Two-dimensional thermoelastic wave propagation in inhomogeneous media

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Abstract

A novel approach to the modelling of thermoelastic wave propagation is presented based on the thermodynamics of discrete systems. The first novelty includes the representation of integral balance laws for thermoelasticity in terms of contact quantities that describe non-equilibrium state of elements. The next new aspect is a modification of the recently proposed wave-propagation algorithm, which is used as a tool for determining the contact quantities in a finite-volume scheme for the numerical simulation. Such a modification is needed to provide the satisfaction of the thermodynamic consistency conditions between adjacent discrete elements.The test problem shows the efficiency of the algorithm based on the clear physical background.

1 Introduction

The present paper shows that the ideas of (i) the finite-volume method and (ii) the thermodynamics of discrete systems can be combined into one framework to provide a physically consistent algorithm for the numerical simulation of two-dimensional thermoelastic wave propagation in inhomogeneous media.

It should be noted that the derivation, analysis, and implementation of approximate solutions to conservation laws and related problems were all the major foci of an enormous amount of activities in recent decades [1],[2]. Modern algorithms were developed that achieve high resolution, stability and efficiency of numerical schemes. Nevertheless, it is well known that monotone (or positive) approximations are at most first-order accurate [3]. The lack of monotonicity for higher order methods is reflected by spurious oscillations in the vicinity of jump discontinuities. The usual way to supress these oscillations is the introduction of certain limiter functions. However, the use of non-linear wave limiters may destroy the superposition principle (at least for linear thermoelasticity). In addition, the limiters may lead to problems of the consistency between adjacent discrete elements in numerical simulation. In particular, the satisfaction of the thermodynamic consistency conditions which follow from the thermodynamics of discrete systems cannot be controlled immediately. These conditions are formulated, however, in terms of so-called contact quantities that are basic notions in the thermodynamics of discrete systems [4]. Therefore, in order to provide the satisfaction of these conditions, a certain re-formulation of the governing equations of thermoelasticity in terms of parameters related to the discrete elements is needed. In addition, the balance laws for discrete elements should have the

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integral form, which leads to a finite-volume algorithm in a natural way. The extension of the concepts of the thermodynamics of discrete systems to the thermoelastic case allows us to formulate the integral balance laws in terms of contact quantities. A modification of the wave-propagation algorithm for conservation laws [5] is successfully used for the connection between bulk and contact quantities in the case of elastic waves. At last, it is possible to eliminate the source terms in the equation of thermoelasticity following the method of balancing source terms [6]. Results of the numerical simulation using the derived algorithm are presented by examples of propagation of a thermoelastic wave in a medium with smoothly varying and piece-wise constant material properties.

2 Basic equations of thermoelasticity

We shall consider here the classical thermoelasticity of heat conductors. Neglecting geometrical nonlinearities, the main equations of thermoelasticity are the *local balance of momentum* at each regular material point (in the absence of body force) [7],[8]:

$$\frac{\partial\left(\bar{\rho}_0(\mathbf{x})v_i\right)}{\partial t}-\frac{\partial\sigma_{ij}}{\partial x_j}=0, \tag{1}$$

the *heat conduction equation*

$$\frac{\partial\left(c(\mathbf{x})T\right)}{\partial t}=\frac{\partial}{\partial x_i}\left(k(\mathbf{x})\frac{\partial T}{\partial x_i}\right)+m(\mathbf{x})\frac{\partial v_k}{\partial x_k}. \tag{2}$$

and the Duhamel-Neumann thermoelastic constitutive equation, of which the time derivative has following form for the particular example of *linear isotropic thermoelasticity*

$$\frac{\partial\sigma_{ij}}{\partial t}=\lambda(\mathbf{x})\frac{\partial v_k}{\partial x_k}\delta_{ij}+\mu(\mathbf{x})\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)+m(\mathbf{x})\frac{\partial T}{\partial t}\delta_{ij}, \tag{3}$$

where $\rho_0=\bar{\rho}_0(\mathbf{x})$, S is the entropy, T is the absolute temperature, $\mathbf{q}$ is the heat flux vector, and $\mathbf{v}$ and $\boldsymbol{\sigma}$ are the velocity and Cauchy stress tensor, respectively. The indicated explicit dependence on the point $\mathbf{x}$ means that the body is materially inhomogeneous in general. The *dilatation coefficient* α is related to the thermoelastic coefficient m, and the Lamé coefficients λ and μ by $m=-\alpha(3\lambda+2\mu)$.

