Nonlinear deformation waves in solids and dispersion ⋆

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Abstract

In contemporary technology, high-speed loading, high frequencies and high-amplitude excitations are widely used. For proper analysis, the microstructure of solids must then be taken into account. Together with nonlinear effects, the dispersive effects due to the microstructure are of importance. In this paper, the Mindlin-type model based on the theory of continua is used for describing the microstructure. On the other hand, it is shown how straight-forward numerical methods, like the finite volume methods, permit to describe the material properties for every discrete element used in numerical simulation. Several phenomena like the existence of solitary waves, the emergence of solitary wave trains, and waves in piece-wise nonlinear laminated materials are briefly discussed and further problems indicated.

Key words: nonlinearity, microstructure, solitary waves

1 Introduction

In principle, all materials behave non-linearly and linear models are appropriate only within moderate loading and low frequency scales (cf. [1]). Nonlinear phenomena in mechanics like shock waves, solitons, spectral changes, etc. have currently gained wide attention. It is, however, clear that all the physical effects of the same order should be taken into account consistently, that is why besides nonlinear effects, the effects of dispersion, dissipation, etc. should be described within the framework of one model. In this paper, the attention is focused on microstructural materials that are characterized by the existence

⋆ Based on the talk at the International Symposium on Mechanical Waves in Solids, May 15-18, 2006, Hangzhou of intrinsic space-scales in matter, like the lattice period, the size of a crystalline or a grain, the thickness of layers in laminates, the distance between the microcracks, etc. These scales introduce a certain scale-dependence into the governing equations that in terms of wave dynamics gives rise to dispersive effects. The classical Korteweg-de Vries (KdV) equation combines the simplest nonlinear (quadratic non-linearity) and dispersive (cubic dispersion) effects resulting in soliton formation (see, for example, [2]). In microstructured materials, however, the situation is much more complicated and the character of nonlinearity/ies and dispersion needs a careful analysis.

One of the important problems is to clarify dispersive effects due to the microstructure. The early studies by Achenbach et al [3] and Sun et al [4] were devoted to the dynamic behaviour of laminated composites proposing the effective stiffness theory close to the Bloch wave expansion ([5]). Later, for the same problem the important results on stopping bands of harmonic waves were obtained by Ziegler [6], while Santosa and Symes [7] derived an effective linear wave equation with a fourth-order term describing dispersion.

In more general terms, the starting point for describing a microstructure could be either the discrete or the continuum approach. In the discrete approach the volume elements of the matter are treated as point masses with a proposed topological structure and some interaction between the discrete masses (see [8] and references therein). This gives a good chance to model crystal lattices with certain symmetries, vacancies, impurities, defects, walls, etc. The continuum limits of the initial governing equations in the form of ODEs are widely used in the analysis.

From the viewpoint of continua, the straight-forward modelling of microstructured solids leads to assigning all the physical properties to every volume element dV in a solid, introducing the dependence on space coordinates. Then the governing equations include automatically space-dependency, and the most effective way to solve the governing equations is numerical integration (see later Sections 3.1 and 3.3).

Another way is to separate macro- and microstructure in continua. Then the conservation laws for both structures should be either separately formulated [9–11] or the microstructural quantities separately taken into account in one set of conservation laws [12]. Here we follow the approach of Mindlin [9], resulting in a very clear structure of the governing equations.

Our early studies were focused on the derivation of the governing equations and dispersion analysis [13,14], and on the numerical simulation of wave distortion [15,16], but also on the analysis of nonlinearities in microscale [17]. Here we focus on explaining the effects of nonlinearities and dispersion using both approaches mentioned above - (i) the straight-forward modelling and the finite volume method and (ii) the governing equations derived from the continuum theory.

