

Internal Variables and Scale Separation in Dynamics of Microstructured Solids

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Abstract Internal variables are introduced in the framework of canonical thermo-mechanics on the material manifold. The canonical equations for energy and pseudomomentum cannot be separated by means of the scale separation because these equations should concern all fields together and, therefore, all scales together. However, the intrinsic interaction force requires a kinetic relation for internal variables. This kinetic relation depends on representation of the internal variable as “internal variable of state” or “internal degree of freedom”.

Keywords Internal variables · kinetic relation · material formulation

1 Introduction

Numerical simulation of wave propagation in heterogeneous solids suggests an approximation of fields inside computational cells. In finite volume methods, all the fields are averaged over the cell volume. If the mesh is sufficiently fine, then the medium inside a cell can be considered as homogeneous. However, in the case of media with microstructure, cells remain inhomogeneous for any size of the mesh. This means that the influence of the microstructure on the global behavior of material should be taken into account.

Theoretical concepts of continua with microstructure based on notions of continuum mechanics were developed in [1–10]. In practice, however, much more simple approaches are used because of the lack of constitutive information needed for the full theory.

Homogenization methods in micromechanics rely upon on the prediction of overall properties of heterogeneous materials [11] on the basis of properties of constituents and microstructure geometry. The result yields governing equations for “effective” media. Though homogenization methods can be quite complicated [12,

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13], the effective media theory is applied usually to quasi-static problems. Clearly, it is not sufficient for the description of local dynamics in microstructured materials.

Another approach to describe the influence of the microstructure is based on the introduction of internal variables [14–17]. As pointed out in [15], “No actual inelastic behavior is likely to be so simply characterized.” However, “the state of the material sample at any given time in the deformation history is taken to be fully characterized by corresponding values of the strain and temperature and the collection of internal variables which mark the extent of microstructural rearrangement within the sample” [15].

The main difficulty in the internal variable theory is the derivation of the evolution (kinetic) equations for internal variables. In this paper, it is shown that the evolution equation for an internal variable describing the influence of microstructure depends on representation of the internal variable as “internal variable of state” or “internal degree of freedom”.

We start with standard formulation of balance laws in material setting in Section 2. Then, in Section 3, the canonical equations of thermomechanics on material manifold are remembered. Internal variables are introduced in Section 4, where the case of non-zero extra entropy flux is also considered. Scale separation and consequent kinetic equations for internal variables are discussed in Section 5. At last, the one-dimensional case is considered as an example in Section 6.

2 Local Balance Laws

The material body is considered as an open, simply connected subset \mathcal{B} of the material manifold \mathcal{M}^3 of material points (or particles), which are referred to a position \mathbf{X} in a reference configuration K_R . Let K_t be the actual (at time t) configuration of the solid body \mathcal{B} in physical space E^3 . The direct time-parameterized motion of \mathbf{X} is given by the sufficiently regular function

$$\mathbf{x} = \chi(\mathbf{X}, t), \quad (1)$$

which represents the time sequence of physical configurations occupied by the material point \mathbf{X} in \mathcal{M}^3 as time goes on. The physical velocity \mathbf{v} and the direct-motion deformation gradient \mathbf{F} are defined by

$$\mathbf{v} := \left. \frac{\partial \chi}{\partial t} \right|_{\mathbf{X}}, \quad \mathbf{F} := \left. \frac{\partial \chi}{\partial \mathbf{X}} \right|_t \equiv \nabla_R \chi. \quad (2)$$

