Internal Variables and Generalized Continuum Theories

Arkadi Berezovski, Jüri Engelbrecht and Gérard A. Maugin

Abstract The canonical thermomechanics on the material manifold is enriched by the introduction of dual weakly non-local internal variables and extra entropy fluxes. In addition to the dissipative reaction-diffusion equation for a single internal variable of state, a hyperbolic evolution equation for the internal degree of freedom can be also recovered in the non-dissipative case. It is demonstrated that the Mindlin micromorphic theory can be represented in terms of dual internal variables in a natural way in the framework of the canonical thermomechanics.

1 Motivation

The description of any phenomenon depends on how many details we take into account. Any description can be improved, e.g., by the transition to finer space and time scales. Though such a transition may be desirable for the understanding of a process at microscopic or quantum level, it is hardly acceptable from the practical point of view. Fortunately, there exists a possibility to include the influence of microstructural effects into the description of a phenomenon without changing of space and time scales. This is the introduction of internal variables.

The use of internal variables in the description of the behavior of materials with microstructure has a long tradition [1–11], and nowadays it is practically commonly accepted. However, there are two clearly distinctive types of internal variables: internal degrees of freedom and internal variables of state [5, 12]. By definition, internal variables of state must have no inertia, and they produce no external work.

Arkadi Berezovski · Jüri Engelbrecht
Centre for Nonlinear Studies, Institute of Cybernetics at Tallinn University of Technology, Akadeemia tee 21, 12618 Tallinn, Estonia; e-mail: arkadi.berezovski@cs.ioc.ee

Gérard A. Maugin
Institut Jean Le Rond d'Alembert, Université Pierre et Marie Curie, UMR 7190, Case 162, 4 Place Jussieu, 75252, Paris Cédex 05, France; e-mail: gerard.maugin@upmc.fr

P. Steinmann (ed.), IUTAM Symposium on Progress in the Theory and Numerics of Configurational Mechanics, 149–158.

The internal variables of state are not governed by a field equation, i.e., by their own balance law; the power expended by internal variables will be only of the dissipated type. From another side, internal degrees of freedom are endowed with both inertia and flux, where the latter is not necessarily purely dissipative (on the contrary, it could be purely non-dissipative) [5, 12].

Starting the modeling of dynamics of a microstructured material, we do not know definitely *a priori* what kind of internal variables is more suitable in the particular case. It is useful therefore to have a procedure which formalizes the choice. The main idea of such a formalization can be illustrated on the simple example of linear elasticity in one dimension.

One-dimensional elastodynamics is described by a Lagrangian density $\mathcal{L}$ that depends on displacement $u(x, t)$ and its first derivatives, which we denote by u_t for the time derivative and by u_x for the spatial derivative. This leads to the Euler–Lagrange equation of motion,

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial u_t}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial u_x}\right) - \frac{\partial \mathcal{L}}{\partial u} = 0. \tag{1}$$

In the linear case, the Lagrangian density has the form

$$\mathcal{L}(u, u_x, u_t) = \frac{1}{2}\left(\rho_0 {u_t}^2 - E u_x^2\right), \tag{2}$$

and we obtain the second-order wave equation for the single field variable u

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \tag{3}$$

where $c = \sqrt{E/\rho_0}$, E is the Young's modulus, and ρ_0 is the density.

Introducing velocity and strain by

$$v = u_t, \quad \varepsilon = u_x, \tag{4}$$

we can represent the wave equation as the system of two first-order equations for the two field variables

$$\frac{\partial \varepsilon}{\partial t} = \frac{\partial v}{\partial x}, \tag{5}$$

$$\rho_0 \frac{\partial v}{\partial t} = E \frac{\partial \varepsilon}{\partial x}. \tag{6}$$

The two variables, v and ε, are dual ones in the sense that the evolution of one of them is governed by another and vice versa. Just the same underlying idea is used for the introduction of dual internal variables, thermodynamics of which is given in [13].

In what follows, we will introduce two internal variables (which may have distinct tensorial nature) in the material formulation of thermomechanics and analyze the conditions that are necessary to classify the internal variables as internal degrees of freedom or internal variables of state. As an example, the micromorphic elasticity theory is presented as a particular case of the obtained formalization.

