients in the right hand side of Eqs. (50)-(51) constitute the matrix $\mathbf{R}$ which we call following Onsager [52] the matrix of conductances

$$\mathbf{R} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \tag{52}$$

These coefficients may depend on state variables. For simplicity, we consider here constant coefficients.

3.3 Interpretation of internal variables

The interpretation of internal variables relates to properties of the matrix of conductances. The matrix of conductances can be symmetric, if Onsager symmetry relations are valid between its coefficients, or antisymmetric in the case of Casimir relations.

Using the fact that internal variables are not specified yet, we may assume that a symmetric matrix of conductances is represented in its diagonal form

$$\mathbf{R} = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \tag{53}$$

where r_1, r_2 are real and distinct eigenvalues of the matrix $\mathbf{R}$. In this case, the intrinsic part of the dissipation inequality (48) is a quadratic form

$$\varPhi = r_2(\tau - \eta_x)^2 + r_3(\xi - \zeta_x)^2 \geq 0 \tag{54}$$

and its positive definiteness is provided by the non-negativity of the eigenvalues. The absence of a coupling between internal variables takes the introduction of the second one superfluous. Therefore, the second internal variable can be omitted without the loss of generality. The situation is reduced to the theory of single internal variable of a dissipative nature described above.

In the case of complex eigenvalues, the antisymmetric matrix of conductances has the Jordan normal form

$$\mathbf{R} = \begin{pmatrix} c_2 & d_2 \\ -d_2 & c_2 \end{pmatrix} \tag{55}$$

with two complex eigenvalues $c_2 \pm d_2 i$. In this case, the intrinsic part of the dissipation inequality (48) still has a quadratic form

$$\varPhi = c_2(\tau - \eta_x)^2 + c_2(\xi - \zeta_x)^2 \geq 0 \tag{56}$$

If $c_2 = 0$, then internal variables do not contribute to the entropy production. Such non-dissipative internal variables φ and ψ can be interpreted as a microdeformation and its conjugate momentum, as it shown in [33, 39].

The less standard example is delivered by the matrix of conductances of the form

$$\mathbf{R} = \begin{pmatrix} 0 & R_{12} \\ -R_{21} & R_{22} \end{pmatrix} \tag{57}$$

The matrix of conductances is still antisymmetric, but now with a contribution to the entropy production. The intrinsic part of the dissipation inequality (48)

$$\varPhi = R_{22}(\xi - \zeta_x)^2 \geq 0 \tag{58}$$

is satisfied in this case by the non-negativity of coefficient R_{22}, but internal variables cannot be considered relating to a microdeformation due to dissipation. Taking into account the parabolic, diffusive form of the evolution equation of the primary internal variable and also the coupling to the Fourier law, one can regard it to microtemperature as shown in [35].

4 Double dual internal variables

In order to take into account the effect of both microdeformation and microtemperature simultaneously, we need to make a step further. We suppose that the free energy depends on the internal variables α, β, φ, ψ and their space derivatives

$$W = \overline{W}(u_x, \theta, \alpha, \alpha_x, \beta, \beta_x, \varphi, \varphi_x, \psi, \psi_x) \tag{59}$$

Constitutive equations are written as usual

$$\begin{gathered} \sigma := \frac{\partial \overline{W}}{\partial u_x}, \quad S := -\frac{\partial \overline{W}}{\partial \theta}, \\ a := -\frac{\partial \overline{W}}{\partial \alpha}, \quad \mathcal{A} := -\frac{\partial \overline{W}}{\partial \alpha_x}, \quad b := -\frac{\partial \overline{W}}{\partial \beta}, \quad \mathcal{B} := -\frac{\partial \overline{W}}{\partial \beta_x}, \\ \tau := -\frac{\partial \overline{W}}{\partial \varphi}, \quad \eta := -\frac{\partial \overline{W}}{\partial \varphi_x}, \quad \xi := -\frac{\partial \overline{W}}{\partial \psi}, \quad \zeta := -\frac{\partial \overline{W}}{\partial \psi_x} \end{gathered} \tag{60}$$

