Hierarchies of waves in nonclassical materials

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Abstract

Wave propagation in microstructured materials is directly affected by the existence of internal space scale(s) in the compound matter. In this case the classical continuum theory cannot be used. In this paper based on the Mindlin model, the balance laws for macro- and microstructure are formulated separately. Using the slaving principles relating macro- and microdisplacements, the governing equations are derived for a single- and two-scale (scale within scale) cases. These equations exhibit hierarchical properties assigning the wave operators to internal scales. In terms of macrodisplacements, higher-order dispersive terms appear having a clear physical background (microinertia, wave speed in microstructure) related to the scale of the microstructure. Full, approximated (corresponding to hierarchical models), and simplified dispersion relations are derived and analysed to demonstrate the validity of the hierarchical governing equations. Linear theory is based on the quadratic free energy function, in nonlinear theory the cubic terms should also be taken into account. The corresponding governing equation includes nonlinearities in both macro- and microscale. Such consistent modelling opens up new possibilities to Nondestructive Testing (NDT) of material properties.

1 Introduction

Materials used in contemporary high technology are characterized often by their complex structure in order to satisfy many requirements in practice. This concerns polycrystalline solids, ceramic composites, alloys, functionally graded materials, granular materials, etc. Often one should also account for the damage effects, ie materials are still usable when they have microcracks. All that shows the existence of intrinsic space-scales in matter, like the lattice period, the size of a crystalline or a grain, and the distance between microcracks. This scale-dependence should also be taken into account in governing equations. The classical theory of the continuous media is built up using the assumption of the smoothness of continua. The continua (materials) we are interested in, contain irregularities with one or more internal scales and therefore the notion "microstructured materials" is used. Clearly the complex dynamic behaviour of such microstructured materials cannot be explained by the classical theory of continua.

The more detailed description between classical and nonclassical theories of continua is given by Pastrone [elsewhere in this volume], here we restrict ourselves only to basic principles needed for modelling dynamical processes.

The cornerstones for describing dynamical processes of microstructured materials at intensive and high-speed deformations are the following:

(i) non-classical theory of continua able to account for internal scales;

(ii) hierarchical structure of waves due to the scales in materials;

(iii) nonlinearities caused by large deformation and character of stress-strain relations.

Within the theories of continua the problems of irregularities of media have been foreseen a long time ago by the Cosserats and Voigt and more recently by Mindlin (1964), Eringen (1966) and others. The elegant mathematical theories of continua with voids or with vector microstructure, of continua with spins of Cosserat continum or micromorphic continuum, etc. have been elaborated since, see overviews by Capriz (1989), Eringen (1999). Clearly every irregularity (or inclusion) creates an additional stress field around it. Consequently the most general approach in modelling should be the presentation of all the conservation laws and constitutive equations taking this stress field into account.

The straight-forward modelling of microstructured solids leads to assigning all the physical properties to every volume element dV in a solid introducing so the dependence on material coordinates X^k. Then the governing equations include implicitly space-dependence parameters but due to the complexity of the system the governing equations should be solved numerically. Another probably much

more effective way is to separate macro- and microstructure in continua. Then the conservation laws for both structures should be separately formulated (Mindlin 1964, Eringen 1966, Eringen 1999), or the microstructural quantities are separately taken into account in one set of conservation laws (Maugin 1993). The last case uses the concept of pseudo-momentum and material in homogeneity force. Separating the macro- and microstructure gives two possibilities: either to consider both structures inertial or to suppose the microstructural quantities to behave noninertially. The first case is exactly what has been done by Mindlin (1964) and Eringen (1966, 1999), the second case leads to the formalism of internal variables (Maugin 1990, Maugin and Muschik 1994).

The second pillar mentioned above is the hierarchy of waves. The concept of hierarchy of waves is introduced by Whitham (1974).

High intensities of external forces and high deformation rates (high speed of deformation) dictate the need to consider nonlinearities in governing equations. One should distinguish between geometrical (large deformation) and physical (stress-strain relation) nonlinearities (see Engelbrecht 1997). The physical nonlinearities are also called material nonlinearities and may be described by the approximation of the strain energy including to the usual second powers the higher order (the third, the fourth ...) terms. These problems for microstructured solids have been analysed, for example, by Erofeeyev (2003) - see also references therein. The nonlinear theory needs also a clear distinction between material and spatial coordinates.

In terms of wave characteristics, there are many physical effects due to microstructure and its possible structural changes in the wave field. In addition, the influence of nonlinearities causes nonadditivity of other physical effects. Leaving aside more complicated effects like phase transition, kinetic localization of damage, shear bands, etc., even the basic dissipative and dispersive effects are strongly influenced by nonlinearities. There are many studies concerning the dissipative effects combined with nonlinearities (Nunziato, Janno ?).

Dispersive and nonlinear effects combined may lead to celebrated solitary waves. The Korteweg-de Vries equation includes quadratic nonlinearity and cubic dispersion and served for more than 100 years as a model case for the balance of dispersion and nonlinearity. The soliton concept has formed a new paradigm in mathematical physics. When we come to microstructured materials then situation is not so simple. There still seem to be discrepancies between various mathematical models concerning the dispersion relation. In this context also the discrete modelling of crystal lattices is used (Brillouin 1953, Askar 1985, Maugin 1999). The continuum models (Erofeeyev 2003, Porubov 2003) have been elaborated with various levels of accuracy.