2 Canonical Thermomechanics on the Material Manifold

First, we need to recall certain basic definitions. A motion of a body is considered as a time-parametrized sequence of mappings χ between the reference configuration and the actual configuration: $\mathbf{x} = \chi(\mathbf{X}, t)$, where t is time, $\mathbf{X}$ represents the position of a material point in the reference configuration, and $\mathbf{x}$ is its position in the actual configuration. The deformation gradient is defined by

$$\mathbf{F} = \left.\frac{\partial \chi}{\partial \mathbf{X}}\right|_t = \nabla_R \chi. \tag{7}$$

If the constitutive relation for free energy has the form $W = \overline{W}(\mathbf{F}, \ldots, \mathbf{X}, t)$, then the first Piola–Kirchhoff stress tensor $\mathbf{T}$ is defined by

$$\mathbf{T} = \frac{\partial \overline{W}}{\partial \mathbf{F}}. \tag{8}$$

The local balance laws for sufficiently smooth fields at any regular material point $\mathbf{X}$ in the body read (cf. [14]):

$$\left.\frac{\partial \rho_0}{\partial t}\right|_{\mathbf{X}} = 0, \tag{9}$$

$$\left.\frac{\partial (\rho_0 \mathbf{v})}{\partial t}\right|_{\mathbf{X}} - \mathrm{Div}_R \mathbf{T} = \mathbf{f}_0, \tag{10}$$

$$\left.\frac{\partial (K + E)}{\partial t}\right|_{\mathbf{X}} - \nabla_R \cdot (\mathbf{T} \cdot \mathbf{v} - \mathbf{Q}) = \mathbf{f}_0 \cdot \mathbf{v}, \tag{11}$$

where ρ_0 is the mass density in the reference configuration, $\mathbf{v} = \partial \chi / \partial t|_X$ is the physical velocity, $\mathbf{f}_0$ is a body force per unit reference volume, $K = \rho_0 \mathbf{v}^2/2$ is the kinetic energy, E is the internal energy per unit reference volume, $\mathbf{Q}$ is the material heat flux, $d/dt = \partial/\partial t|_X$ or a superimposed dot denotes the material time derivative.

The second law of thermodynamics is written as

$$\left.\frac{\partial S}{\partial t}\right|_{\mathbf{X}} + \nabla_R \cdot \mathbf{S} \geq 0, \quad \mathbf{S} = (\mathbf{Q}/\theta) + \mathbf{K}, \tag{12}$$

where S is the entropy density per unit reference volume, θ is the absolute temperature, $\mathbf{S}$ is the entropy flux, and the “extra entropy flux” $\mathbf{K}$ vanishes in most cases, but this is not a basic requirement.

The canonical form of the energy conservation has the form [12, 15]

$$\left.\frac{\partial (S\theta)}{\partial t}\right|_{\mathbf{X}} + \nabla_R \cdot \mathbf{Q} = h^{\mathrm{int}}, \quad h^{\mathrm{int}} := \mathbf{T} : \dot{\mathbf{F}} - \left.\frac{\partial \overline{W}}{\partial t}\right|_{\mathbf{X}}, \tag{13}$$

where the right-hand side of Eq. $(13)_1$ is formally an internal heat source.

Then the second law can be represented in the form

$$S\dot{\theta} + \mathbf{S} \cdot \nabla_R \theta \le h^{\mathrm{int}} + \nabla_R \cdot (\theta \mathbf{K}). \tag{14}$$

Correspondingly, the canonical (material) momentum conservation equation is obtained as [12, 15]

$$\left.\frac{\partial \mathbf{P}}{\partial t}\right|_{\mathbf{X}} - \mathrm{Div}_R \mathbf{b} = \mathbf{f}^{\mathrm{int}} + \mathbf{f}^{\mathrm{ext}} + \mathbf{f}^{\mathrm{inh}}, \tag{15}$$

where the *material momentum* $\mathbf{P}$, the material *Eshelby stress* $\mathbf{b}$, the material *inhomogeneity force* $\mathbf{f}^{\mathrm{inh}}$, the material *external* (or body) force $\mathbf{f}^{\mathrm{ext}}$, and the material *internal* force $\mathbf{f}^{\mathrm{int}}$ are defined by

$$\mathbf{P} := -\rho_0 \mathbf{v} \cdot \mathbf{F}, \quad \mathbf{b} = -\left(L\mathbf{I}_R + \mathbf{T} \cdot \mathbf{F}\right), \quad L = K - \overline{W}, \tag{16}$$

