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On the interaction of deformation waves in microstructured solids

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Abstract. The modelling of wave propagation in microstructured materials should be able to account for various scales of microstructure. In the present paper governing equations for 1D waves in microstructured material are presented, based on the Mindlin model and the hierarchical approach. The governing equation under consideration has an analytical solution only in limit cases, therefore numerical analysis is carried out. Numerical solutions are found in the case of localized initial conditions by employing the pseudospectral method. Special attention is paid to the solitonic character of the solution.

Key words: microstructured solids, Mindlin model, solitary waves, solitons.

1. INTRODUCTION AND MODEL EQUATIONS

The modelling of wave propagation in microstructured materials (alloys, crystallites, ceramics, functionally graded materials, etc.) should be able to account for various scales of microstructure [1–3]. The scale-dependence involves dispersive as well as nonlinear effects. It is widely known that the balance between these two effects may result in the emergence of solitary waves and solitons.

Propagation of solitary waves in microstructured solids is analysed by making use of different models (see [3–5] and references therein). However, the crucial point related to the derivation of governing equations is the distinction between nonlinearities on macro- and microlevel, together with proper modelling of dispersive effects. In [6–8] the Mindlin model [9] and hierarchical approach by

Engelbrecht and Pastrone [3] is used in order to derive governing equations. Basic model equations for 1D waves in microstructured material are

$$\rho u_{tt} = \sigma_x, \quad I\psi_{tt} = \eta_x - \tau. \tag{1}$$

Here u denotes the macrodisplacement, ψ the microdeformation, ρ the macrodensity, I the microinertia, σ the macrostress, η the microstress, τ the interactive force, x space coordinate, and t time. The free energy function is considered in the following form:

$$W = \frac{1}{2}au_x^2 + \frac{1}{2}B\psi^2 + \frac{1}{2}C\psi_x^2 + D\psi u_x + \frac{1}{6}Nu_x^3 + \frac{1}{6}M\psi_x^3, \tag{2}$$

where a, B, C, D, M, and N are constants. Then, using the formulae

$$\sigma = \frac{\partial W}{\partial u_x}, \qquad \eta = \frac{\partial W}{\partial \psi_x}, \qquad \tau = \frac{\partial W}{\partial \psi}, \tag{3}$$

Eqs (1) are expressed in terms of variables u and ψ:

$$\rho u_{tt} = au_{xx} + Nu_x u_{xx} + D\psi_x, \quad I\psi_{tt} = C\psi_{xx} + M\psi_x\psi_{xx} - Du_x - B\psi. \tag{4}$$

After introducing dimensionless variables $X = x/L$, $T = tc_0/L$, $U = u/U_0$, the scale parameter $\delta = l^2/L^2$ (L and U_0 can be amplitude and wavelength of the initial excitation, respectively; $c_0^2 = a/\rho$ and l is the scale of the microstructure) and making use of the slaving principle [3], the following hierarchical model equation is obtained from Eqs (4):

$$U_{TT} - bU_{XX} - \frac{\mu}{2}\left(U_X^2\right)_X - \delta\left(\beta U_{TT} - \gamma U_{XX} - \delta^{1/2}\frac{\lambda}{2}U_{XX}^2\right)_{XX} = 0, \tag{5}$$

where b, μ, β, γ, and λ are constants (see [7,8] for details). If the scale parameter δ is small, then the wave process is governed by properties of the macrostructure, and vice-versa, if δ is large, then properties of the microstructure govern the process. For future analysis Eq. (5) is expressed in terms of deformation $v = U_X$ and lower-case letters x and t are used for dimensionless coordinate and time:

$$v_{tt} = bv_{xx} + \frac{\mu}{2}\left(v^2\right)_{xx} + \delta\left(\beta v_{tt} - \gamma v_{xx}\right)_{xx} - \delta^{3/2}\frac{\lambda}{2}\left[(v_x)^2\right]_{xxx}. \tag{6}$$