We have previously analysed dissipative effects in microstructured materials (Engelbrecht, Cermelli, Pastrone 1999), nonlinearities in microscale (Engelbrecht

Pastrone 2003) and general dispersive effects (Engelbrecht et al 2004).

Here in this paper we concentrate our attention on the description of dispersive effects in microstructured solids following the consistent theory of nonclassical continua. This allows us to unite two important concepts – influence of microstructure on dispersion from one side and the concept of hierarchies from another side. The third pillar – nonlinearities – will take more space and its consistent description will be published elsewhere. Here we touch this problem only briefly.

The paper is organized as follows. Section 2 involves the derivation of the basic single-scaled model. The Mindlin (1964) assumption on strain in microstructure is used and the governing equations derived using the Euler-Lagrange formalism. It is shown that the model is consistent within the framework of pseudomomentum (Maugin, 1993). Section 3 describes the modelling for the case of two-scale (scale within the scale) microstructure. In Section 4 the hierarchy of waves is explained following Whitham's (1974) idea. Section 5 is devoted to the dispersion analysis and Section 6 - to nonlinear models. A discussion and further prospects are given in Section 7.

2 The basic single-scaled model

2.1 Governing equations

Here we follow Mindlin (1964) who has interpreted the microstructure "as a molecule of a polymer, a crystallite of a polycrystal or a grain of a granular material". This microelement is taken as a deformable cell. Note that if this cell is rigid, then the Cosserat model follows. The displacement u of a material particle in terms of macrostructure is defined by its components $u_i \equiv x_i - X_i$, where $x_i, X_i (i = 1, 2, 3)$ are the components of the spatial and material position vectors, respectively. Within each material volume (particle) there is a microvolume and the microdisplacement u' is defined by its components $u'_i \equiv x'_i - X'_i$, where the origin of the coordinates x'_i moves with the displacement u. The displacement gradient assumed to be small. This leads to the *basic assumption* of Mindlin (1964) – "the microdisplacement can be expressed as a sum of products of specified functions of x'_i and arbitrary functions of x_i and t". The first approximation is then

$$u'_j = x'_k \, \varphi_{kj} \, (x_i, t). \tag{1}$$

The *microdeformation* is

$$\frac{\partial u'_j}{\partial x'_i} = \partial'_i \, u'_j = \varphi_{ij}. \tag{2}$$

Further we consider the simplest 1D case and drop the indices i, j dealing with u and φ only. The indices t and x denote differentiation.

The fundamental balance laws for microstructured materials can be formulated separately for macroscopic and microscopic scales (Eringen 1999). We show here how the balance laws can be derived from the Lagrangian (Mindlin 1964, Pastrone 2003)

$$\mathcal{L} = K - W, \tag{3}$$

formed from the kinetic and potential energies

$$K = \frac{1}{2}\rho u_t^2 + \frac{1}{2} I \varphi_t^2, \quad W = W(u_x, \varphi, \varphi_x), \tag{4}$$

where ρ is the density and I - microinertia.

The corresponding Euler-Lagrange equations have the general form

$$\left(\frac{\partial L}{\partial u_t}\right)_t + \left(\frac{\partial L}{\partial u_x}\right)_x - \frac{\partial L}{\partial u} = 0, \tag{5}$$

$$\left(\frac{\partial L}{\partial \varphi_t}\right)_t + \left(\frac{\partial L}{\partial \varphi_x}\right)_x - \frac{\partial L}{\partial \varphi} = 0. \tag{6}$$

Inserting the partial derivatives

$$\begin{aligned} &\frac{\partial L}{\partial u_t} = \rho u_t, \quad \frac{\partial L}{\partial u_x} = -\frac{\partial W}{\partial u_x}, \quad \frac{\partial L}{\partial u} = 0, \\ &\frac{\partial L}{\partial \varphi_t} = I \varphi_t, \quad \frac{\partial L}{\partial \varphi_x} = -\frac{\partial W}{\partial \varphi_x}, \quad \frac{\partial L}{\partial \varphi} = -\frac{\partial W}{\partial \varphi}, \end{aligned} \tag{7}$$

into eq (5), (6) we arrive to equations of motion

$$\rho u_{tt} - \left(\frac{\partial W}{\partial u_x}\right)_x = 0, \quad I \varphi_{tt} - \left(\frac{\partial W}{\partial \varphi_x}\right)_x + \frac{\partial W}{\partial \varphi} = 0. \tag{8}$$

Here we denote

$$\sigma = \frac{\partial W}{\partial u_x}, \quad \eta = \frac{\partial W}{\partial \varphi_x}, \quad \tau = \frac{\partial W}{\partial \varphi}, \tag{9}$$

and recognise σ as the macrostress (Piola stress), η as the microstress and τ as the interactive force.

The equations of motion (8) take now the form

$$\rho u_{tt} = \sigma_x, \tag{10}$$

$$I\varphi_{tt} = \eta_x - \tau. \tag{11}$$

These equations can be compared with the analogous equations deduced in a different way by Capriz (1989).

The simplest potential energy function describing the influence of a microstructure is a quadratic function

$$W = \frac{1}{2}\alpha u_x^2 + A\varphi u_x + \frac{1}{2}B\varphi^2 + \frac{1}{2}C\varphi_x^2 \tag{12}$$

with α, A, B, C – constants. Inserting (12) into (9) and the result into eqs (10), (12), the governing equations take the form

$$\rho u_{tt} = \alpha u_{xx} + A\,\psi_x, \tag{13}$$

$$I\varphi_{tt} = C\,\varphi_{xx} - A\,u_x - B\varphi. \tag{14}$$

Equations (13), (14) with proper initial and boundary conditions form the basis for the further analysis.