$$\mathbf{f}^{\mathrm{inh}} := \left.\frac{\partial L}{\partial \mathbf{X}}\right|_{\mathrm{expl}} \equiv \left.\frac{\partial L}{\partial \mathbf{X}}\right|_{\text{fixed fields}} = \left(\frac{1}{2}\mathbf{v}^2\right) \nabla_R \rho_0 - \left.\frac{\partial \overline{W}}{\partial \mathbf{X}}\right|_{\mathrm{expl}}, \tag{17}$$

$$\mathbf{f}^{\mathrm{ext}} := -\mathbf{f}_0 \cdot \mathbf{F}, \quad \mathbf{f}^{\mathrm{int}} = \mathbf{T} : (\nabla_R \mathbf{F})^T - \left.\nabla_R \overline{W}\right|_{\mathrm{impl}}. \tag{18}$$

Here the subscript notations *expl* and *impl* mean, respectively, the material gradient keeping the fields fixed (and thus extracting the explicit dependence on $\mathbf{X}$), and taking the material gradient only through the fields present in the function.

3 Dual Internal Variables

Our goal is to show how the dual internal variables can be introduced in canonical thermomechanics. The corresponding theory with a single internal variable was recently presented in [12, 15]. The generalization of the internal variable theory to the case of two internal variables is straightforward. Let us consider the free energy W as a function of two internal variables, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, each of which is a second-order tensor

$$W = \overline{W}(\mathbf{F}, \theta, \boldsymbol{\alpha}, \nabla_R \boldsymbol{\alpha}, \boldsymbol{\beta}, \nabla_R \boldsymbol{\beta}). \tag{19}$$

In this case, the equations of state are given by

$$\mathbf{T} = \frac{\partial \overline{W}}{\partial \mathbf{F}}, \quad S = -\frac{\partial \overline{W}}{\partial \theta}, \quad \mathbf{A} := -\frac{\partial \overline{W}}{\partial \boldsymbol{\alpha}}, \quad \mathcal{A} := -\frac{\partial \overline{W}}{\partial \nabla_R \boldsymbol{\alpha}}, \tag{20}$$

$$\mathbf{B} := -\frac{\partial \overline{W}}{\partial \boldsymbol{\beta}}, \quad \mathcal{B} := -\frac{\partial \overline{W}}{\partial \nabla_R \boldsymbol{\beta}}. \tag{21}$$

We include into consideration the non-zero extra entropy flux according to the case of the single internal variable [12, 15]

$$\mathbf{K} = -\theta^{-1}\mathcal{A} : \boldsymbol{\alpha} - \theta^{-1}\mathcal{B} : \boldsymbol{\beta}. \tag{22}$$

The canonical equations of momentum and energy keep their form

$$\frac{\partial \mathbf{P}}{\partial t} - \mathrm{Div}_R \widetilde{\mathbf{b}} = \mathbf{f}^{th} + \widetilde{\mathbf{f}}^{\mathrm{intr}}, \quad \frac{\partial (S\theta)}{\partial t} + \nabla_R \cdot \widetilde{\mathbf{Q}} = h^{th} + \widetilde{h}^{\mathrm{intr}}, \tag{23}$$

with the modified Eshelby stress tensor

$$\widetilde{\mathbf{b}} = -(L\mathbf{1}_R + \mathbf{T} \cdot \mathbf{F} - \mathcal{A} : (\nabla_R \boldsymbol{\alpha})^T - \mathcal{B} : (\nabla_R \boldsymbol{\beta})^T), \tag{24}$$

and intrinsic source terms

$$\widetilde{\mathbf{f}}^{\mathrm{intr}} := \widetilde{\mathcal{A}} : \nabla_R \boldsymbol{\alpha} + \widetilde{\mathcal{B}} : \nabla_R \boldsymbol{\beta}, \quad \widetilde{h}^{\mathrm{intr}} := \widetilde{\mathcal{A}} : \dot{\boldsymbol{\alpha}} + \widetilde{\mathcal{B}} : \dot{\boldsymbol{\beta}}. \tag{25}$$