To be more specific, we prescribe explicitly the dependence of the free energy on state variables. We still use a quadratic free energy function

$$\begin{aligned}\overline{W} &= \frac{1}{2}\left(\lambda+2\mu\right)u_x^2 - \frac{\rho_0 c_p}{2\theta_0}\left(\theta-\theta_0\right)^2 + m\left(\theta-\theta_0\right)u_x + \\ &+ A\alpha u_x + \frac{1}{2}B\alpha^2 + \frac{1}{2}C\alpha_x^2 + \frac{1}{2}D\beta^2 + M\varphi_x u_x + \frac{1}{2}N\varphi_x^2 + \frac{1}{2}P\psi^2\end{aligned} \tag{61}$$

Here A, B, C, D and M, N, P are material parameters. As one can see, only contributions of gradients of the primary internal variable and those of the secondary internal variables themselves are included here. Constitutive relations (60) determine the macrostress σ

$$\sigma := \frac{\partial\overline{W}}{\partial u_x} = \left(\lambda+2\mu\right)u_x + m\left(\theta-\theta_0\right) + A\alpha + M\varphi_x \tag{62}$$

microstresses $\mathcal{A}$ and η

$$\mathcal{A} = -\frac{\partial\overline{W}}{\partial\alpha_x} = -C\alpha_x, \quad \eta := -\frac{\partial\overline{W}}{\partial\varphi_x} = -N\varphi_x - Mu_x \tag{63}$$

interactive internal forces

$$\tau := -\frac{\partial\overline{W}}{\partial\varphi} = 0, \quad a = -\frac{\partial\overline{W}}{\partial\alpha} = -Au_x - B\alpha \tag{64}$$

and auxilary quantities related to secondary internal variables

$$\zeta = -\frac{\partial\overline{W}}{\partial\psi_x} = 0, \quad \xi = -\frac{\partial\overline{W}}{\partial\psi} = -P\psi, \quad \mathcal{B} = -\frac{\partial\overline{W}}{\partial\beta_x} = 0, \quad b = -\frac{\partial\overline{W}}{\partial\beta} = -D\beta \tag{65}$$

correspondingly. The extra entropy flux is selected in the same way as previously

$$J = -\theta^{-1}\eta\varphi_t - \theta^{-1}\zeta\psi_t - \theta^{-1}\mathcal{A}\alpha_t - \theta^{-1}\mathcal{B}\beta_t = -\theta^{-1}\eta\varphi_t - \theta^{-1}\mathcal{A}\alpha_t \tag{66}$$

Dissipation inequality (8) reduces to the following one:

$$\Phi = (\tau-\eta_x)\varphi_t + \xi\psi_t + (a-\mathcal{A}_x)\alpha_t + b\beta_t - \left(\frac{Q-\eta\varphi_t-\mathcal{A}\alpha_t}{\theta}\right)\theta_x \geq 0 \tag{67}$$

Accordingly, the Fourier law is modified

$$Q - \eta\varphi_t - \mathcal{A}\alpha_t = -k\theta_x \tag{68}$$

to satisfy the thermal part of the dissipation inequality.

4.1 Microdeformation

Suppose that internal variables α and β are coupled as in the non-dissipative case

$$\alpha_t = R_{12}(b - \mathcal{B}_x) \tag{69}$$

$$\beta_t = -R_{12}(a - \mathcal{A}_x) \tag{70}$$

It follows from Eqs. (69) and ($65_{3,4}$) that

$$\alpha_t = -R_{12} D\beta \tag{71}$$

and Eq. (70) can be represented as a hyperbolic equation

$$\alpha_{tt} = R_{12}^2 D(a - \mathcal{A}_x) \tag{72}$$

Due to the definitions of microstress (Eq. $(63)_1$) and interactive force (Eq. $(64)_2$), we arrive at

$$I\alpha_{tt} = C\alpha_{xx} - Au_x - B\alpha \tag{73}$$

where $I = 1/R_{12}^2 D > 0$ represents the microinertia. The latter is the hyperbolic evolution equation for the microdeformation.