In the case of thermoelastic conductors of heat, we have the following *local balance laws* at any regular point \mathbf{X} [18, 19]

$$\left. \frac{\partial}{\partial t} \rho_0 \right|_{\mathbf{X}} = 0, \quad (3)$$

$$\left. \frac{\partial}{\partial t} \mathbf{p} \right|_{\mathbf{x}} - \nabla_R \cdot \mathbf{T} = \mathbf{f}, \quad (4)$$

$$\left. \frac{\partial}{\partial t} \mathcal{H} \right|_{\mathbf{x}} - \nabla_R \cdot (\mathbf{T}\mathbf{v} - \mathbf{Q}) = 0, \quad (5)$$

and the second law of thermodynamics

$$\left. \frac{\partial S}{\partial t} \right|_{\mathbf{x}} + \nabla_R \cdot \mathbf{S} \geq 0, \quad \mathbf{S} = (\mathbf{Q}/\theta) + \mathbf{K}. \quad (6)$$

Here t is time, $\rho_0(\mathbf{X})$ is the matter density in the reference configuration, $\mathbf{p} = \rho_0 \mathbf{v}$ is the linear momentum, \mathbf{T} is the first Piola-Kirchhoff stress tensor, $\mathcal{H} = K + E$, $K = 1/2 \rho_0 \mathbf{v}^2$ is the kinetic energy per unit volume in the reference configuration, E is the corresponding internal energy, S is the entropy per unit volume, θ is temperature, \mathbf{Q} is the material heat flux, \mathbf{S} is the entropy flux, and the “extra entropy flux” \mathbf{K} vanishes in most cases, but this is not a basic requirement.

3 Canonical Thermomechanics on the Material Manifold

Inhomogeneities in solids evolve due to the configurational forces first derived and discussed by Eshelby [20]. The role played by configurational forces on (macro) defects has attracted increased interest in the recent years [18, 21–24].

3.1 A Canonical Form of the Energy Conservation

By taking the scalar product of eqn. (4) with \mathbf{v} , we traditionally obtain the so-called *theorem of the kinetic energy*:

$$\left. \frac{\partial K}{\partial t} \right|_{\mathbf{x}} - \nabla_R \cdot (\mathbf{T}\mathbf{v}) + \mathbf{T} : \dot{\mathbf{F}} - \mathbf{f} \cdot \mathbf{v} = 0. \quad (7)$$

Combining this with the first law of thermodynamics (5) we obtain the so-called *theorem of internal energy*:

$$\left. \frac{\partial E}{\partial t} \right|_{\mathbf{x}} - \mathbf{T} : \dot{\mathbf{F}} + \nabla_R \cdot \mathbf{Q} = 0. \quad (8)$$

A most interesting form of the energy equation for subsequent developments reads (with $E := W + S\theta$)

$$\left. \frac{\partial(S\theta)}{\partial t} \right|_{\mathbf{x}} + \nabla_R \cdot \mathbf{Q} = h^{int}, \quad h^{int} := \mathbf{T} : \dot{\mathbf{F}} - \frac{\partial W}{\partial t}, \quad (9)$$

where the right-hand side of eqn. (9)₁ is formally an *internal heat source*.

3.1.1 Zero Extra Entropy Flux

In the case where $\mathbf{K} = \mathbf{0}$, introducing the Helmholtz free energy function by $W = E - S\theta$, the second law (6) multiplied by θ yields the celebrated *Clausius-Duhem inequality*

$$-\left(\frac{\partial W}{\partial t} + S\frac{\partial\theta}{\partial t}\right) + \mathbf{T} : \dot{\mathbf{F}} - \mathbf{S}\nabla_R\theta \geq 0. \quad (10)$$

We also note the following other form of the Clausius-Duhem inequality:

$$S\dot{\theta} + \mathbf{S}\nabla_R\theta \leq h^{int}. \quad (11)$$

3.1.2 Non-Zero Extra Entropy Flux

In the case where $\mathbf{K} \neq \mathbf{0}$, instead of (10) we shall obtain the more general expression

$$-\left(\frac{\partial W}{\partial t} + S\frac{\partial\theta}{\partial t}\right) + \mathbf{T} : \dot{\mathbf{F}} + \nabla_R \cdot (\theta\mathbf{K}) - \mathbf{S}\nabla_R\theta \geq 0, \quad (12)$$