The paper is organized as follows. Section 2 describes the mathematical models used for the analysis: the hierarchical Mindlin-type model and the straightforward numerical model used in the finite volume method. In Section 3 several results are presented: first, the existence of solitary pulses in microstructured solids is shown; second, the emergence of solitary waves is described on the basis of the full 1D equation with higher order dispersion due to nonlinearity and on the basis of the finite volume method applied to waves in laminated compound material; third, the importance of structural nonlinearities (mismatch of impedance in laminated materials) is demonstrated. Section 4 includes conclusions and a brief discussion on the nature of dispersion.

2 Mathematical models

2.1 Hierarchical governing equations

As said above, we follow [9] and derive a 1D governing equation for longitudinal deformation waves [13], distinguishing macrostress σ and microstress η and the interactive force τ. The free energy function W is given in the following form: $W = W_2 + W_3$, where W_2 is the simplest quadratic function

$$W_2 = \frac{1}{2}au_x^2 + \frac{1}{2}B\psi^2 + \frac{1}{2}C\psi_x^2 + D\psi u_x \tag{1}$$

and W_3 includes nonlinearities on both the macro- and microlevel

$$W_3 = \frac{1}{6}Nu_x^3 + \frac{1}{6}M\psi_x^3. \tag{2}$$

Here u is the macrodisplacement, ψ is the microdeformation, a, B, C, D, N and M are constants and right sub-indices denote differentiation. The basic 1D model for longitudinal waves is

$$\rho u_{tt} = \sigma_x, \quad I\psi_{tt} = \eta_x - \tau, \tag{3}$$

where ρ and I are macrodensity and microinertia, respectively. Then we use the formulae

$$\sigma = \frac{\partial W}{\partial u_x}, \quad \eta = \frac{\partial W}{\partial \psi_x}, \quad \tau = \frac{\partial W}{\partial \psi}, \tag{4}$$

and introduce the dimensionless variables $X = x/L, T = tc_0/L, U = u/U_0$, where U_0 and L are certain constants (e.g. amplitude and wavelength of the initial excitation) and also geometric parameters $\delta = l^2/L^2, \varepsilon = U_0/L$, where l is the scale of microstructure. By using an asymptotic procedure (for details see [13]), we arrive at the governing equation in the following form

$$U_{TT} = bU_{XX} + \frac{\mu}{2}\left(U_X^2\right)_X + \delta\left(\beta U_{TT} - \gamma U_{XX}\right)_{XX} - \delta^{3/2}\frac{\lambda}{2}\left(U_{XX}^2\right)_{XX}, \quad (5)$$

where $b, \mu, \beta, \gamma, \lambda$ are constants. Equation (5) actually involves hierarchically two nonlinear wave operators

$$L_{ma} = U_{TT} - bU_{XX} - \frac{\mu}{2}\left(U_X^2\right)_X$$

and (6)

$$L_{mi} = \delta\left(\beta U_{TT} - \gamma U_{XX} - \delta^{1/2}\frac{\lambda}{2}U_{XX}^2\right)_{XX},$$

characteristic of macro- and microstructure, respectively (cf. [2]).

2.2 Straight-forward approach

This approach is extremely powerful for heterogeneous materials composed of alternating layers of two different materials [18] but also for FGMs [15]. Again, let us consider 1D longitudinal waves. Then in every element (i), the waves are described by the governing equations

$$\rho_i(w_i)_t = (\sigma_i)_x, \quad (u_i)_t = (w_i)_x, \quad w_i = (u_i)_t, \quad (7)$$

where ρ_i is the density of the element and there is no need to distinguish between macro- and microstresses. The constitutive equation is taken in the form

$$\sigma_i = \rho_i c_i^2 (u_i)_x (1 + A_i (u_i)_x), \quad (8)$$

where c_i is the longitudinal wave speed and A_i is a parameter of nonlinearity (cf. [19]). For solving the system (7) with (8) and suitable initial and boundary conditions, the finite volume algorithm is used [18,20].