4.2 Microtemperature

Now we turn to next internal variables which we associate with a microtemperature. According to evolution equations (50)-(51) and accounting the form of the matrix of conductances for microtemperature (Eq. (57)), we have for internal variables φ and ψ

$$\begin{aligned} \varphi_t &= R_{12}(\xi - \zeta_x), \\ \psi_t &= -R_{12}(\tau - \eta_x) + R_{22}(\xi - \zeta_x) \end{aligned} \tag{74}$$

Due to Eq. $(65)_{1,2}$ evolution equation for the primary internal variable $(74)_1$ is reduced to

$$\varphi_t = -R_{12} P\psi \tag{75}$$

and Eq. $(74)_2$ becomes (denoting $L = 1/R_{12}^2 P$)

$$L\varphi_{tt} + \frac{R_{22}}{R_{12}^2}\varphi_t = N\varphi_{xx} + Mu_{xx} \tag{76}$$

which is a Cattaneo-Vernotte-type hyperbolic equation [6] for the internal variable φ. Correspondingly, energy conservation equation (4) determines the heat conduction equation for small deviations from the reference temperature

$$\rho_0 c_p\, \theta_t - (k\theta_x)_x = m\theta_0 u_{xt} + \frac{R_{22}}{R_{12}^2}\varphi_t^2 \tag{77}$$

which is influenced by source terms depending on the internal variable φ. Equation of motion (1) is supplemented by a source term as well

$$\rho_0 u_{tt} = (\lambda + 2\mu)\, u_{xx} + m\theta_x + A\alpha_x + M\varphi_{xx} \tag{78}$$

due to definition of macrostress (62). All four governing equations (73), (76) - (78) are coupled. This coupling can induce wave-like propagation for macrotemperature [37, 38].

4.3 Boundary conditions

As it is mentioned in [31], natural boundary conditions for microstructured solids should provide zero value of the extra entropy flux at boundaries. According to definition of the extra entropy flux (66), the latter condition is equivalent to $\alpha_t = 0, \beta_t = 0$ and $\varphi_t = 0, \psi_t = 0$ at boundaries, which provides zero boundary conditions for internal variables under zero initial conditions. A non-trivial solution for internal fields will appear due to the coupling even if initial and boundary conditions for them are equal to zero.

5 Conclusions

To make the description of the influence of microstructure on a global motion apparent, it is useful to compare the governing equations for each considered case. The comparison of equations of motion is shown in Table 1.

Classical thermoelasticity	$\rho_0 u_{tt} =$	$(\lambda + 2\mu)\, u_{xx}$	$+m\theta_x$		
Single internal variable	$\rho_0 u_{tt} =$	$(\lambda + 2\mu)\, u_{xx}$	$+m\theta_x$	$+A\alpha_x$	
Dual internal variable (microdeformation)	$\rho_0 u_{tt} =$	$(\lambda + 2\mu)\, u_{xx}$	$+m\theta_x$	$+A\alpha_x$	
Dual internal variable (microtemperature)	$\rho_0 u_{tt} =$	$(\lambda + 2\mu)\, u_{xx}$	$+m\theta_x$		$+M\varphi_{xx}$
Double dual internal variables	$\rho_0 u_{tt} =$	$(\lambda + 2\mu)\, u_{xx}$	$+m\theta_x$	$+A\alpha_x$	$+M\varphi_{xx}$

Table 1. Balance of linear momentum

As one can see, the influence of microdeformation gives the same contribution both in the single and in the dual internal variables (for microdeformation) theories. The main difference between the two cases is in the evolution equations for internal variables: in the case of the single internal variable theory it is a parabolic one, but in the case of dual internal variable theory it is a hyperbolic one (Table 3). The equation of motion in the case of the double dual internal variables combines contributions from both dual internal variables cases, namely, for microdeformation as well as for microtemperature.