where \mathbf{S} is still given by the general expression (6)₂ and (12) differs from (10) only by the divergence term. Equations (9) are left unchanged:

$$\frac{\partial(S\theta)}{\partial t} + \nabla_R \cdot \mathbf{Q} = h^{int}, \quad h^{int} = \mathbf{T} : \dot{\mathbf{F}} - \frac{\partial W}{\partial t}, \quad (13)$$

but eqn. (11) takes on the form

$$S\dot{\theta} + \mathbf{S}\nabla_R\theta \leq h^{int} + \nabla_R \cdot (\theta\mathbf{K}). \quad (14)$$

3.2 Canonical (Material) Momentum Conservation

Applying \mathbf{F} to the right of eqn. (4) and noting that (T = transpose)

$$\left(\frac{\partial(\rho_0\mathbf{v})}{\partial t}\right) \cdot \mathbf{F} = -\frac{\partial\mathbf{P}}{\partial t}\Big|_{\mathbf{x}} - \nabla_R \left(\frac{1}{2}\rho_0\mathbf{v}^2\right) + \left(\frac{1}{2}\mathbf{v}^2\right)\nabla_R\rho_0, \quad (15)$$

and

$$(\nabla_R \cdot \mathbf{T}) \cdot \mathbf{F} = \nabla_R \cdot (\mathbf{T}\mathbf{F}) - \mathbf{T} : (\nabla_R\mathbf{F})^T, \quad (16)$$

where we have set

$$\mathbf{P} := -\rho_0\mathbf{v} \cdot \mathbf{F}, \quad (17)$$

the *material momentum*, and introducing plus and minus the material gradient of an (*unspecified*) free energy density $W = \overline{W}(\cdot, \cdot, \cdot, \cdot, \mathbf{X})$, we then check that eqn. (4) yields the following material balance of momentum [24]

$$\frac{d\mathbf{P}}{dt} - \nabla_R \cdot \mathbf{b} = \mathbf{f}^{int} + \mathbf{f}^{ext} + \mathbf{f}^{inh}, \quad (18)$$

where the material *Eshelby stress* \mathbf{b} , the material *inhomogeneity force* \mathbf{f}^{inh} , the material *external* (or body) force \mathbf{f}^{ext} , and the material *internal* force \mathbf{f}^{int} are defined by

$$\mathbf{b} = -(L_W \mathbf{1}_R + \mathbf{T} \cdot \mathbf{F}), \quad L_W = K - W, \quad (19)$$

$$\mathbf{f}^{inh} := \left. \frac{\partial L_W}{\partial \mathbf{X}} \right|_{expl} \equiv \left. \frac{\partial L_W}{\partial \mathbf{X}} \right|_{fixed\ fields} = \left(\frac{1}{2} \mathbf{v}^2 \right) \nabla_R \rho_0 - \left. \frac{\partial \bar{W}}{\partial \mathbf{X}} \right|_{expl}, \quad (20)$$

$$\mathbf{f}^{ext} := \mathbf{f} \cdot \mathbf{F}, \quad \mathbf{f}^{int} = \mathbf{T} : (\nabla_R \mathbf{F})^T - \nabla_R W|_{impl}. \quad (21)$$

Here the subscript notations *expl* and *impl* mean, respectively, the material gradient keeping the fields fixed (and thus extracting the explicit dependence on \mathbf{X}), and taking the material gradient only through the fields present in the function.

The general expressions of eqns. (9)₁ and (18)

$$\frac{\partial(S\theta)}{\partial t} + \nabla_R \cdot \mathbf{Q} = h^{int}, \quad h^{int} = \mathbf{T} : \dot{\mathbf{F}} - \frac{\partial W}{\partial t}, \quad (22)$$

$$\frac{d\mathbf{P}}{dt} - \nabla_R \cdot \mathbf{b} = \mathbf{f}^{int} + \mathbf{f}^{ext} + \mathbf{f}^{inh}, \quad (23)$$

are the most general *canonical equations for momentum and energy* we can write down without a postulate of the full dependency of W . It is remarkable that eqn. (18) reads formally just the same whether the extra entropy vector \mathbf{K} vanishes or does not vanish, so that eqns. (22) and (23) are not only consistent with one another, but they are also independent on whether we remain in the standard thermodynamic framework or we deviate from it by the postulate of existence of an extra entropy flux.