2.2 Balance of pseudomomentum

The analysis above is based on two balance laws of momentum, expressed by eqs (10), (11). Here we show that this presentation is equivalent to the balance of pseudomomentum according to Maugin (1993).

We multiply eq (13) by u_x and eq (14) by φ_x that yields after rearranging

$$(\rho\,u_t\,u_x)_t = \frac{1}{2}(\rho\,u_t^2)_x + \frac{1}{2}(\alpha\,u_x^2)_x + A\,\varphi_x\,u_x, \tag{15}$$

$$(I\,\varphi_t\,\varphi_x)_t = \frac{1}{2}(I\,\varphi_t^2)_x + \frac{1}{2}(C\,\varphi_x^2)_x - A\,u_x\,\varphi_x - \frac{1}{2}(B\,\varphi^2)_x. \tag{16}$$

Summing up eq (15) and (16), we obtain

$$\begin{aligned}(\rho\,u_t\,u_x + I\,\varphi_t\,\varphi_x)_t = \frac{1}{2}(\rho\,u_t^2 + I\,\varphi_t^2)_x + \frac{1}{2}(\alpha\,u_x^2)_x + \\ + \frac{1}{2}(C\,\varphi_x^2)_x - \frac{1}{2}(B\,\varphi^2)_x.\end{aligned} \tag{17}$$

After rearranging, eq (17) yields

$$\begin{aligned}(\rho\,u_t\,u_x + I\,\varphi_t\,\varphi_x)_t = \frac{1}{2}(\rho\,u_t^2 + I\,\varphi_t^2)_x + \frac{1}{2}(\alpha\,u_x^2 + \\ +2\,A\,\varphi\,u_x + B\,\varphi^2 + C\,\varphi_x^2)_x - (A\,\varphi\,u_x)_x - (B\,\varphi^2)_x.\end{aligned} \tag{18}$$

Last equations can be represented in terms of pseudomomentum and Lagrangian density as follows

$$-\mathcal{P}_t = \mathcal{L}_x + 2W_x - (A\,\varphi\,u_x)_x - (B\,\varphi^2)_x, \tag{19}$$

where the pseudomomentum is defined as $\mathcal{P} = -(\rho\ u_t\ u_x + I\ \varphi_t\ \varphi_x)$, and the Lagrangian density is $\mathcal{L} = \frac{1}{2}(\rho\,u_t^2 + I\ \psi_t^2) - W$.

From another side,

$$(u_x\,\sigma + \varphi\,\tau + \varphi_x\,\eta)_x = (\alpha\,u_x^2 + 2\,A\,\varphi\,u_x + B\,\varphi^2 + C\,\varphi_x^2)_x = 2W_x. \tag{20}$$

This means that

$$(u_x\,\sigma + \varphi_x)_x = (\alpha\,u_x^2 + 2\,A\,\varphi\,u_x + B\,\varphi^2 + C\,\varphi_x^2)_x - (\varphi\,\tau)_x = 2W_x - (A\,\varphi\,u_x)_x + (B\,\varphi^2)_x. \tag{21}$$

Substituting the last relation into equation (19), we will have

$$-\mathcal{P}_t = \mathcal{L}_x + (u_x\,\sigma + \varphi_x\,\eta)_x. \tag{22}$$

If we now define the Eshelby stress as

$$b = -(\mathcal{L} + u_x\,\sigma + \varphi_x\,\eta), \tag{23}$$

we can represent the equation (22) in the form of balance of pseudomomentum

$$\mathcal{P}_t - b_x = 0. \tag{24}$$

3 The two-scale model

We follow now the same idea as in Section 2 but generalise it for a two-scale situation. In physical terms it means that every deformable cell of the microstructure includes new deformable cells at a smaller scale. So instead of the system macrostructure-microstructure (see Section 2), the material is supposed to be composed by the macrostructure including microstructure 1 at a certain scale that includes microstructure 2 at a certain smaller scale. A qualitative sketch of such a material is shown in Fig 1. Then the corresponding displacements are

$$u_j\ (x_i, t), \tag{25}$$

$$u_j^{'}\ =\ x_k^{'}\ \varphi_{kj}\ (x_i, t), \tag{26}$$

$$u_j^{''}\ =\ x_k^{''}\ \bar{\psi}_{kj}\ (x_i^{'}, t), \tag{27}$$

respectively, where x'_k, x''_k correspond to the local coordinates within respective cells. As far as we are interested in motion on the macrolevel, we assume relation (27) to be replaced by

$$u'' \;=\; x''_k \, \psi_{kj} \, (x_i, t). \tag{28}$$

The gradients, i.e. deformations are then easily found

$$\frac{\partial \, u'_j}{\partial \, x'_i} = \varphi_{ij}, \qquad\qquad \frac{\partial \, u''_j}{\partial \, x''_i} \;= \psi_{ij}. \tag{29}$$

The balance laws for a 1D case (cf. (10), (11)) are

$$\rho u_{tt} = \sigma_x, \tag{30}$$

$$I_1 \, \varphi_{tt} = \eta_{1x} - \tau_1, \tag{31}$$

$$I_2 \, \psi_{tt} = \eta_{2x} - \tau_2, \tag{32}$$

where indices are dropped. In this case we have to deal with two microstresses η_1, η_2, and two interactive forces τ_1, τ_2. Microinertias are I_1, I_2, respectively. All the stress components and forces are determined from the free energy W by relations:

$$\sigma \;=\; \frac{\partial W}{\partial u_x}, \;\; \eta_1 \;=\; \frac{\partial W}{\partial \varphi_x}, \;\; \eta_2 \;=\; \frac{\partial W}{\partial \psi_x}, \;\; \tau_1 \;=\; \frac{\partial W}{\partial \varphi}, \;\; \tau_2 \;=\; \frac{\partial W}{\partial \psi}. \tag{33}$$