Equation (6) admits the analytic solitary wave solution

$$v(x - ct) = A\,\mathrm{sech}^2\,\frac{\varkappa(x - ct)}{2}, \quad A = \frac{3(c^2 - b)}{\mu}, \quad \varkappa = \sqrt{\frac{c^2 - b}{\delta(\beta c^2 - \gamma)}} \tag{7}$$

only if $\lambda = 0$ [7,8]. If $\lambda \neq 0$, one can find a travelling wave solution $v(x - ct)$ for Eq. (6) in the form of an asymmetric solitary wave by numerical integration under asymptotic boundary conditions. The analytic conditions for the existence of solitary waves modelled by Eq. (6) are given by Janno and Engelbrecht in [7,8]:

$$\mu \neq 0, \;\; \beta c^2 - \gamma \neq 0, \;\; c^2 - b \neq 0, \;\; \frac{c^2 - b}{\beta c^2 - \gamma} > 0, \;\; \left(\frac{\beta c^2 - \gamma}{c^2 - b} \right)^3 > \frac{4\lambda^2}{\mu^2}. \tag{8}$$

In the present paper the interaction of sech^2-shaped waves (7) is analysed numerically.

2. STATEMENT OF THE PROBLEM AND NUMERICAL TECHNIQUE

The main goals of the present paper are (i) to simulate numerically the interactions between solitary waves (7); (ii) to estimate the influence of the microlevel nonlinear parameter λ on the solution, and (iii) to examine the solitonic character of the solution. For this reason Eq. (6) is integrated numerically under the initial conditions

$$v(x,0) = \sum_{i=1}^{2} A_i \operatorname{sech}^2 \frac{\varkappa_i (x - \xi_i)}{2}, \qquad 0 \leq x < 2k\pi. \tag{9}$$

Here amplitudes A_i and the widths $\varkappa_i$ $(i = 1, 2)$ correspond to different initial speeds $c_1 \neq c_2$ and ξ_i are initial phase shifts. It is clear that if $c_1 c_2 < 0$, head-on collision and if $c_1 c_2 > 0$, the overtaking interaction takes place, and the lower the value of $\varkappa_i$, the wider the initial solitary wave.

For numerical integration the pseudospectral method (PsM) based on the discrete Fourier transform (DFT) [10,11] is used and therefore periodic boundary conditions

$$v(x,t) = v(x + 2k\pi, t) \tag{10}$$

are applied. The idea of the PsM is to approximate space derivatives making use of the DFT and then to use standard ODE solvers for integration with respect to time. Due to the mixed partial derivative term $\delta\beta v_{ttxx}$, the model Eq. (6) cannot be directly integrated by the PsM. Therefore we introduce a new variable Φ and apply properties of the DFT:

$$\begin{gathered} \Phi = v - \delta\beta v_{xx} = F^{-1}[(1 + \delta\beta\omega^2) F(v)], \\ v = F^{-1}\left[\frac{F(\Phi)}{1 + \delta\beta\omega^2}\right], \quad \frac{\partial^n v}{\partial x^n} = F^{-1}\left[\frac{(i\omega)^n F(\Phi)}{1 + \delta\beta\omega^2}\right]. \end{gathered} \tag{11}$$

Here $\omega = 0, \pm 1, \pm 2, \ldots, \pm (N/2 - 1), -N/2$, i is the imaginary unit, N denotes the number of space-grid points, F the DFT, and F^{-1} the inverse DFT. Finally the equation

$$\Phi_{tt} = \left[bv + \frac{\mu}{2} v^2 - \delta\gamma v_{xx} - \delta^{3/2} \frac{\lambda}{2} (v_x^2)_x \right]_{xx} \tag{12}$$

is solved numerically by the PsM under initial and boundary conditions (9) and (10), respectively. Calculations are carried out using the SciPy package [12]: for the DFT the FFTW [13] library and for the ODE solver the F2PY [14] generated Python interface to the ODEPACK Fortran code [15] are used.

3. RESULTS AND DISCUSSION

Three different interaction cases are considered in the present section. Travelling wave solutions in the form of an asymmetric solitary wave can exist for all considered sets of parameters, i.e., parameters for Eq. (6) and initial condition (9) are chosen according to conditions (8).