In the present approach, in order to proceed further we need to specify the full functional dependence of free energy W .

4 Internal Variables

In the classical thermoelasticity of conductors, the constitutive equations (laws of state) are given in terms of free energy per unit volume, W , by

$$W = W(\mathbf{F}, \theta), \quad \mathbf{T} = \frac{\partial W}{\partial \mathbf{F}}, \quad S = -\frac{\partial W}{\partial \theta}. \quad (24)$$

Equations (3), (4), (5), (6), (7), (8), (9), (10), (11), (12), (13), (14), (15), (16), (17), (18), (19), (20), (21), (22), (23) and (24), together with a more precise expression for W and some of its mathematical properties (e.g., convexity), are those to be used in studying sufficiently regular nonlinear dynamical processes in thermoelastic conductors.

Let φ the internal variable of state whose tensorial nature is not specified. Then W is specified as the general sufficiently regular function

$$W = \overline{W}(\mathbf{F}, \theta, \varphi, \nabla_R \varphi). \quad (25)$$

4.1 Zero Extra Entropy Flux

First we assume that \mathbf{K} vanishes. The *equations of state* (in a sense, mere definition of the partial derivatives of the free energy) are given by *Gibbs' equation* as

$$\mathbf{T} = \frac{\partial \overline{W}}{\partial \mathbf{F}}, \quad S = -\frac{\partial \overline{W}}{\partial \theta}, \quad \tau := -\frac{\partial \overline{W}}{\partial \varphi}, \quad \boldsymbol{\eta} := -\frac{\partial \overline{W}}{\partial \nabla_R \varphi}. \quad (26)$$

Accordingly, we find that the “internal” material force and heat source each split in two terms according to

$$\mathbf{f}^{int} = \mathbf{f}^{th} + \mathbf{f}^{intr}, \quad h^{int} = h^{th} + h^{intr}, \quad (27)$$

where the *thermal sources* have already been defined and the “*intrinsic*” sources are given by

$$\mathbf{f}^{intr} := \tau(\nabla_R \varphi)^T + \boldsymbol{\eta} \cdot \nabla_R(\nabla_R \varphi)^T, \quad h^{intr} := \tau \dot{\varphi} + \boldsymbol{\eta} \cdot (\nabla_R \dot{\varphi})^T, \quad (28)$$

so that we have the following consistent (obviously non-Hamiltonian) system of canonical balance laws:

$$\frac{d\mathbf{P}}{dt} - \nabla_R \cdot \mathbf{b} = \mathbf{f}^{th} + \mathbf{f}^{intr}, \quad \frac{\partial(S\theta)}{\partial t} + \nabla_R \cdot \mathbf{Q} = h^{th} + h^{intr}, \quad (29)$$

while the dissipation reads

$$\Phi = h^{intr} - S \nabla_R \theta \geq 0, \quad \mathbf{K} \equiv \mathbf{0}. \quad (30)$$

Here the thermodynamical forces τ and $\boldsymbol{\eta}$ are purely dissipative by virtue of the “internal” character of the state variable φ .

4.2 Non-Zero Extra Entropy Flux

A more *field-theoretic viewpoint* is to envisage the set of eqns. (12), (13) and (14) as holding true and selecting the non-zero \mathbf{K} such that the total divergence term obtained in (12) in its exploitation with (13) and (14), be identically zero, i.e.,

$$\mathbf{K} = -\theta^{-1} \boldsymbol{\eta} \dot{\varphi}. \quad (31)$$

This follows the scheme originally developed [25] for materials with *diffusive* dissipative processes described by means of internal variables of state.