In order to start explaining the dispersive effects, we assume the quadratic free energy function

$$W = \frac{1}{2} \, \alpha u_x^2 \;+\; A\varphi u_x + \frac{1}{2} B_1 \varphi^2 + \frac{1}{2} C_1 \varphi_x^2 + \\ + A_1 \varphi_x \psi + \frac{1}{2} B_2 \psi^2 + \frac{1}{2} C_2 \psi_x^2, \tag{34}$$

where $\alpha, A_i, B_i, C_i, i \;=\; 1, 2$ are constants. Introducing (33) and (34) into (30), (31), (32), the system of governing equations takes the form:

$$\rho u_{tt} = \alpha u_{xx} + A\varphi_x, \tag{35}$$

$$I_1 \, \varphi_{tt} = C_1 \varphi_{xx} - A u_x - B_1 \varphi + A_1 \psi_x, \tag{36}$$

$$I_2 \, \psi_{tt} = C_2 \psi_{tt} - A_1 \varphi_x - B_2 \psi. \tag{37}$$

4 Hierarchies of waves

4.1 Preliminaries

Whitham (1974) has described the complicated wave systems where a scale parameter δ plays an important role. Depending on its limit values ($\delta \to \infty,\ \ \delta \to 0$), one or another wave operator governs the process asymptotically. The full system so includes a hierarchy of waves with certain stability conditions (see Whitham, 1974). Here we show that waves in microstructured materials exhibit the hierarchical behaviour governed by a parameter which is the ratio of the characteristic scale of a microstructure over the wavelength of the excitation.

4.2 Single scale

Let the scale of the microstructure be l and the excitation characterised by its amplitude U_0 and wavelength L. Then we can introduce the following dimensionless variables and parameters

$$U = uU_0^{-1}, \quad X = xL^{-1}, \quad T = tc_0L^{-1}, \quad \delta = l^2L^2, \quad \epsilon = U_0L^{-1}, \tag{38}$$

where $c_0^2 = \alpha/\rho$. We also suppose that $I = \rho l^2 I^*, \quad C = l^2 C^*$, where I^* is dimensionless and C^* has the dimension of stress. Note I is scaled against ρ and the difference of densities is embedded in I^*.

Next, the system (13), (14) is rewritten in its dimensionless form and then the slaving principle (Christiansen et al. 1992, Porubov 2003) is used. It means in principle that we determine φ in terms of U_x using a series representation. Indeed, eq (14) yields

$$\varphi = -(\epsilon A/B)\, U_X - (\delta/B)(\alpha I^* \varphi_{TT} - C^* \varphi_{XX}). \tag{39}$$

If we consider $\varphi = \varphi_0 + \delta\varphi_1 + ...$, we get

$$\varphi_0 = -(\epsilon A/B)\, U_X, \tag{40}$$

$$\varphi_1 = (\epsilon\alpha A I^*/B^*) U_{XTT} - (\epsilon A C^*/B^2) U_{XXX}). \tag{41}$$

Inserting (40), (41) into the governing system in its dimensionless form, we get finally in terms of U as follows (cf. Engelbrecht and Pastrone 2003)

$$U_{TT} = \left(1 - \frac{c_A^2}{c_0^2}\right)\, U_{XX} + \frac{c_A^2}{c_B^2}\left(U_{TT}\ -\ \frac{c_1^2}{c_0^2}\, U_{XX}\right)_{XX} \tag{42}$$

where $c_1^2 = C/I,\ c_A^2 = A^2/B\,\rho,\ c_B^2 = B\,L^2/I$. Note that c_B^2 involves the scales L and l and c_A^2 includes the interaction effects between the macro- and microstructure (through the parameter A). It means that

$$\frac{c_A^2}{c_B^2} = \frac{A^2}{B^2} I^* \delta. \tag{43}$$

So, eq (42) reflects clearly the hierarchical nature of wave propagation in microstructured solids: if c_A^2/c_B^2 is small then waves are governed by the properties of macrostructure, if however c_A^2/c_B^2 is large then waves "feel" more microstructure. Note that in absence of the interaction between macro- and microstructure (i.e. when A = 0), in terms of U the wave operator is simply $U_{TT} - U_{XX}$.

It is of interest to restore the dimensions in order to compare the various approximations. First, the system (13), (14) of two 2nd order equations can be represented also in the form of one 4th order equation:

$$u_{tt} = \left(c_0^2 - c_A^2\right)\ u_{xx} - p^2 \left(u_{tt} - c_0^2\, u_{xx}\right)_{tt} + p^2\, c_1^2 \left(u_{tt} - c_0^2\, u_{xx}\right)_{xx}, \tag{44}$$

where $p^2 = I/B$.