3 Results and discussion

3.1 *Existence of solitary pulses*

A well-known manifestation of the balance between nonlinear and dispersive effects is the existence of solitary waves. The celebrated KdV-model exhibits solitons, i.e., solitary waves which interact with each other elastically. Here the balance is much more complicated because of nonlinearities on both macro- and microlevel and complicated dispersion. For further analysis we rewrite Eq. (5) in terms of $v = U_X$

$$v_{tt} = v_{xx} + \frac{\mu}{2}(v^2)_{xx} + \delta(\beta v_{tt} - \gamma v_{xx})_{xx} - \delta^{3/2}\frac{\lambda}{2}(v_x^2)_{xxx}. \tag{9}$$

Janno and Engelbrecht [21,22] have proved the existence of a solitary wave solution to Eq. (9) provided

$$\left(\frac{\beta c_1^2 - \gamma}{c_1^2 - b}\right)^3 > \frac{4\lambda^2}{\mu^2}, \quad \frac{c_1^2 - b}{\beta c_1^2 - \gamma} > 0, \quad \beta c_1^2 - \gamma \neq 0, \quad c_1^2 - b \neq 0, \quad \mu \neq 0, \tag{10}$$

where c_1 is the velocity of the solitary wave $v(x,t) = v(x - c_1 t)$. If $\lambda = 0$, i.e. the nonlinearity exists only on the macroscale ($\mu \neq 0$), then the sech^2-type solitary wave exists. If however $\lambda \neq 0$ then the solitary wave is asymmetric, i.e. the nonlinearity on the microscale affects the process. The direct solution of Eq. (9) by the pseudospectral method (see [23,24]) shows clearly that a solitary wave obeying the condition (10) propagates in the medium. The initial condition is chosen as a soliton for Eq. (9) with $\lambda = 0 : v = A\,\mathrm{sech}^2(x/\Delta), A = 0.6413, \Delta = 1.638$. In the course of time, the solitary wave propagates with a small change in its shape (Fig.1) that is obvious from the phase-plane (Fig.2) which demonstrates clear asymmetry. The analytic conditions for the existence of solitary waves modelled by Eq. (9) are given in [21].

3.2 *Formation of solitary waves*

The classical paper of Zabusky and Kruskal [25] demonstrated on the basis of the KdV-equation the formation of a soliton train from a harmonic input. Here the situation is much more complicated, but follows the same patterns. The governing equation (9) has a main wave operator of the second order and that describes both right- and left-going waves. Consequently, an initial excitation should also generate both those waves. Figure 3 demonstrates the situation for a solitary-type $v(0,x) = A\,\mathrm{sech}^2(x/\Delta), A = 1.0, \Delta = 100$. Two wave-trains

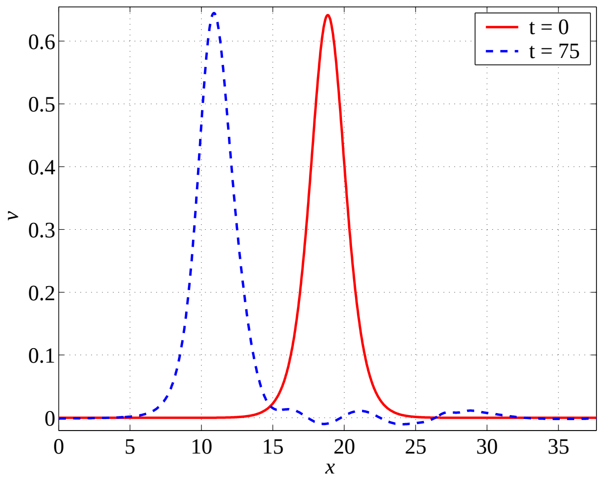


Fig. 1. Propagation of a solitary wave described by Eq. (9) with $\lambda = 1.6681; \mu = 1.45, b = 0.5, \beta = 1.32, \gamma = 0.2376, \delta = 0.25$.