From the mathematical viewpoint, the problem is to find a solution to the system (1)-(3) of equations with corresponding initial and boundary conditions. This is conceively difficult in general so that an efficient and robust numerical method is required. Obviously, numerical schemes deal with *discrete* elements representing the continuous body. The formulation (1)-(3) paves the way for a re-formulation of the governing equations in terms of parameters related to the discrete elements. It happens that the latter can be viewed from both the numerical and thermodynamic viewpoints.

The main idea of the thermodynamics of discrete systems is the extension of the thermodynamic state space by virtue of so-called contact quantities for the description of non-equilibrium states of a system [4].

In the required extension of the concepts of the thermodynamics of discrete systems to the *thermoelastic case*, we divide the body into a *finite number of identical elements.* The state of each element is then identified with the thermodynamic state of a discrete system associated with that element, each element being assumed in local equilibrium. In thermoelasticity, in addition to the contact temperature Θ, we must define a *contact* *dynamic stress tensor* Σ_{ij} since the state space includes the deformation. It remains now to make the link between the *bulk* quantities that appear in eqns.(1)-(3) and the *contact* quantities. The first step in this direction is the following one. The contact stress tensor Σ_{ij} being defined at the boundary ∂V of the volume element V, and V_i denoting, by duality, the contact deformation velocity at that boundary of unit outward normal n_i, integration over the finite volume element of eqns.(1)-(3) and of the definition of the strain rate yields the following set of integral forms:

$$\frac{\partial}{\partial t}\int_V \rho_0 v_i dV = \int_{\partial V} \Sigma_{ij} n_j dA, \tag{4}$$

$$\frac{\partial}{\partial t}\int_V \varepsilon_{ij} dV = \int_{\partial V} H_{ijk} n_k dA, \tag{5}$$

$$\frac{\partial}{\partial t}\int_V \sigma_{ij} dV = \int_{\partial V} (2\mu H_{ijk} n_k + \lambda \delta_{ij} V_k n_k) dA + \varphi_{ij}, \tag{6}$$

$$\frac{\partial}{\partial t}\int_V cT dV = \int_{\partial V} k(\mathbf{n}\cdot\nabla)\Theta dA. \tag{7}$$

where $H_{ijk} = 1/2(\delta_{ik}V_j + \delta_{jk}V_i)$ and source terms due to material inhomogeneities (labelled "inh") and thermoelastic couplings (labelled "te") are given by

$$\varphi_{ij} = \varphi_{ij}^{te} + \varphi_{ij}^{inh}, \qquad \varphi_{ij}^{te} = \int_V m\delta_{ij}\frac{\partial T}{\partial t} dV,$$

$$\varphi_{ij}^{inh} = -\int_V \left(v_k \frac{\partial \lambda}{\partial x_k}\delta_{ij} + v_i \frac{\partial \mu}{\partial x_j} + v_j \frac{\partial \mu}{\partial x_i} \right) dV,$$

Equations (4)-(7) - are in a form suitable for a numerical approach by the method of *finite volumes.* It remains, however, to make the second step by determining contact quantities.

3 Links between bulk and contact quantities

The connection between bulk and contact quantities in the elastic case will be established by means of the wave-propagation algorithm [5]. The general form for the integral equations (4)-(7) in the considering case is

$$\frac{\partial}{\partial t}\int_V q dV = \int_{\partial V} Q\cdot n dA, \tag{8}$$

A finite-volume scheme corresponding to these equations in two space dimensions can be represented in the form

$$q_{ij}^{k+1} = q_{ij}^{k} - \frac{\Delta t}{\Delta x}(AQ_{ij}^{+} - AQ_{ij}^{-}) - \frac{\Delta t}{\Delta y}(BQ_{ij}^{+} + BQ_{ij}^{-}). \tag{9}$$

where $AQ^{\pm}$ and $BQ^{\pm}$ are the contact quantities in the horizontal and vertical directions, repectively, which include the fluctuations arising from Riemann problems in the $x-$ and $y-$directions, respectively, second-order correction terms and transverse fluctuations in the same way as in [5].

It should be noted that these definitions of the contact stresses satisfy the thermodynamic consistency conditions in their isothermal form.