In the above equations the following definitions are used

$$\widetilde{\mathcal{A}} \equiv -\frac{\delta \overline{W}}{\delta \boldsymbol{\alpha}} := -\left(\frac{\partial \overline{W}}{\partial \boldsymbol{\alpha}} - \mathrm{Div}_R \frac{\partial \overline{W}}{\partial (\nabla_R \boldsymbol{\alpha})}\right) = \mathbf{A} - \mathrm{Div}_R \mathcal{A}, \tag{26}$$

$$\widetilde{\mathcal{B}} \equiv -\frac{\delta \overline{W}}{\delta \boldsymbol{\beta}} := -\left(\frac{\partial \overline{W}}{\partial \boldsymbol{\beta}} - \mathrm{Div}_R \frac{\partial \overline{W}}{\partial (\nabla_R \boldsymbol{\beta})}\right) = \mathbf{B} - \mathrm{Div}_R \mathcal{B}, \tag{27}$$

$$\widetilde{\mathbf{S}} = \theta^{-1}\widetilde{\mathbf{Q}}, \quad \widetilde{\mathbf{Q}} = \mathbf{Q} - \mathcal{A} : \dot{\boldsymbol{\alpha}} - \mathcal{B} : \dot{\boldsymbol{\beta}}, \tag{28}$$

$$\mathbf{f}^{th} = S\nabla_R \theta, \quad h^{th} = S\dot{\theta}, \tag{29}$$

which are similar to those in the case of the single internal variable [12, 15].

The corresponding dissipation inequality

$$\Phi = \widetilde{h}^{\mathrm{intr}} - \widetilde{\mathbf{S}} \nabla_R \theta \geq 0, \tag{30}$$

is reduced in the isothermal case to

$$\widetilde{h}^{\mathrm{intr}} := \widetilde{\mathcal{A}} : \dot{\boldsymbol{\alpha}} + \widetilde{\mathcal{B}} : \dot{\boldsymbol{\beta}} \geq 0. \tag{31}$$

The introduction of the second internal variable results in a more general form of evolution equations for the internal variables $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ than in the case of a single internal variable [12,15]. In accordance with (31) these evolution equations are chosen as

$$\begin{pmatrix} \dot{\boldsymbol{\alpha}} \\ \dot{\boldsymbol{\beta}} \end{pmatrix} = \mathbf{L} \begin{pmatrix} \widetilde{\mathcal{A}} \\ \widetilde{\mathcal{B}} \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} \dot{\boldsymbol{\alpha}} \\ \dot{\boldsymbol{\beta}} \end{pmatrix} = \begin{pmatrix} \mathbf{L}^{11} & \mathbf{L}^{12} \\ \mathbf{L}^{21} & \mathbf{L}^{22} \end{pmatrix} \begin{pmatrix} \widetilde{\mathcal{A}} \\ \widetilde{\mathcal{B}} \end{pmatrix}, \tag{32}$$

where components $\mathbf{L}^{11}, \ldots, \mathbf{L}^{22}$ of the linear operator $\mathbf{L}$ are dependent on state variables. Representing the linear operator $\mathbf{L}$ as the sum of symmetric and skew-symmetric components $\mathbf{L} = (\mathbf{L} + \mathbf{L}^T)/2 + (\mathbf{L} - \mathbf{L}^T)/2$, i.e.

$$\begin{pmatrix} \dot{\boldsymbol{\alpha}} \\ \dot{\boldsymbol{\beta}} \end{pmatrix} = \begin{pmatrix} \mathbf{L}^{11} & (\mathbf{L}^{12} + \mathbf{L}^{21})/2 \\ (\mathbf{L}^{21} + \mathbf{L}^{12})/2 & \mathbf{L}^{22} \end{pmatrix} \begin{pmatrix} \widetilde{\mathscr{A}} \\ \widetilde{\mathscr{B}} \end{pmatrix} + \\ + \begin{pmatrix} 0 & (\mathbf{L}^{12} - \mathbf{L}^{21})/2 \\ (\mathbf{L}^{21} - \mathbf{L}^{12})/2 & 0 \end{pmatrix} \begin{pmatrix} \widetilde{\mathscr{A}} \\ \widetilde{\mathscr{B}} \end{pmatrix}, \tag{33}$$

we can see that the symmetry of the linear operator $\mathbf{L}$, which is equivalent to the Onsagerian reciprocity relations $\mathbf{L}^{12} = \mathbf{L}^{21}$, leads to the elimination of the antisymmetric part of the linear operator $\mathbf{L}$. However, we have no reasons to assume the symmetry of the linear operator $\mathbf{L}$ in the case of arbitrary internal variables.