Then it is readily shown that the canonical equations of momentum and energy read:

$$\frac{d\mathbf{P}}{dt} - \nabla_R \cdot \tilde{\mathbf{b}} = \mathbf{f}^{th} + \tilde{\mathbf{f}}^{intr}, \quad \frac{\partial(S\theta)}{\partial t} + \nabla_R \cdot \tilde{\mathbf{Q}} = h^{th} + \tilde{h}^{intr}, \quad (32)$$

$$\Phi = \tilde{h}^{intr} - \tilde{\mathbf{S}} \nabla_R \theta \geq 0, \quad \tilde{h}^{intr} := \tilde{\tau} \dot{\varphi} \quad (33)$$

where we have introduced the new definitions

$$\tilde{\tau} \equiv -\frac{\delta \bar{W}}{\delta \varphi} := -\left(\frac{\partial \bar{W}}{\partial \varphi} - \nabla_R \cdot \frac{\partial \bar{W}}{\partial (\nabla_R \varphi)} \right) = \tau - \nabla_R \cdot \boldsymbol{\eta}, \quad \tilde{\mathbf{S}} = \theta^{-1} \tilde{\mathbf{Q}}, \quad (34)$$

$$\tilde{\mathbf{Q}} = \mathbf{Q} - \boldsymbol{\eta} \dot{\varphi}, \quad \tilde{\mathbf{b}} = -(\mathbf{L} \mathbf{1}_R + \mathbf{T} \mathbf{F} - \boldsymbol{\eta} \cdot (\nabla_R \varphi)^T), \quad \tilde{\mathbf{f}}^{intr} := \tilde{\tau} \nabla_R \varphi. \quad (35)$$

In this formulation the Eshelby stress complies with its role of grasping all effects presenting gradients since the material gradient of φ plays a role parallel to that of the deformation gradient \mathbf{F} .

5 Scale Separation

It follows from (35) that $\tilde{\tau}$ represents a physical body force (per unit volume).

If we consider that the macroscopic motion is characterized by a scale with a characteristic length l_{macro} and the micromotion by a characteristic length l_{micro} , then the length-scales are assumed to differ by an order of magnitude. This scale separation requires

$$l_{micro}/l_{macro} \ll 1. \quad (36)$$

The latter means that the force $\tilde{\tau}$ and, therefore, $\boldsymbol{\eta}$ have different order of magnitude than the macroscopic stress tensor \mathbf{T} , while $\tilde{\mathbf{f}}^{intr}$ remains of the order of unity. It follows that the force $\tilde{\tau}$ should not appear in the balance of linear momentum (4) at the length scale l_{macro} . Fortunately, the same interactive force $\tilde{\tau}$ together with the time derivative of the internal variable φ determines the intrinsic heat source in the dissipation inequality (33). Therefore, the evolution equation of the internal variable is dependent on the interactive force $\tilde{\tau}$.

Here we should emphasize the distinction between the notions of internal variables of state and of internal degrees of freedom (cf. [16]). Essentially, an internal variable of state has no inertia, but it dissipates. Accordingly, the thermodynamical force $\tilde{\tau}$ acting on it must be proportional to the first-order time derivative of the variable. This is in the correspondence with standard irreversible thermodynamics. If we keep the idea of an internal variable of state then according to the dissipation (33),

$$\tilde{\tau} = k \dot{\varphi}. \quad (37)$$

The dissipation inequality (33) is automatically satisfied in the isothermal case with $k \geq 0$ since

$$\Phi = k\dot{\varphi}^2 \geq 0. \quad (38)$$

In this case, the *kinetic equation for internal variable* reads

$$k \left. \frac{\partial}{\partial t} \varphi \right|_{\mathbf{x}} = \tilde{\tau}, \quad (39)$$

or

$$k \left. \frac{\partial}{\partial t} \varphi \right|_{\mathbf{x}} + \nabla_R \cdot \boldsymbol{\eta} = \tau. \quad (40)$$

An internal degree of freedom normally owns an inertia and, consequently, its equation of motion should contain a second-order time derivative of the said variable.