In order to analyse *head-on collision of solitary waves with equal amplitudes*, the case where parameters for Eq. (6) are $b = 0.7683$, $\mu = 0.125$, $\delta = 9$, $\beta = 7.6452$, $\gamma = 6.1817$, $\lambda = 0$, solitary wave speeds $c_1 = 0.9$ and $c_2 = -0.9$, the corresponding amplitudes $A_1 = A_2 = 1.0$ and widths $\varkappa_1 = \varkappa_2 = 0.65$ is considered. Numerical integration is carried out for $0 \leq t \leq 500$ and the length of the space period is 24π. The amplitudes of the waves increase during interactions and initial amplitudes are restored after interactions (Fig. 1a) like in the case of Boussinesq models [16]. The amplitude of the wave profile attains a value close to the double initial amplitude at every "peak" of the interaction in the considered time interval. However, the behaviour of the amplitude curve between interactions varies essentially – the more interactions have taken place, the more distinctive the changes are. Analysis of trajectories of single waves demonstrates that unlike Boussinesq models, solitary waves are not phase shifted during interactions in the present case. This phenomenon is reflected in Fig. 1b – after eleven interactions two solitary waves are still in the same phase. However, the shape of initial waves (9) is altered during interactions and it is clear that, instead of initial symmetric bell-like waves, asymmetric solitary waves are formed, shown at $t = 460.8$. Therefore the

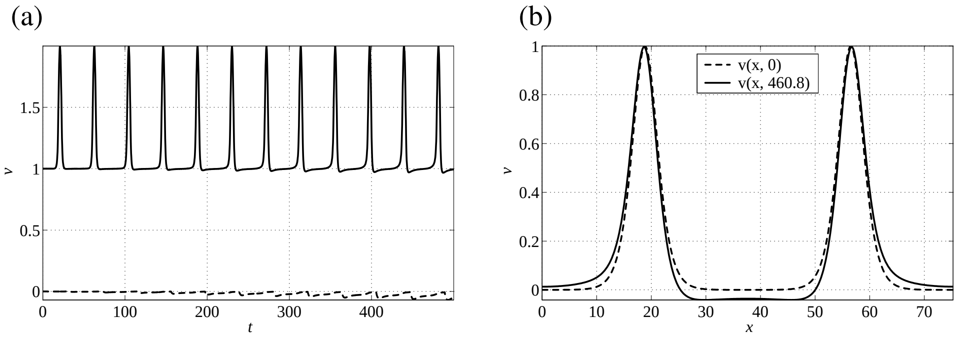

Fig. 1. Head-on collision of solitary waves with equal amplitudes. (a) Waveprofile minima and maxima against time. (b) The initial waveprofile and the waveprofile after eleven interactions at $t = 460.8$.

interaction process is near elastic (i.e. the height of waves is restored but the initial shape is slightly altered after interactions) in the present case.

In order to examine *head-on collision of solitary waves with nonequal amplitudes*, the case where parameters for Eq. (6) are $b = 0.7683$, $\mu = 0.125$, $\delta = 9$, $\beta = 7.6452$, $\gamma = 6.1817$, $\lambda = 0$, solitary wave speeds $c_1 = 0.9$ and $c_2 = -0.9115$, the corresponding amplitudes $A_1 = 1.0$, $A_2 = 1.0$ and widths $\varkappa_1 = 0.65$, $\varkappa_2 = 0.202$ is considered. Numerical integration is carried out for $0 \leq t \leq 500$ and the length of the space period is 96π. In the present case the behaviour of the waveprofile maxima and minima is similar to the case considered above, i.e., during the interaction the amplitude attains the value close to $A_1 + A_2$, between interactions both solitary waves restore initial values, and the behaviour of the amplitude curves between interactions varies depending on the number of passed interactions (Fig. 2a). The analysis of trajectories of solitary waves demonstrates that in the present case solitary waves are phase shifted during interactions and in Fig. 2b one can see that after three interactions at $t = 335.2$ the distance between solitary waves is changed compared to that at $t = 0$. Like in the previous case, both solitary waves are asymmetric after several interactions (more distinctive asymmetry can be detected for the lower one). Nevertheless, one can conclude that the interaction process is near elastic in the present case.

During *overtaking interaction* both solitary waves are phase shifted but do not restore their shape after the interaction. This case is not analysed in the present paper.

Numerical experiments with $\lambda \neq 0$ were carried out in order to estimate the *influence of microlevel nonlinearity*. Analysis of solutions for $\lambda = 0$ and $\lambda = 0.005$ demonstrates that in the case of head-on interaction, both solutions practically coincide – maximal differences between corresponding waveprofiles, i.e. $\max_{t,x}(v(t,x)|_{\lambda=0} - v(t,x)|_{\lambda=0.005})$, are of order 0.01. For $\lambda = 0.5$ the microlevel nonlinear effects are stronger and they are able to change the character of interactions. In Fig. 3 the waveprofile maxima and minima reflect head-on collision, which is less elastic than for $\lambda = 0$.