Equation (42), however, can be rewritten as

$$u_{tt} = \left(c_0^2 - c_A^2\right)\ u_{xx} - p^2\, c_A^2 \left(u_{tt} - c_1^2\, u_{xx}\right)_{xx}. \tag{45}$$

It is obvious that the approximated model (45) which displays clearly the hierarchical structure, neglects u_{tttt} completely while the influence of u_{ttxx} is different in (44) and (45). What is important – in this approximation the effects of inertia of microstructure and wave velocity in pure microstructure are taken into account. There are certainly other approximations possible. From lattice theory (see, for example Maugin 1999) the governing equation in its simplest form is

$$u_{tt} = c_0^2\, u_{xx} + 1/12\, c_0^2\, a^2\, u_{xxxx}. \tag{46}$$

where a is the distance between the particles. This must be compared with

$$u_{tt} = \left(c_0^2 - c_A^2\right)\ u_{xx} - p^2\, c_A^2\, c_1^2 u_{xxxx} \tag{47}$$

resulting from eq (45). In case of periodic structures (Santosa and Symes 1991) the governing equation corresponds to the dispersion relation

$$\omega(k) = \Omega_1\, k + \frac{1}{6}\, \epsilon^2\, \Omega_3\, k^3 + ..., \tag{48}$$

and takes the form

$$u_{tt} = \Omega_1^2\, u_{xx} + \frac{1}{3}\, \Omega_1\, \Omega_3\, p^2\, u_{xxxx}. \tag{49}$$

Here ω and k are the frequency and the wave number, respectively, Ω_1 and Ω_3 are related to densities, elastic moduli, and impedances $(\Omega_1 < c_1,\ \Omega_3 < 0)$ and c_1 is the velocity in the leading structure while ϵ is the small parameter and p is here related to the cell size (cf. with l in our case). If only the effect of microinertia is retained (Wang and Sun 2002) then in our notations the governing equation reads

$$u_{tt} = \left(c_0^2 - c_A^2\right)\ u_{xx} - p^2\ c_A^2\ u_{ttxx}. \tag{50}$$

The dispersion analysis below (Section 5) shows the difference between the various models.

4.3 Multiple scales

We apply now the same reasoning as above to the system (35), (36), (37) - the balance laws in terms of u (macrostructure), φ (microstructure 1), and ψ (microstructure 2). In order to do so, dimensionless variables are introduced

$$\begin{array}{lll} U = u/U_0, & X = x/L, & T = tc_0/L, \\ \epsilon = U_0/L, & \delta_1 = l_1^2/L^2, & \delta_2 = l_2^2/L^2, \\ I_1 = \rho\, l_1^2\, I_1^*, & I_2 = \rho\, l_2^2\, I_2^*, & \\ C_1 = l_1^2\, C_1^*, & C_2 = l_2^2\, C_2^*, & A_1 = l\, A_1^*, \end{array} \tag{51}$$

where U_0 and L are the amplitude and the characteristic length of the excitation while l_1 and l_2 are the scales of microstructures while $c_0^2 = \alpha/\rho$. Here I_1^*, I_2^* are dimensionless and A_1^*, C_1^*, C_2^* have the dimensions of stress. Note that I_1 and I_2 are scaled against ρ and possible differencies on densities of microstructures are embedded in I_1^* and I_2^*, respectively.

Substituting (51) into eqs (35), (36), (37) we obtain

$$U_{TT} = \ U_{XX} + \frac{A}{\alpha\varepsilon}\varphi_X, \tag{52}$$

$$\varphi_{TT} = \frac{C_1^*}{\alpha\ I_1^*}\varphi_{XX} - \frac{1}{\delta_1}\ \frac{A\epsilon}{\alpha I_1^*}\ U_X - \frac{B_1}{\alpha I_1^*}\varphi + \frac{1}{\delta_1}\frac{A_1^*\delta_2^{1/2}}{\alpha I_1^*}\psi_X, \tag{53}$$

$$\psi_{TT} = \frac{C_2^*}{\alpha\ I_2^*}\psi_{XX} - \frac{1}{\delta_2^{1/2}}\ \frac{A_1^*}{\alpha I_2^*}\varphi_X - \frac{1}{\delta_2}\frac{B_2}{\alpha I_2^*}\psi, \tag{54}$$

Using the presence of small parameters δ_1 and δ_2, we use slaving principle (Christiansen et al. 1992, Porubov 2003) for determining first $\psi(\varphi)$ from eq (54) and then $\varphi(U_X)$ from eq (53). Assuming

$$\psi = \psi_0 + \delta_2\ \psi_1 + ... \tag{55}$$

we obtain from eq (54)

$$\psi = -\delta_2^{1/2}\ \frac{A_1^*}{B_2}\varphi_x + \delta_2^{3/2}\ \frac{\alpha A_1}{B_2^2}\varphi_{TTX} - \delta_2^{3/2}\ \frac{C_2^* A_1^*}{B_2^2}\varphi_{xxx}. \tag{56}$$

Now we assume

$$\varphi = \varphi_0 + \delta_1\ \varphi_1 + ... \tag{57}$$

and by making use of $\psi(\varphi)$ determined by relation (56) we obtain from (53)

$$\begin{aligned}\varphi = &-\ \frac{A\epsilon}{B_1}\ U_X + \delta_1\ \frac{\alpha\epsilon I_1^* A}{B_1^2}\left(U_{TTX} - \frac{C_1^*}{\alpha I_1^*}\ U_{XXX}\right) + \\ &+\ \delta_2\ \frac{\epsilon(A_1^*)^2 A}{B_1^2 B_2}\ U_{XXX} - \delta_2^2\ \frac{\alpha\epsilon(A_1^*)^2 A I_2^*}{B_1^2 B_2^2}\ U_{TTXXX} + \\ &+\ \delta_2^2\ \frac{\epsilon(A_1^*)^2 A C_2^*}{B_1^2 B_2^2}\ U_{XXXXX}\end{aligned} \tag{58}$$