To provide the satisfaction of the dissipation inequality

$$\widetilde{h}^{\text{intr}} := \widetilde{\mathscr{A}} : \dot{\boldsymbol{\alpha}} + \widetilde{\mathscr{B}} : \dot{\boldsymbol{\beta}} = \widetilde{\mathscr{A}} : (\mathbf{L}^{11} \cdot \widetilde{\mathscr{A}} + \mathbf{L}^{12} \cdot \widetilde{\mathscr{B}}) + \widetilde{\mathscr{B}} : (\mathbf{L}^{21} \cdot \widetilde{\mathscr{A}} + \mathbf{L}^{22} \cdot \widetilde{\mathscr{B}}) \geq 0, \tag{34}$$

we may require that

$$\widetilde{\mathscr{A}} : (\mathbf{L}^{12} \cdot \widetilde{\mathscr{B}}) = -\widetilde{\mathscr{B}} : (\mathbf{L}^{21} \cdot \widetilde{\mathscr{A}}). \tag{35}$$

If $\widetilde{\mathscr{A}} \cdot \widetilde{\mathscr{B}}^T$ is symmetric, the latter relation is reduced to the Casimir reciprocity relations

$$\mathbf{L}^{12} = -\mathbf{L}^{21}. \tag{36}$$

On account of the relation (35), we arrive at the decomposition of evolution equations into dissipative and non-dissipative parts

$$\begin{pmatrix} \dot{\boldsymbol{\alpha}} \\ \dot{\boldsymbol{\beta}} \end{pmatrix} = \begin{pmatrix} \mathbf{L}^{11} & 0 \\ 0 & \mathbf{L}^{22} \end{pmatrix} \begin{pmatrix} \widetilde{\mathscr{A}} \\ \widetilde{\mathscr{B}} \end{pmatrix} + \begin{pmatrix} 0 & \mathbf{L}^{12} \\ -\mathbf{L}^{12} & 0 \end{pmatrix} \begin{pmatrix} \widetilde{\mathscr{A}} \\ \widetilde{\mathscr{B}} \end{pmatrix}, \tag{37}$$

and the dissipation inequality is reduced to

$$\widetilde{h}^{\text{intr}} = \widetilde{\mathscr{A}} : (\mathbf{L}^{11} \cdot \widetilde{\mathscr{A}}) + \widetilde{\mathscr{B}} : (\mathbf{L}^{22} \cdot \widetilde{\mathscr{B}}) = (\widetilde{\mathscr{A}} \cdot \widetilde{\mathscr{A}}^T) : \mathbf{L}^{11} + (\widetilde{\mathscr{B}} \cdot \widetilde{\mathscr{B}}^T) : \mathbf{L}^{22} \geq 0. \tag{38}$$

As it is seen, the form of evolution equations is determined by components of the linear operator $\mathbf{L}$. To analyze the possible forms of the evolution equations, we consider two limiting cases, corresponding to pure symmetric and pure skew-symmetric linear operator $\mathbf{L}$.

The most remarkable feature of the considered approach is its applicability to nondissipative processes. It is clear that in the skew-symmetric case ($\mathbf{L}^{11} = \mathbf{L}^{22} = \mathbf{0}$) the dissipation $\widetilde{h}^{\text{intr}}$ vanishes, while evolution equations for the two internal variables are fully coupled

$$\dot{\boldsymbol{\alpha}} = \mathbf{L}^{12} \cdot \widetilde{\mathscr{B}}, \quad \dot{\boldsymbol{\beta}} = -\mathbf{L}^{12} \cdot \widetilde{\mathscr{A}}. \tag{39}$$

In this case, the evolution of one internal variable is driven by another one that means the duality between the internal variables.

To be more specific, let us consider a simple case with $\mathscr{B} = \mathbf{0}$. In this case, the free energy function $\overline{W}$ is independent of $\nabla_R \boldsymbol{\beta}$, and the kinetic relations (39) are reduced to

$$\dot{\boldsymbol{\alpha}} = \mathbf{L}^{12} \cdot \mathbf{B}, \quad \dot{\boldsymbol{\beta}} = -\mathbf{L}^{12} \cdot \widetilde{\mathscr{A}}. \tag{40}$$