If we introduce an inertia, the variable φ becomes an internal degree of freedom. In this case, we expect that the interactive force $\tilde{\tau}$ is balanced by time variation of an internal momentum, $\mathbf{p}' = \mathbf{I}\dot{\varphi}$, where \mathbf{I} is an appropriate tensor of internal inertia

$$\left. \frac{\partial}{\partial t} \mathbf{p}' \right|_{\mathbf{x}} = \tilde{\tau}, \quad (41)$$

or

$$\left. \frac{\partial}{\partial t} \mathbf{I}\dot{\varphi} \right|_{\mathbf{x}} + \nabla_R \cdot \boldsymbol{\eta} = \tau. \quad (42)$$

The choice of kinetic equation for internal variables should depend on a particular problem.

Often the distinction between internal variables of state and internal degrees of freedom is related to a time scale (which is not considered here). In truth if high frequency phenomena are considered we better base on the consideration the “internal degree of freedom” version, because inertia plays a role. Otherwise, if dissipation (e.g., viscosity) dominates, then inertia of the variable can be neglected, and there remains the “internal variable of state” view. Microscopically, where everything is thermodynamically reversible, only internal degrees of freedom exist. But sometimes they are perceived macroscopically only as “internal variables of state” depending on the time-scaling conditions.

6 Example: Microstructure in One-Dimension

In one-dimensional case, the governing equation is the balance of linear momentum

$$\frac{\partial}{\partial t}(\rho_0 v) = \frac{\partial \sigma}{\partial x}. \quad (43)$$

To be able to describe the microstructure influence, we introduce an internal variable φ which we associate with the integral effect of microdeformation. We suppose that the free energy depends on the internal variable φ and its space derivative

$$W = \overline{W}(u_x, \varphi, \varphi_x), \quad (44)$$

where u is the displacement. Then the constitutive equations follow

$$\sigma := \frac{\partial \overline{W}}{\partial u_x}, \quad \tau := -\frac{\partial \overline{W}}{\partial \varphi}, \quad \eta := -\frac{\partial \overline{W}}{\partial \varphi_x}. \quad (45)$$

We choose the kinetic equation for the internal variable φ in the form (42)

$$\frac{\partial}{\partial t}(I\dot{\varphi}) + \frac{\partial \eta}{\partial x} = \tau. \quad (46)$$

The latter means that φ is actually the internal degree of freedom.

The simplest free energy dependence is a quadratic function [26]

$$\overline{W} = \frac{\rho_0 c^2}{2} u_x^2 + A\varphi u_x + \frac{1}{2} B\varphi^2 + \frac{1}{2} C\varphi_x^2, \quad (47)$$

where u is the displacement, c is the elastic wave speed. The first two terms in the right hand side of (47) correspond to the macroscopic part of the free energy function

$$\tilde{W} = \frac{\rho_0 c^2}{2} u_x^2 + A\varphi u_x, \quad (48)$$

while the rest represents its ‘‘internal’’ part

$$W' = \frac{1}{2} B\varphi^2 + \frac{1}{2} C\varphi_x^2. \quad (49)$$

This means that the stress components are determined as follows:

$$\sigma = \frac{\partial \overline{W}}{\partial u_x} = \rho_0 c^2 u_x + A\varphi, \quad \eta = -\frac{\partial \overline{W}}{\partial \varphi_x} = -C\varphi_x. \quad (50)$$

As previously, τ is coincided with the interactive internal force

$$\tau = -\frac{\partial \overline{W}}{\partial \varphi} = -A u_x - B\varphi. \quad (51)$$