It is well known that the Lax-Wendroff scheme produces oscillations behind discontinuities [9]. The usual way for reducing spurious oscillations is to introduce limiter functions for modifying the second-order corrections near discontinuities. However, the application of limiters still seems to be more of an art than a science. Moreover, the fulfilling of the thermodynamic consistency conditions cannot be controlled in this case. It seems that the recently proposed composite schemes [9] are more convenient, because of using filters that are consistent with differential equations. Here the composite scheme is obtained by application of the Godunov step after each three second-order Lax-Wendroff steps. Obviously, the thermodynamic consistency conditions remain satisfied at each step.

In the thermoelastic case, the equation for the temperature in two dimensions is solved by means of the algorithm for the heat conduction equation in an inhomogeneous medium [10]. According to the method of balancing source terms [6], the modified bulk velocities both in x- and y-directions are used for the calculation of normal dynamic stresses and corresponding contact velocities, which allows one to eliminate the source terms. The first-order Godunov method as well as transverse propagation, second-order correction, and composition are constructed then as above.

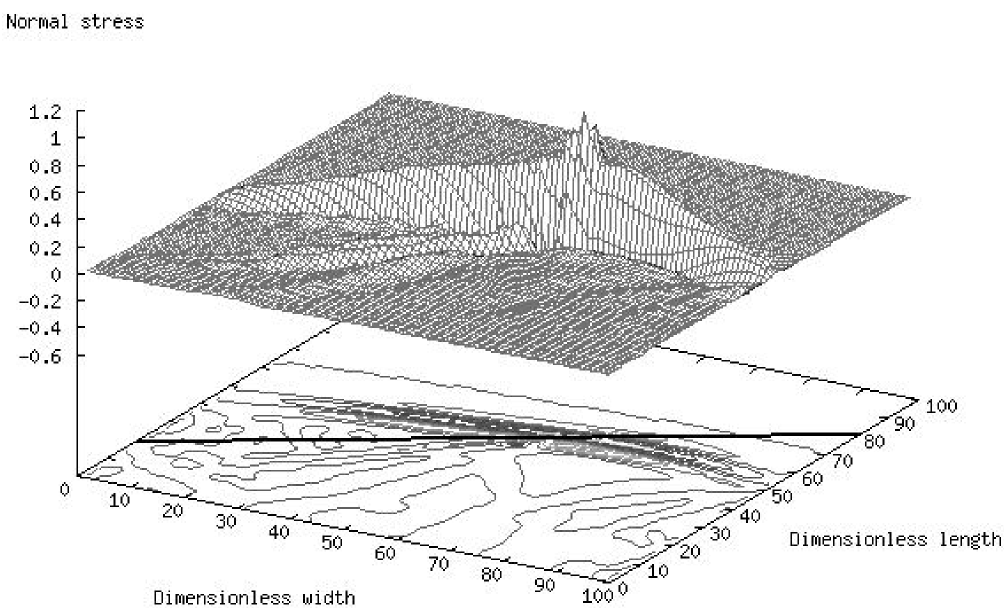

FIG. 1. *Interaction of a thermoelastic wave with an interface between two distinct media, 70 time steps, the Courant number 0.9.*

4 Numerical results

Results of simulation for thermoelastic wave propagation in an inhomogeneous medium are demonstrated in Figure 1, which represents a snapshot of the interaction of a thermoelastic wave with an *interface* between two distinct, but otherwise spatially homogeneous, thermoelastic materials. The interface is placed along the straight line, which is inclined to the bottom boundary by a certain angle and crossed the center of the computational domain (thick line in the plane of level curves). The wave was excited by a purely thermal shock at the main part of the bottom boundary ($20 \leq x \leq 80$), all other boundaries being stress free. Figure 1 shows the first time wave hits the interface.

Thus, the thermodynamic description of interacting discrete elements leads to the formulation of integral balance laws for thermoelasticity in terms of contact quantities and to the thermodynamic consistency conditions, which should be fulfilled for each pair of adjacent elements. The corresponding finite-volume numerical scheme consists of two steps, because the values of the contact quantities are needed for the updating of the state for each element. Fortunately, the contact quantities can be determined by means of the recently proposed wave-propagation technique [5] in the inhomogeneous thermoelastic case. Moreover, the satisfaction of the thermodynamic consistency conditions is provided by using a composite scheme, where three Lax-Wendroff steps are followed by one Godunov step repeatedly. The combination keeps the advantage of the wave-propagation algorithm, which is stable up to the Courant number 1, and allows us to avoid the application of limiter functions that are not consistent with differential equations.

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