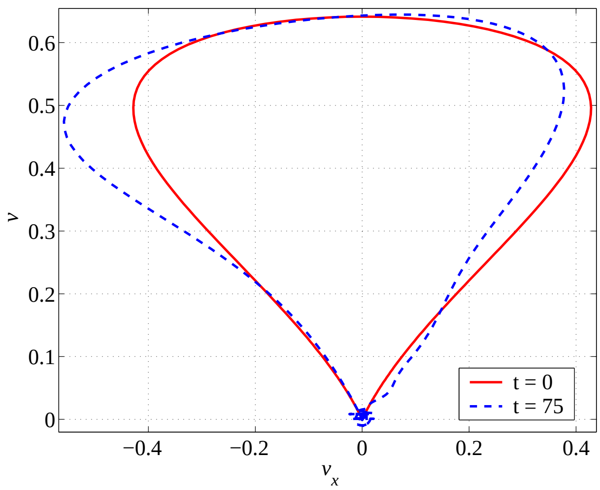


Fig. 2. The phase-plane of a solitary wave (data from Fig. 1).

form, propagating to the left and to the right. The similar situation for a rod, where geometrical dispersion is taken into account is analyzed in [26]. But be it an evolution equation [25] or a second order wave operator (Eq. (9)), if dispersive and nonlinear effects are balanced, then solitary waves emerge.

As mentioned in Section 2.1, the dispersive effects are captured also by the straight-forward approach, similarly to [7], where the Bloch expansion is simplified to an effective medium model. We include also the nonlinearity as described by Eq. (8) and use the finite-volume algorithm [15] for the numerical simulation. The laminated material [18] is used made by alternating polycarbonate and aluminium layers. Each layer has a thickness $4\Delta x$, where Δx is the computational grid size. The material parameters were for polycarbonate ρ_1 =1190 kg/m^3, c_1 =3914 m/s, A_1 =0.8 and for aluminium ρ_2 =2710 kg/m^3, c_2 =5386 m/s, A_1 =0.8 (see Eq. (7), (8)). The boundary condition (excitation)