Assuming further a quadratic dependence of the free energy function with respect to the internal variable $\boldsymbol{\beta}$

$$\mathbf{B} := -\frac{\partial \overline{W}}{\partial \boldsymbol{\beta}} = -\boldsymbol{\beta}, \tag{41}$$

we reduce Eq. $(40)_1$ to

$$\dot{\boldsymbol{\alpha}} = -\mathbf{L}^{12} \cdot \boldsymbol{\beta}, \tag{42}$$

while Eq. $(40)_2$ is not changed

$$\dot{\boldsymbol{\beta}} = -\mathbf{L}^{12} \cdot \widetilde{\mathscr{A}}. \tag{43}$$

Substituting Eq. (42) into Eq. (43), we obtain a hyperbolic evolution equation for the internal variable $\boldsymbol{\alpha}$:

$$\ddot{\boldsymbol{\alpha}} = (\mathbf{L}^{12} \cdot \mathbf{L}^{12}) \cdot \widetilde{\mathscr{A}}. \tag{44}$$

This means that the introduced internal variable $\boldsymbol{\alpha}$ now is practically an internal degree of freedom, and the structure of Eqs. (42), (43) and (44) is similar to that in the case of elasticity.

If, vice versa, $\mathbf{L}^{11} \neq \mathbf{0}, \mathbf{L}^{22} \neq \mathbf{0}$, while $\mathbf{L}^{12} = \mathbf{0}$, we return to the classical situation, where internal variables are fully independent:

$$\dot{\boldsymbol{\alpha}} = \mathbf{L}^{11} \cdot \widetilde{\mathscr{A}}, \quad \dot{\boldsymbol{\beta}} = \mathbf{L}^{22} \cdot \widetilde{\mathscr{B}}. \tag{45}$$

Therefore, the classical internal variable theory implicitly includes the Onsagerian reciprocity relations. In the fully dissipative case we are dealing with true internal variables of state.

4 Example: Micromorphic Linear Elasticity

In the framework of the Mindlin micromorphic theory [16], each material point is endowed with three translational degrees of freedom u_i and a full microdeformation tensor ψ_{ij} with nine independent components. Three strain tensors are deduced: the classical strain tensor ε_{ij}

$$\varepsilon_{ij} \equiv \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right), \tag{46}$$

the relative deformation tensor γ_{ij}

$$\gamma_{ij} \equiv \partial_i u_j - \psi_{ij}, \tag{47}$$

and the microdeformation gradient $\varkappa_{ijk}$ defined by

$$\varkappa_{ijk} \equiv \partial_i \psi_{jk}. \tag{48}$$

The free energy density $\overline{W}$ is supposed to be a homogeneous, quadratic function of 42 variables $\varepsilon_{ij}, \gamma_{ij}, \varkappa_{ijk}$ [16]

$$\overline{W} = \frac{1}{2} c_{ijkl} \varepsilon_{ij} \varepsilon_{kl} + \frac{1}{2} b_{ijkl} \gamma_{ij} \gamma_{kl} + \frac{1}{2} a_{ijklmn} \varkappa_{ijk} \varkappa_{lmn} + \\ + d_{ijklm} \gamma_{ij} \varkappa_{klm} + f_{ijklm} \varkappa_{ijk} \varepsilon_{lm} + g_{ijkl} \gamma_{ij} \varepsilon_{kl}. \tag{49}$$

As it was emphasized, only 903 of the 1764 coefficients in the former equation are independent. In the case of centrosymmetric, isotropic materials the number of independent coefficients is greatly reduced [16]

$$\begin{aligned} \overline{W} = \frac{1}{2} \lambda \varepsilon_{ii} \varepsilon_{jj} + \mu \varepsilon_{ij} \varepsilon_{ij} + \frac{1}{2} b_1 \gamma_{ii} \gamma_{jj} + \frac{1}{2} b_2 \gamma_{ij} \gamma_{ij} + \\ + \frac{1}{2} b_3 \gamma_{ij} \gamma_{ji} + g_1 \gamma_{ii} \varepsilon_{jj} + g_2 \left(\gamma_{ij} + \gamma_{ji} \right) \varepsilon_{ij} + \\ + a_1 \varkappa_{iik} \varkappa_{kjj} + a_2 \varkappa_{iik} \varkappa_{jkj} + \frac{1}{2} a_3 \varkappa_{iik} \varkappa_{jjk} + \frac{1}{2} a_4 \varkappa_{ijj} \varkappa_{ikk} + \\ + a_5 \varkappa_{ijj} \varkappa_{kik} + \frac{1}{2} a_8 \varkappa_{iji} \varkappa_{kjk} + \frac{1}{2} a_{10} \varkappa_{ijk} \varkappa_{ijk} + a_{11} \varkappa_{ijk} \varkappa_{jki} + \\ + \frac{1}{2} a_{13} \varkappa_{ijk} \varkappa_{ikj} + \frac{1}{2} a_{14} \varkappa_{ijk} \varkappa_{jik} + \frac{1}{2} a_{15} \varkappa_{ijk} \varkappa_{kij}. \end{aligned} \tag{50}$$