Classical thermoelasticity	$\rho_0 c_p \theta_t = (k\theta_x)_x + m\theta_0 u_{xt}$
Single internal variable	$\rho_0 c_p \theta_t = (k\theta_x)_x + m\theta_0 u_{xt} + \frac{1}{d}\varphi_t^2$
Dual internal variable (microdeformation)	$\rho_0 c_p \theta_t = (k\theta_x)_x + m\theta_0 u_{xt}$
Dual internal variable (microtemperature)	$\rho_0 c_p \theta_t = (k\theta_x)_x + m\theta_0 u_{xt} + \frac{R_{22}}{R_{12}^2}\varphi_t^2$
Double dual internal variables	$\rho_0 c_p \theta_t = (k\theta_x)_x + m\theta_0 u_{xt} + \frac{R_{22}}{R_{12}^2}\varphi_t^2$

Table 2. Balance of energy

The heat conduction equation remains parabolic for all considered cases, and in the conservative case of the dual internal variable theory for microdeformation it is the same as in the case of classical thermoelasticity (Table 2).

Single internal variable	$\frac{1}{d}\varphi_t =$	$C\varphi_{xx} - Au_x - B\varphi$
Dual internal variable (microdeformation)	$I\alpha_{tt} =$	$C\alpha_{xx} - Au_x - B\alpha$
Dual internal variable (microtemperature)	$L\varphi_{tt} + \frac{R_{22}}{R_{12}^2}\varphi_t =$	$N\varphi_{xx} + Mu_{xx}$
Double dual internal variables	$I\alpha_{tt} =$ $L\varphi_{tt} + \frac{R_{22}}{R_{12}^2}\varphi_t =$	$C\alpha_{xx} - Au_x - B\alpha$ $N\varphi_{xx} + Mu_{xx}$

Table 3. Evolution equations for internal variables

For convenience, we collect all the governing equations for thermoelasticity with coupled microdeformation and microtemperature.
Balance of linear momentum:

$$\rho u_{tt} = (\lambda + 2\mu)\, u_{xx} + m\theta_x + A\alpha_x + M\varphi_{xx} \tag{79}$$

Balance of energy:

$$\rho c_p \theta_t = (k\theta_x)_x + m\theta_0 u_{xt} + \frac{R_{22}}{R_{12}^2}\varphi_t^2 \tag{80}$$

Evolution equation (microdeformation):

$$I\alpha_{tt} = C\alpha_{xx} - Au_x - B\alpha \tag{81}$$

Evolution equation (microtemperature):

$$L\varphi_{tt} + \frac{R_{22}}{R_{12}^2}\varphi_t = N\varphi_{xx} + Mu_{xx} \tag{82}$$

To our best knowledge, system of Eqs. (79) - (82) is the first attempt to model the effects of microdeformation and microtemperature simultaneously under the internal variable approach. The double dual internal variable theory contains two hyperbolic evolution equations: one for the microdeformation (Eq. (81)) and another for the microtemperature (Eq. (82)). These evolution equations are not coupled directly, but both of them are coupled with the balance of linear momentum (Eq. (79)). The heat conduction equation (Eq. (80)) is affected only by the microtemperature field because of the non-dissipative microdeformation.

We have solved earlier system of equations (79), (81) with $M = 0$ [34] and system of equations (79) - (82) with $A = 0$ [37] separately. The calculations for full coupled system of governing equations (79) - (82) are in progress. In summary, the introduction of double dual internal variables provides a complete extension of the classical thermoelasticity theory onto the case of microstructured solids.

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