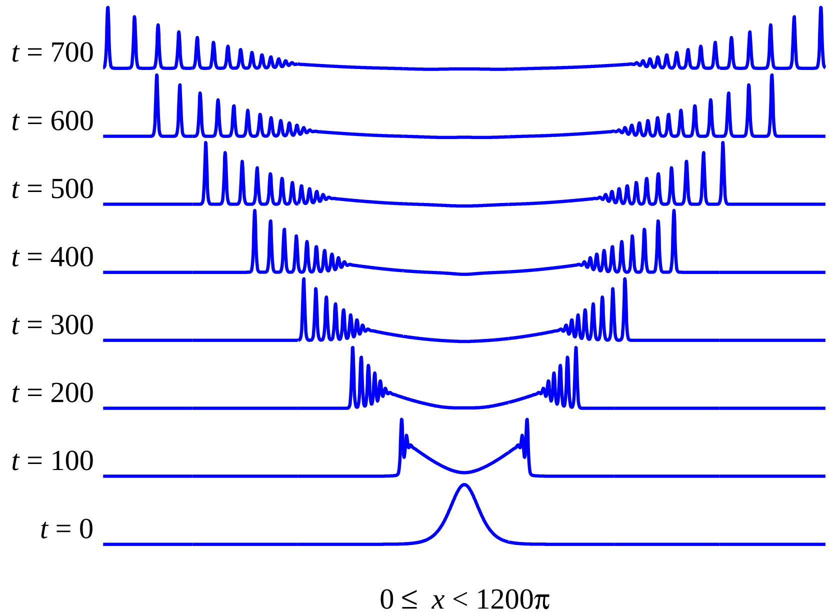
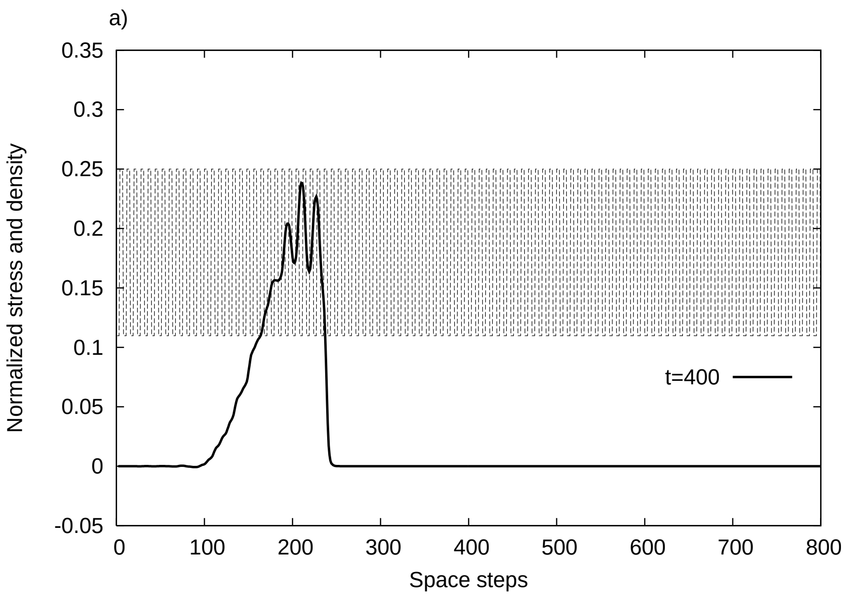
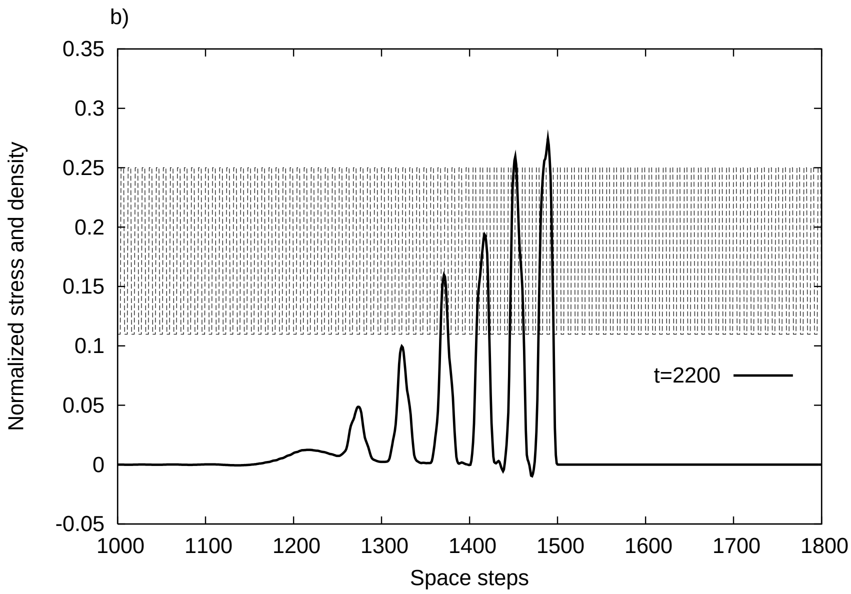


Fig. 3. Generation of two trains of solitons (Eq. 9) from an initial excitation. Here $\lambda = 2.0833; \mu = 16.667, b = 0.775, \beta = 45.0405, \gamma = 7.5, \delta = 0.25$.

is given in terms of a velocity pulse with definite zeros at $t = 0$ and $t = 240\Delta t$: $v(0,t) = a(1+cos(\pi(t-120)/120), a = 0.45)$. The stress is normalized against $\rho_2 c_2^2$ and the density against $4\rho_2$. The results are given in Figs. 4 a, b, c, where the normalized density is shown by dashed lines. The train of solitary waves is formed in due time exhibiting a similar distribution like in [25]. The question whether these solitary waves are solitons or not needs special attention. This was stressed already by LeVeque and Yong in [18] and [27] who focused their attention to the similarity with the Toda lattice. As the solitary waves expand over several layers, their detailed shape differs from that of the classical