As a result, we can represent the equations of motion (43), (46) in the form

$$\rho_0 u_{tt} = \rho_0 c^2 u_{xx} + A\varphi_x, \quad I\varphi_{tt} = C\varphi_{xx} - A u_x - B\varphi. \quad (52)$$

In terms of stresses introduced by (45), the same system of equations is represented as

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x}, \quad I \frac{\partial^2 \varphi}{\partial t^2} = -\frac{\partial \eta}{\partial x} + \tau. \quad (53)$$

It is worth to note that the same equations are derived in [27] based on a different consideration. We can determine the first space derivative of the internal variable from the equation (52)₂

$$\varphi_x = -\frac{I}{B} \varphi_{ttx} + \frac{C}{B} \varphi_{xxx} - \frac{A}{B} u_{xx}, \quad (54)$$

and its third derivatives from eqn. (52)₁

$$\frac{A}{\rho_0} \varphi_{xxx} = (u_{tt} - c^2 u_{xx})_{xx}, \quad \frac{A}{\rho_0} \varphi_{ttx} = (u_{tt} - c^2 u_{xx})_{tt}. \quad (55)$$

Inserting the results into the balance of linear momentum, we obtain a more general equation

$$u_{tt} = c^2 u_{xx} + \frac{C}{B} (u_{tt} - c^2 u_{xx})_{xx} - \frac{I}{B} (u_{tt} - c^2 u_{xx})_{tt} - \frac{A^2}{\rho_0 B} u_{xx}, \quad (56)$$

which covers all the generalizations of wave equation obtained on the basis of continualization procedure [28, 29] as well as on the basis of homogenization methods [30, 31].

This model clearly exhibits the hierarchical structure of microstructured solids. The dispersive effects are obviously governed by the higher-order derivatives in eqn. (56). However, even if the dispersive effects are neglected, the wave velocity is influenced by the microstructure (the last term in the right hand side of eqn. (56)).

It may be also instructive to compare the developed approach with second gradient elasticity theory. In the one-dimensional case, the second gradient elasticity correspond with the choice of the internal variable in the form

$$\varphi = l^2 u_{xxx}, \quad (57)$$

where l is an internal length scale.

The the equations of motion (52) are rewritten then as follows

$$\rho_0 u_{tt} = \rho_0 c^2 u_{xx} + Al^2 u_{xxxx}, \quad Il^2 u_{ttxxx} = Cl^2 u_{xxxxx} - Au_x - Bl^2 u_{xxx}. \quad (58)$$

Summing latter equations (after integration the second equation over x), we obtain

$$(\rho_0 u + Il^2 u_{xx})_{tt} = (\rho_0 c^2 u + Cl^2 u_{xx})_{xx} - (Bl^2 u - Al^2 u_{xx})_{xx} - Au. \quad (59)$$

The one-dimensional version of second gradient elasticity with the appropriate choice of coefficients b and d can be represented as (cf. [32])

$$(u + bl^2u_{xx})_{tt} = c^2(u + dl^2u_{xx})_{xx}. \quad (60)$$

It is clear that the second gradient theory fully corresponds to our one-dimensional model with $A = B = 0$.

7 Conclusions

Scale separation cannot be directly applied to the canonical equations for energy and material momentum even if internal variables are introduced to describe the microstructure influence on the overall behavior. Nevertheless, the canonical equations show clearly that evolution equations for internal variables depend on the interactive force which is hidden in the standard formulation of continuum mechanics. This is a clear advantage of the approach over effective medium approximation.

Hyperbolic evolution equation for the internal variable provides the generalization of elastic wave equation, which covers all the models proposed on the basis of continualization and homogenization methods as well as on the second gradient elasticity theory.

The mathematical model like (56) can also be used for determining of the material properties, i.e. the coefficients of the equation. In linear case, the dependence of phase velocity on microstructure gives a solid ground for solving such an inverse problem [33]. In nonlinear case, the distortion of solitary waves due to microstructure can be used [34].

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