Finally, by making use of $\varphi(U)$ from (58), eq (52) yields:

$$\begin{aligned}U_{TT} = &(1-b)\ U_{XX} + \delta_1\ \left[a\, U_{TT} - (d - \delta_2\ f)U_{XX}\right]_{XX} - \\ &- \delta_2^2\ \left[h\, U_{TT} - g\, U_{XX}\right]_{XXXX},\end{aligned} \tag{59}$$

where

$$\begin{aligned} b &= \frac{A^2}{\alpha\ B_1}, & a &= \frac{A^2\ I_1^*}{B_1^2}, \\ d &= \frac{A^2\ C_1^*}{B_1^2\ \alpha}, & f &= \frac{(A_1^*)^2\ A^2}{B_1^2\ B_2\ \alpha}, \\ h &= \frac{A^2\ (A_1^*)^2\ I_2^*}{B_1^2\ B_2^2}, & g &= \frac{A^2\ C_2^*\ A_1^*}{B_1^2\ B_2^2}. \end{aligned}$$

Equation (59) is the sought hierarchical equation in terms of macrodisplacement U where microstructures are accounted for by special wave operators. In order to compare the result with the basic system (35), (36), (37), and also with the results of Section 4.2 we represent here also the hierarchical equation in terms of dimensional variables and parameters. This has the following form:

$$\begin{aligned}u_{tt}\ =\ &\left(c_0^2 - c_A^2\right)\ u_{xx} + \\ &+ c_A^2\ p_1^2\left(u_{tt} - (c_1^2 - c_B^2)\ u_{xx}\right)_{xx} - \\ &- c_A^2\ p_2^2\ q^2\left(u_{tt} - c_2^2\ u_{xx}\right)_{xxxx}\end{aligned} \tag{60}$$

where $c_A^2 = A^2/(B_1\ \rho)$, $c_B^2 = A_1^2/(B_2\ I_1)$, $p_1^2 = I_1/B_1$, $p_2^2 = I_2^2/B_2$, $q^2 = A_1^2/(B_1\ B_2)$. Clearly $c_B^2 = q^2/p^2$.

5 Dispersion

5.1 General

Internal scales of microstructured solids lead to the dispersive effects. It is also quite clear from governing equation derived in Section 4. The presence of higher-order derivatives in governing equations is a clear sign about dispersion. Below we demonstrate how the various combinations of material parameters and wave characteristics are reflected in dispersion relations. We start from the models with dimensions and introduce then dimensionless wave number and frequency. The solution is assumed in the form of a wave

$$u(x,t) = \hat{u}\, exp\ \left[i\ (kx - \omega t)\right], \tag{61}$$

with wave number k and frequency ω, here $\hat{u}$ is the amplitude.

5.2 Single scale

The corresponding mathematical models are presented in Section 4.2. Introducing now (61) into equation (44), the dispersion relation is obtained.

$$\omega^2 = \left(c_0^2 - c_A^2\right)\ k^2 + p^2\ \left(\omega^2 - c_0^2\, k^2\right)\ \left(\omega^2 - c_1^2 k^2\right) = 0. \tag{62}$$

The parameters involved are a time constant p and three characteristic velocities c_0, c_A, c_1. Instead of c_A the velocity $c_R = \left(c_0^2 - c_A^2\right)^{1/2}$ could be introduced as a parameter since it has an obvious meaning for given wave process. Waves of very low frequencies $(\omega \ll p^{-1})$ are propagated at the velocity c_R. The velocity c_A does not occur explicitly as a limit velocity. In order to reduce the number of independent variables we introduce dimensionless quantities

$$\xi = p\, c_0\, k, \qquad \eta\ = p\, \omega, \tag{63}$$

and dimensionless parameters

$$\gamma_1 = c_1/c_0, \qquad \gamma_A = c_A/c_0. \tag{64}$$

Using these new quantities the full dispersion relation (44) assumes the form

$$\eta^2 = \left(1 - \gamma_A^2\right)\ \xi^2 + \left(\eta^2 - \xi^2\right)\ \left(\eta^2 - \gamma_1^2\, \xi^2\right). \tag{65}$$

In the same way, the approximate differential equation (45) yields the dimensionless dispersion relation

$$\eta^2 = \left(1 - \gamma_A^2\right)\ \xi^2 - \gamma_A^2\left(\eta^2 - \gamma_1^2\, \xi^2\right)\ \xi^2. \tag{66}$$

Eventually the simplified differential equations (47) and (50) yield

$$\eta^2 = (1 - \gamma_A)\ \xi^2 + \gamma_A^2\ \gamma_1^2\ \xi^4, \tag{67}$$

$$\eta^2 = \left(1 - \gamma_A^2\right)\ \xi^2 - \gamma_A^2\ \eta^2\ \xi^2, \tag{68}$$

respectively.

The full dispersion relation (65) represents two branches which, in general, are distinct (see Fig. 2, 3). The upper, or "optical" branch starts in the $\xi - \eta$ plane at $\eta = 1$ with zero scope, while the lower, or "acoustical" branch starts at the origin with slope $\gamma_R = c_R 7 c_0 = (1 - c_A)^{1/2}$. In the short wave limit $\xi \gg 1$ the branches asymptotically approach the lines $\eta = \xi$ and $\eta = \gamma_1 \xi$. If $\gamma_A = 0, \gamma_1 < 1$ then in this exceptional case the branches meet in one point. Since the free energy (12) should be positive definite, we have always $\gamma_A > 0$. There is, however, no physical restriction on the magnitude of γ_1. Figure 2 shows an example where $\gamma_1 < \gamma_R < 1$, and Fig. 3 - where $\gamma_R < \gamma_1 < 1$.