The corresponding stress tensors are the following ones [16]:
Cauchy stress

$$\sigma_{ij} \equiv \frac{\partial \overline{W}}{\partial \varepsilon_{ij}} = \sigma_{ji} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij} + g_1 \delta_{ij} \gamma_{kk} + g_2 (\gamma_{ij} + \gamma_{ji}), \tag{51}$$

relative stress

$$\tau_{ij} \equiv \frac{\partial \overline{W}}{\partial \gamma_{ij}} = g_1 \delta_{ij} \varepsilon_{kk} + 2 g_2 \varepsilon_{ij} + b_1 \delta_{ij} \gamma_{kk} + b_2 \gamma_{ij} + b_3 \gamma_{ji}, \tag{52}$$

and double stress

$$\mu_{ijk} \equiv \frac{\partial \overline{W}}{\partial \varkappa_{ijk}}. \tag{53}$$

The equations of motion in terms of stresses have the form (no body force) [16]

$$\rho \ddot{u}_j = \partial_i \left(\sigma_{ij} + \tau_{ij} \right), \tag{54}$$

$$\frac{1}{3} \rho' d_{ji}^2 \ddot{\psi}_{ik} = \partial_i \mu_{ijk} + \tau_{jk} + \Phi_{jk}, \tag{55}$$

where $\rho' d_{ji}^2$ is a microinertia tensor, Φ_{jk} is a volume double force.

In order to apply the internal variable theory, we need to represent the constitutive relations in the Mindlin theory in terms of distortion $\partial_j u_i$ and microdeformation tensor ψ_{ji}. Accordingly, the stresses are represented as

$$\begin{aligned}\sigma'_{ij} \equiv \frac{\partial \overline{W}}{\partial(\partial_i u_j)} &= \lambda \delta_{ij} \partial_k u_k + \mu(\partial_i u_j + \partial_j u_i) + \\ &+ g_1 \delta_{ij}(\partial_k u_k - \psi_{kk}) + g_2 \left(\partial_i u_j - \psi_{ij} + \partial_j u_i - \psi_{ji}\right) + \\ &+ b_1 \delta_{ij}(\partial_k u_k - \psi_{kk}) + b_2(\partial_i u_j - \psi_{ij}) + b_3(\partial_j u_i - \psi_{ji}),\end{aligned} \quad (56)$$

$$\begin{aligned}\tau'_{ij} \equiv \frac{\partial \overline{W}}{\partial \psi_{ij}} &= -g_1 \delta_{ij} \partial_k u_k - g_2(\partial_i u_j + \partial_j u_i) - \\ &- b_1 \delta_{ij}(\partial_k u_k - \psi_{kk}) - b_2(\partial_i u_j - \psi_{ij}) - b_3(\partial_j u_i - \psi_{ji}).\end{aligned} \quad (57)$$

The double stress remains unchanged. At last, equations of motion take on the form

$$\rho \ddot{u}_j = \partial_i \sigma'_{ij}, \quad (58)$$

$$\frac{1}{3} \rho' d_{ij}^2 \ddot{\psi}_{ik} = \partial_i \mu_{ijk} - \tau'_{jk} + \Phi_{jk}. \quad (59)$$

Now we consider the microdeformation tensor ψ_{ij} as an internal variable $\boldsymbol{\alpha}$ and apply the formalism developed in Section 3. The microdeformation gradient $\varkappa_{ijk}$ plays the role of the gradient of the internal variable $\boldsymbol{\alpha}$, and we introduce a dual internal variable $\boldsymbol{\beta}$ in the same way as in Section 3.