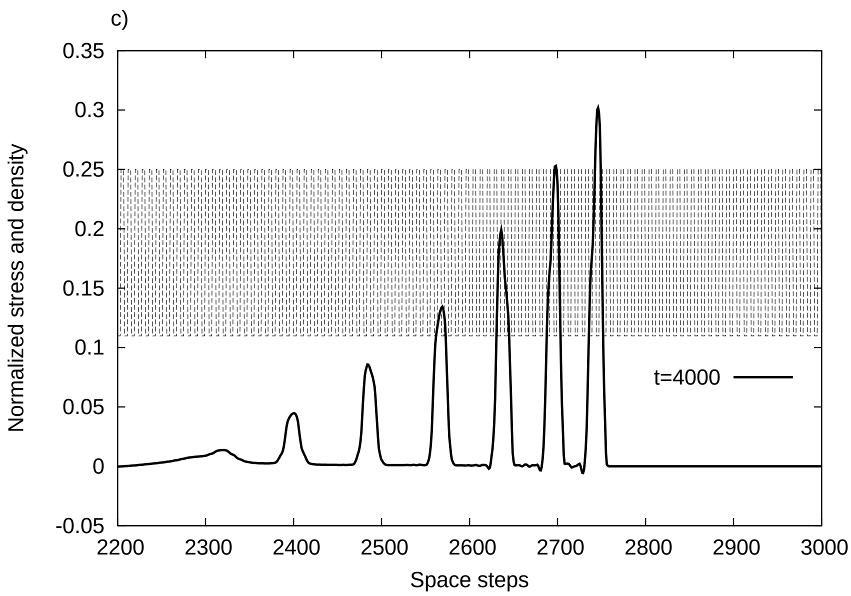


Fig. 4. Formation of a train of solitons in a laminated material (explanation in the text) a) $t = 400$, b) $t = 2200$, c) $t = 4000$. Solid line - normalized stress, dashed line - normalized density.

sech^2-type solution. What is actually important, is the similarity of solitary wave generation processes in different cases. Here, however, dispersion occurs because of the successive reflections at each interface as shown by LeVeque and Yong [27]. The finite-volume algorithm [15] captures this effect by solving the Riemann problem at each interface between discrete elements.

In addition to the usual physical nonlinearities, the mismatch of constitutive equations can also play a decisive role. In terms of acoustics, this effect can be handled as an impedance mismatch [28]. Here we show how this situation can be described in the case of a two-layered composite made of soft (polycarbonate) and hard (steel) alternating layers. An experiment by Zhuang et al [29] is taken as the basis with corresponding material parameters and impact conditions. We have used the finite-volume algorithm and used the constitutive equation (8) with nonlinear parameter $A = 0$ for hard layers and $A \neq 0$ for soft layers. The results are extremely good - in Fig. 5 the stress history is shown for the experiment [29] and for the numerical simulations in the case where nonlinearity is taken into account and in the linear case. The specimen