The important question is how the hierarchical model describes the situation. The corresponding dispersion relation (66) provides an approximation for the acoustical branch only. The curve starts at $\xi = 0$ with the slope γ_R and, for $\xi \to \infty$, tends asymptotically to the line $\eta = \gamma_1 \xi$ provided $\gamma_A = 0$(see Fig. 3). The special feature of this approximation is that it can be used over the whole range of wave numbers, since it does not represent a short-wave or long-wave approximation. The underlying assumption is that the influence of the microstructure is small. In case in Fig. 3, the full and approximate dispersion relations agree pretty well. The approximation gets worse if the parameter γ_A tends to zero and, for $\gamma_A = 0$, degenerates to the nondispersive wave represented by $\eta = \xi$.

The simplified cases (67) and (68) give rather distorted results. The dispersion curves deviate strongly from the correct course (see Fig. 4).

5.3 Multiple scales

The corresponding mathematical models are represented in Section 4.3. Introducing now (61) into equations (35), (36), (37), we obtain the dispersion relation

$$\begin{aligned}&\left(c_0^2\, k^2 - \omega^2\right)\ \left(c_1^2\, k^2 - \omega^2 + \omega_1^2\right)\ \left(c_2^2\, k^2 - \omega^2 + \omega_2^2\right) - \\ &- c_B^2 \omega_2^2\, k^2\ \left(c_0^2\, k^2 - \omega^2\right) - c_A^2 \omega_1^2\, k^2\ \left(c_2^2\, k^2 - \omega^2 + \omega_2^2\right) = 0.\end{aligned} \tag{69}$$

In addition, the hierarchical governing equation (60) gives, respectively:

$$\begin{aligned}&\left(c_0^2 - c_A^2\right)\ k^2 - \omega^2 + c_A^2\, p_1^2\, k^2 \left[\left(c_1^2 - c_B^2\right) k^2 - \omega^2\right] + \\ &+ c_A^2\, p_2^2\, q^2\, k^4 \left(c_2^2\, k^2 - \omega^2\right) = 0.\end{aligned} \tag{70}$$

Here $\omega_1^2 = 1/p_1^2,\ \omega_2^2 = 1/p_2^2$. In further analysis, the dimensionless quantities

$$\xi = p_1\, c_0\, k, \qquad \eta\ = p_1\, \omega, \tag{71}$$

are used. Introducing (71) into eqs (69) and (70), the following dimensionless dispersion relations are obtained, respectively:

$$\begin{aligned}&\left(\xi^2 - \eta^2\right)\ \left(\gamma_1^2\, \xi^2 - \eta^2 - \eta_1^2\right)\ \left(\gamma_2^2 \xi^2 - \eta^2 + \eta_2^2\right) - \\ &\quad -\gamma_B^2 \left(\eta_2^2/\eta_1^2\right) \xi^2 \left(\xi^2 - \eta^2\right)\ - \gamma_A^2\, \xi^2\ \left(\gamma_2^2 \xi^2 - \eta^2 + \eta_2^2\right) = 0,\end{aligned} \tag{72}$$

$$\begin{aligned}&\left(\ \gamma_R^2\, \xi^2 - \eta^2\right)\ +\ \gamma_A^2\, \xi^2 \left(\gamma_Q^2\, \xi^2 - \eta^2\right) + \\ &\qquad\qquad +\gamma_A^2\, \gamma_{12}^2\, \xi^4 \left(\gamma_2^2\, \xi^2 - \eta^2\right) = 0.\end{aligned} \tag{73}$$

The parameters in (72), (73) denote the ratios of velocities:

$$\begin{aligned}&\gamma_1 = c_1/c_0,\ \ \gamma_2 = c_2/c_0,\ \ \gamma_A = c_A/c_0,\ \ \gamma_B = c_B/c_0, \\ &\gamma_R = c_R/c_0, \quad \gamma_Q = c_Q/c_0, \quad \gamma_{12} = (p_1 q)/(p^2 c_0),\end{aligned} \tag{74}$$

where $c_R = \left(c_0^2 - c_A^2\right)^{1/2}$, $c_Q = \left(c_1^2 - c_B^2\right)^{1/2}$ and the ratios of fixed frequencies

$$\eta_1 = \omega_1^2\, p_1^2 = 1, \quad \eta_2 = \omega_2^2\, p_2^2 = p_2^2/p_1^2. \tag{75}$$

6 Nonlinearities

As said in the Introduction, one should often account for large deformation or complicated stress-strain relations that leads to nonlinear mathematical models. This means that the full deformation tensor involves nonlinear terms and a more complicated free energy function W (higher-order than quadratic) should be used. In this case dispersion effects described in Section 5 are combined with nonlinear effects. Here we represent a brief description of the nonlinear theory based on our earlier results (Engelbrecht. Pastrone 2003, Berezovski et al. 2003, Janno. engelbrecht 2004).