In the non-dissipative case, the evolution equation for the internal variable $\boldsymbol{\alpha}$ can be symbolically written as

$$\ddot{\boldsymbol{\alpha}} = (\mathbf{L}^{12} \cdot \mathbf{L}^{12}) \cdot \widetilde{\mathcal{A}} = (\mathbf{L}^{12} \cdot \mathbf{L}^{12}) \cdot \left(-\frac{\partial \overline{W}}{\partial \boldsymbol{\alpha}} + \operatorname{Div} \frac{\partial \overline{W}}{\partial (\nabla \boldsymbol{\alpha})} \right). \quad (60)$$

In terms of components of the microdeformation tensor ψ_{ij} the latter evolution equation obtains the form

$$\left(\mathbf{L}^{12} \cdot \mathbf{L}^{12}\right)^{-1}_{ji} \ddot{\psi}_{ik} = \left(-\frac{\partial \overline{W}}{\partial \psi_{jk}} + \operatorname{Div} \frac{\partial \overline{W}}{\partial (\nabla \psi_{jk})} \right) = \partial_i \mu_{ijk} - \tau'_{jk}. \quad (61)$$

As one can see, the evolution equation for the microdeformation is practically the same as in the Mindlin theory. The volume double force Φ_{jk} can appear if we consider a more general case than the pure nondissipative one.

5 Conclusions

The internal variables theory is extended to cover both internal variables of state and internal degrees of freedom by the generalization of its formal structure exploiting the possible coupling between the dual internal variables. The canonical thermomechanics provides the best framework for this generalization. It should be emphasized, however, that any new balance laws has not been introduced; only the Clausius–Duhem inequality was exploited for the derivation of evolution equations for internal variables.

References

1. Rice, J.R., Inelastic constitutive relations for solids: An internal-variable theory and its application to metal plasticity. *J. Mech. Phys. Solids* **19**, 1971, 433–455.
2. Mandel, J.P., Thermodynamics and plasticity. In: Domingos, J.J., Nina, M.N.R. and Whitelaw, J.H. (Eds.), *Foundations of Continuum Thermodynamics*. MacMillan, London, 1974, pp. 283–304.
3. Kestin, J., Internal variables in the local-equilibrium approximation. *J. Non-Equilibr. Thermodyn.* **18**, 1993, 360–379.
4. Muschik, W., Comment to J. Kestin: Internal variables in the local-equilibrium approximation. *J. Non-Equilibr. Thermodyn.* **18**, 1993, 380–388.
5. Maugin, G.A. and Muschik, W., Thermodynamics with internal variables. *J. Non-Equilib. Thermodyn.* **19**, 1994, 217–249. (See also Part II. Applications, ibid., 250–289.)
6. Maugin, G.A., *The Thermomechanics of Nonlinear Irreversible Behaviors*. World Scientific, Singapore, 1999.
7. Houlsby, G.T. and Puzrin, A.M., A thermomechanical framework for constitutive models for rate-independent dissipative materials. *Int. J. Plasticity* **16**, 2000, 1017–1047.
8. Benallal, A. and Bigoni, D., Effects of temperature and thermo-mechanical couplings on material instabilities and strain localization of inelastic materials. *J. Mech. Phys. Solids* **52**, 2004, 725–753.
9. Xiao, H., Bruhns, O.T. and Meyers, A., Elastoplasticity beyond small deformations. *Acta Mechanica* **182**, 2006, 31–111.
10. Zhao, J. and Sheng, D., Strain gradient plasticity by internal-variable approach with normality structure. *Int. J. Solids Struct.* **43**, 2006, 5836–5850.
11. Magnenet, V., Rahouadj, R., Ganghoffer, J.-F. and Cunat, C., Continuous symmetries and constitutive laws of dissipative materials within a thermodynamic framework of relaxation. Part I: Formal aspects. *Int. J. Plasticity* **23**, 2007, 87–113.
12. Maugin, G.A., On the thermomechanics of continuous media with diffusion and/or weak nonlocality. *Arch. Appl. Mech.* **75**, 2006, 723–738.
13. Ván, P., Berezovski, A. and Engelbrecht, J., Internal variables and dynamic degrees of freedom. *J. Non-Equilib. Thermodyn.* **33**, 2008, 235–254.
14. Maugin, G.A., *Material Inhomogeneities in Elasticity*. Chapman and Hall, London, 1993.
15. Maugin, G.A., On canonical equations of continuum thermomechanics. *Mech. Res. Comm.* **33**, 2006, 705–710.
16. Mindlin, R.D., Micro-structure in linear elasticity. Arch. Rat. Mech. Anal. **16**, 1964, 51–78.