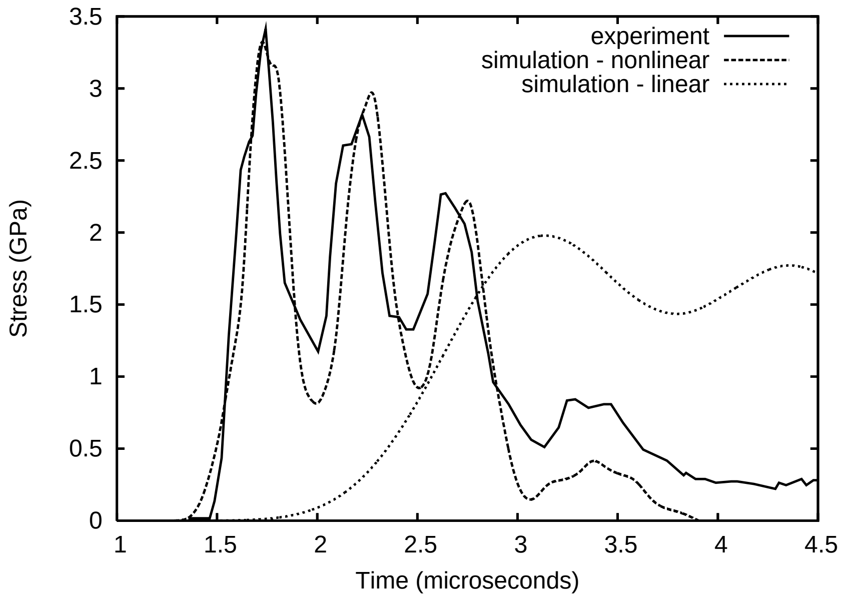


Fig. 5. Stress history for a laminated specimen of Zhuang et al [29] (explanation in the text).

consists of 16 units of polycarbonate, each one 0.39 mm thick and of 16 units of stainless steel, each one 0.19 mm thick. The boundary condition (the velocity of a flyer) is given as a step-function with the amplitude $w_0 = 1043$ m/s. The stress time histories correspond to the distance of 3.44 mm from the impact face. The nonlinear parameter is $A = 2.8$ for polycarbonate and $A = 0$ for stainless steel. The nonlinear model captures all the essential features of the experiment, while the linear model is far from that [16].

4 Conclusions

Nonlinear effects influence considerably the propagation of deformation waves in solids. The generation of higher order harmonics and the tendency to shock wave formation are well-known phenomena in wave dynamics [2,30]. Especially important are nonlinear effects together with other effects of the same order. The corresponding mathematical models should describe the phenomenon with the needed accuracy and then they could also be effectively used either for the prediction of the characteristics of waves (direct problem) or for using the measured characteristics for determining the properties of the material (inverse problem). Here we focused our attention to microstructural materials and the possible balance of nonlinear and dispersive effects. The Mindlin-type model [9] has been developed in [13,14] so that it exhibits the needed accuracy and demonstrates clearly the scaled structure of the material and the dispersive effects related to this structure. Due to the complexity of the governing equation (cf. Eq. (9)), the analytical solutions at arbitrary initial and boundary conditions are not known and therefore numerical simulations are needed. The pseudospectral method ([24,31] and references therein) has proved to have a good accuracy and it shows in addition the spectral changes explicitly. On the other hand, the straight-forward numerical methods could be used which permit to describe the needed material properties for every element of the microstructural material. Here we stress the importance of the finite-volume method [18,15].

Both approaches - derivation of the governing equations together with suitable methods for their integration and straight-forward numerics - should be used in parallel. The governing equations permit a full dispersion analysis [13] and the analytical description of certain solutions [21], while the numerical simulation has a wider area of applications. As shown in Section 3.2, the results describing the emergence of solutions are similar, although here the aim was not to find the best quantitative matching. What should be stressed is the possibility to capture nonlinear piece-wise changing effects by the finite-volume method, as demonstrated in Section 3.3 for the case of structural nonlinearities in laminated composites.

It should be stressed that dispersive effects in general, and also in this paper, can be of a different nature. The dispersion modelled by the free energy function (1) and yielding the governing equation (5) is physical. As a result, two wave operators are present in the mathematical formulation and one of them (6) introduces higher order derivatives responsible for dispersive effects into the governing equation. In the modelling of layered material (see Sec. 2.2) the dispersion is geometric, being due to the constitution of the solid in layers. Geometric dispersion occurs also for longitudinal waves in rods [26], due to the influence of the lateral surface of a rod. Finally we could determine the numerical dispersion related to the discreteness of the computational grid. In this case one should clearly distinguish such an effect from the real dispersion.

Further attention in this field will be focused on 2D problems, multiscale dispersion, quantitative comparison of various methods and solving the inverse problems. The asymmetry of possible solitary waves in microstructural solids might be a good chance to construct practical algorithms [21] for NDE.

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5 Figure captions

Fig. 1. Propagation of a solitary wave described by Eq. (9) with $\lambda = 1.6681; \mu = 1.45, b = 0.5, \beta = 1.32, \gamma = 0.2376, \delta = 0.25$.

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