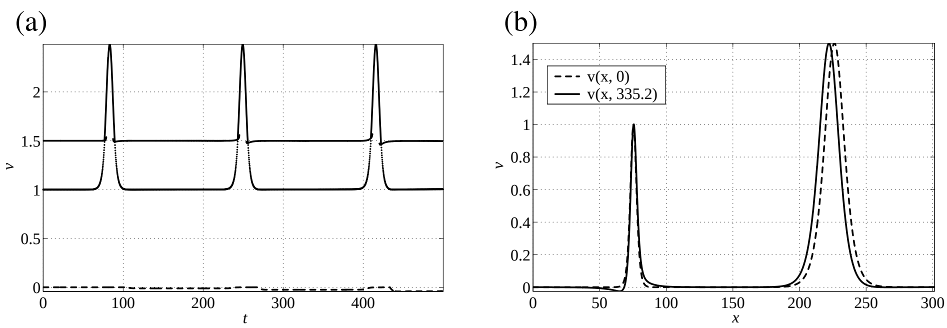

Fig. 2. Head-on collision of solitary waves with nonequal amplitudes. (a) Waveprofile minima and maxima against time. (b) The initial waveprofile and the waveprofile after two interactions at $t = 335.2$.

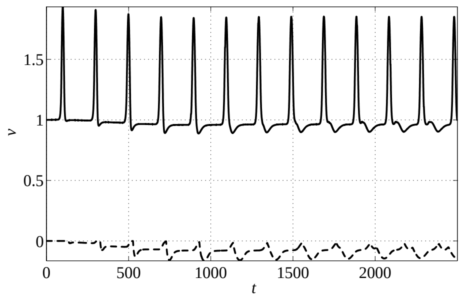

Fig. 3. Head-on collision of solitary waves with equal amplitudes. Waveprofile minima and maxima against time for $\lambda = 0.5$.

4. CONCLUSIONS

The characteristic feature of the governing equation (6) is that, unlike the well-known evolution equations, it describes two waves instead of one. A similar situation occurs for waves in rods [5]. This gives us an opportunity to analyse also head-on collisions of waves.

In the case of $\lambda = 0$, bell-like solitary waves (9) can propagate with constant speed and shape, but during head-on collisions the initial symmetric shape changes to asymmetric. In the case of $\lambda \neq 0$, the initial symmetric shape is altered even before the first interaction. Analysis of the results of our numerical experiments demonstrates that for $\lambda = 0$ and for relatively small values of λ interactions between solitary waves are near elastic. Consequently, the behaviour of solitary waves is very close to solitonic behaviour. If initial waves have speeds $c_1 = -c_2$, then solitons do not become phase-shifted during interactions. The higher the value of λ, the less elastic the head-on collision and therefore the less solitonic the behaviour of interacting waves. The overtaking interaction is not elastic either for $\lambda = 0$ or for $\lambda \neq 0$.

Numerical experiments in order to analyse the long-time behaviour of solutions over a wide range of material parameters and initial conditions are in progress. Clearly, the two-wave governing equation, possessing solitary wave type solutions, needs more attention in the future.

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Deformatsioonilainete interaktsioonist mikrostruktuursetes tahkistes

Andrus Salupere, Kert Tamm, Jüri Engelbrecht ja Pearu Peterson

Mikrostruktuursetes tahkistes toimuva lainelevi modelleerimisel tuleb arvesse võtta erinevaid mikrostruktuuri mastaape. Põhivõrrandite tuletamisel on eriti oluline mikro- ja makrotaseme mittelineaarsete efektide eristamine ning dispersiivsete efektide adekvaatne modelleerimine. Artiklis vaatluse all olevate ühedimensiooniliste lainete levi kirjeldavate võrrandite tuletamisel on kasutatud Mindlini mikrostruktuurse materjali mudelit ja lainehierarhiate teooriat. Kuna kasutatavatele võrranditele eksisteerivad analüütilised lahendid vaid teatavatel piirjuhtudel, siis on lahendite leidmisel ja tulemuste analüüsimisel kasutatud numbrilisi meetodeid. Põhitähelepanu on pööratud lahendi solitonilise iseloomu selgitamisele.