We use here only single-scale model (10), (11). Based on estimations that physical nonlinearity (stress-strain relation) is stronger than geometrical (full strain tensor), we limit ourselves here only with the more complicated free energy function (see estimations in Engelbrecht, 1983). So instead of (12) we assume

$$W = W_2 + W_3, \tag{76}$$

where W_2 is the simplest quadratic function (12)

$$W_2 = \frac{1}{2}\, \alpha\, u_x^2 + A\, \varphi\, u_x + \frac{1}{2}\, B\, \varphi^2 + \frac{1}{2}\, C\, \psi^2 \tag{77}$$

and W_3 includes nonlinearities on both the macro- and microlevel

$$W_3 = \frac{1}{6}\, N\, u_x^3 + \frac{1}{6}\, M\, \varphi_x^3. \tag{78}$$

Using the relations (9) for determining the macrostress, the microstress and the interactive force, we obtain

$$\rho\, u_{tt} \;=\; \alpha\, u_{xx} \;+\; N u_x\, u_{xx} \;+\; A\, \varphi_x \tag{79}$$

$$I\, \varphi_{tt} \;=\; C\, \varphi_{xx} \;+\; M \varphi_x\, \varphi_{xx} \;-\; A\, u_x \;-\; B\, \varphi \tag{80}$$

(cf. with (13), (14)).

We introduce the same dimensionless variables and scaling as in Section 4, adding $M = M^*\, l^3$. Following the same scheme as before, we come to the following hierarchical equation (cf. eq (42)):

$$\begin{aligned} U_{TT} = &\left(1 - \frac{c_A^2}{c_0^2}\right)\, U_{XX} + \frac{1}{2}\, k_1 \left(U_X^2\right)_X + \\ &+ \frac{c_A^2}{c_B^2} \left(U_{TT} - \frac{c_1^2}{c_0^2}\, U_{XX}\right)_{XX} \;+\; \frac{1}{2}\, k_2 \left(U_{XX}^2\right)_{XX}. \end{aligned} \tag{81}$$

Here $k_1 = N\, \epsilon/\alpha,\ k_2 = \delta^{3/2}\, (A^3\, M^* \epsilon)\, /\, (\alpha\, B^3)$ are the parameters expressing the strengths of physical nonlinearities on macro- and microscale, respectively. It has been shown (Janno, Engelbrecht, 2004) that this model may exhibit the balance between nonlinear and dispersive effects and therefore solitary waves may exist. The similar situation arises for nonlinear waves in rods (Samsonov 2001, Porubov 2003) when the governing equation is of the type (81) with $k_1 \neq 0,\ k_2 = 0$.

7 Discussion

Nonclassical theory of continua takes the internal scales into account and is therefore able to describe microstructural effects. In the limit case we could intuitively understand that the microstructure is composed by (different) particles and so we are actually dealing with crystal lattices (Brillouin 1953, Maugin 1999). In crystal lattices the simplest case with identical particles leads to a dispersion relation (Brillouin, 1953)

$$\omega(k) = B|\sin\ \pi k d| \tag{82}$$

in our notation with $B =$ const and d - the distance between the particles. Comparing (82) with dispersion relations of Section 5, it is obvious that our model

grasps the essential convexity of the dispersion curve for smalles wavelengths which correspond to the larger wavelengths in the particle model. Dispersion is then normal, i.e. $v_{ph} > v_{gr}$.

The model we have used for describing the microstructure is rather general: it is based on Euler-Lagrange equations and it could be represented also in terms of the balance of pseudo momentum (Maugin 1993).

The main value of the model is the explicit description of the hierarchy in Whitham's sense - the model is composed by two (or more) wave operators and depending on the characteristics of the initial excitation (the wavelength), a certain wave operator governs the wave motion. The dispersion curves (Section 5) demonstrate the transformation from one operator with its wave speed to another. The wave speed in the material is affected by the microstructure (cf. eq (42) with dimensionless velocity $1 - c_A^2/c_0^2$). Contrary to simplified models, the double dispersion (different terms U_{TTXX} and U_{XXXX} is of importance like in the case of waves in rods (Samsonov 2001, Porubov 2003). In physical terms, the effects of inertia of microstructure and wave velocity in pure microstructure both affect the wave motion in general. This is not possible when deriving the continuum models from the lattice theory using series representation.

The multi-scale model (59) actually prolongs the hierarchical properties of the single-scale model (42). Indeed, the wave operators macro versus micro 1 and micro 1 versus micro 2 are related by similar sign convention and the wave velocity in microstructure 1 is affected by properties of microstructure 2 similarly like wave velocity in macro is affected by properties of microstructure 1. In addition, the scaling goes like $\mathcal{O}(1), \delta_1, \delta_2^2, ...$ indicating the successive hierarchy in the sequence of wave operators. It is clearly seen that higher-order dispersive terms $U_{XXXX},\ U_{XXXXXX}, ...$ coincide with those derived from the lattice theory (Maugin 1999) but mixed derivatives $U_{TTXX},\ U_{TTXXXX}, ...$ reflect the role of microinertias.

The proper modelling is certainly important for solving direct problems, given the initial excitation and calculating the wave field. Not less important are the inverse problems when given the excitation and measured wave field, the material properties must be determined. The methods of Nondestructive Testing (NDT) of materials are all based on solving the inverse problems. Based on the essentially more accurate mathematical model described in this paper, it is possible to determine the properties of the microstructured solids. The preliminary results in this direction are obtained by Janno and Engelbrecht (2004) with regard to eq (42). It is possible to determine 3 material parameters from 3 phase velocities which are measured using various wavelengths.

The results above are also supported by numerical calculations (Berezovski et al. 2003, Engelbrecht et al. 2004) but there are many studies in progress.

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