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On solitons in microstructured solids and granular materials

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Abstract

The problems under consideration are related to wave propagation in nonlinear dispersive media, characterised by higher-order nonlinear and higher-order dispersive effects. Particularly two problems — wave propagation in dilatant granular materials and wave propagation in shape-memory alloys — are studied. Model equations are KdV-like evolution equations in both cases. The types of solutions are determined and analysed. It has been found that waves in granular materials are composed by two concurrent ensembles of solitary waves and there exist a threshold parameter value governing the behaviour of stable waves in alloy type materials.

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Keywords: Solitons; KdV-like evolution equation; Higher order nonlinearity; Higher-order dispersion; Pseudospectral methods

1. Introduction

Waves in microstructured solids are clearly affected by dispersion due to the internal structure of the material. For high-intensity excitations, nonlinearity should also be taken into account. It is widely known that nonlinear and dispersive effects may be balanced and then solitary waves and solitons may emerge.

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The simplest case of quadratic nonlinearity and cubic dispersion is modelled by the celebrated Korteweg–deVries (KdV) equation [1]. Since the discovery of solitons, many studies have been devoted to various generalisations of this simple but rich model.

Indeed, it has been shown that the higher-order dispersive terms enrich the physical picture [2–5]. On the other hand, the existence of higher-order nonlinearit-y/ies may still be balanced by dispersive effects. The consistent derivation of mathematical models, however restricts the simple summing-up the higher-order terms. In [5] it has been shown how the lattice theory can be used for the derivation of wave models. In this case the discrete terms are derived into the Taylor series giving rise to higher-order derivatives in governing equations. Starting from the theory of continua, the balance laws are formulated separately for macro- and microstructure [6,7]. Then the higher-order terms depend on the structure of the energy function and interaction forces between the structural elements. Due to the importance of microstructured materials in contemporary technology, the studies of dynamical processes in such materials become also of importance [8–11].

Here we restrict ourselves to the analysis of the physical effects due to the presence of the fifth order dispersive term and higher-order nonlinearities. We focus on waves in dilatant granular materials [12–14] and in shape-memory alloys [3,4,15,16]. In the first case, the character of grains is directly taken into account, and the result is a hierarchical KdV equation with two KdV operators. In the second case the governing equation is derived on the basis of lattice theory. Despite of a difference of models, the similarity is striking — the cubic and the fifth order dispersive terms in both cases and complicated nonlinearity to balance them. If the inertia of grains is neglected then the dispersive terms of both models coincide. In this paper the attention is paid to understand the mechanisms of the emergence of solitary waves and to the characterisation of possible types of waves emerging from the balance or imbalance of dispersive and nonlinear effects.

The paper is organised as follows. In Section 2 the mathematical models are described and the problems under consideration and the methods are introduced. Results are presented and discussed in Section 3 while in Section 4 concluding remarks are given.

2. The problem, mathematical models and methods

2.1. Wave propagation in dilatant granular materials

One-dimensional motion of suspension of particles (grains) in a compressible fluid is considered. Fluid density is assumed to be small compared to the particle density and rotation of particles is neglected. Corresponding equations of motion are derived by Giovine and Oliveri [12]. In the case of incompressible grains equations of motion result in a KdV equation

$$u_t + uu_x + du_{xxx} = 0 \tag{1}$$

where d is the dispersion parameter. However, the case of compressible grains results in a hierarchical KdV (HKdV) equation

$$u_t + uu_x + \alpha_1 u_{xxx} + \beta(u_t + uu_x + \alpha_2 u_{xxx})_{xx} = 0 \tag{2}$$

Here α_1 and α_2 are dispersion parameters for macro- and microlevel, respectively and β is the microstructure parameter that includes the ratio of wavelength to the grain size. The parameter β can be negative (in the case of high kinetic energy of particles) as well as positive (in the case of small kinetic energy of particles) [12]. Eq. (2) is clearly hierarchical in the Whitham sense [17] — if $|\beta| \to 0$, the influence of microstructure can be neglected and the wave "feels" only the macrostructure and, vice versa, if $1/|\beta| \to 0$ only the influence of the microstructure is "felt" by the wave. For Eq. (2) the limit case $\beta = 0$ reveals the standard KdV equation with standard soliton solutions.

Our main goal is to analyse and characterise the time-space behaviour of the solution in three-dimensional space of material parameters $\alpha_1, \alpha_2, \beta$. In this reason the HKdV equation is integrated numerically under harmonic initial conditions

$$u(x, 0) = \sin x, \quad x \in [0\, 2\pi] \tag{3}$$

and periodic boundary conditions

$$u(x + 2n\pi, t) = u(x, t), \quad n = \pm 1, \pm 2, \dots \tag{4}$$

2.2. Wave propagation in microstructured solids

We are particularly interested in the wave propagation in shape-memory alloys. Such alloys (Fe–C, Ni–Ti, Fe–Ni–Cr, for example) are characterised by austenite-martenste phase transitions. Corresponding model equations, proposed by G.A. Maugin, include higher-order nonlinear and higher-order dispersion terms and can be of Boussinesq [4,5] or KdV type [15,16].

In the present paper the KdV type model equation [15,16]

$$u_t + [P(u)]_x + du_{xxx} + bu_{xxxxx} = 0, \quad P(u) = -0.5u^2 + 0.25u^4 \tag{5}$$

is used. Here the nonlinearity is described by the quartic elastic potential and d and b are the third- and the fifth-order dispersion parameters, respectively. The higher-order KdV-like Eq. (5) is shortly referred as KdV435 below. The character of dispersion depends on the signs of dispersion parameters d and b. If $db > 0$ then one has normal dispersion. However, if $db < 0$ then one has so called mixed dispersion case, i.e., the character of dispersion is normal for some values of wavenumber and anomalous for anothers [16]. Numerical experiments with the harmonic initial conditions (3) have shown that the KdV435 equation

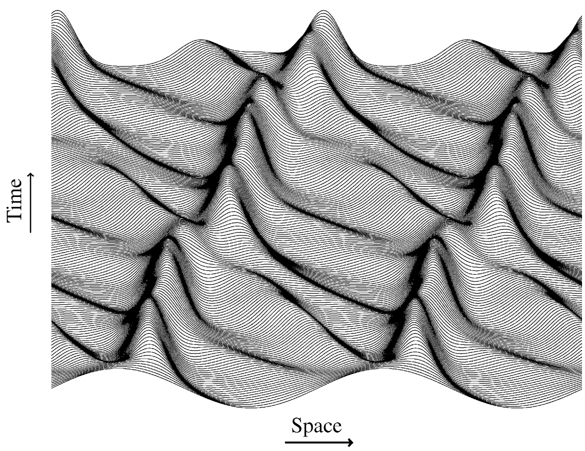

Fig. 1. KdV ensemble of solitons (time-slice plot over two 2π periods in space, $\alpha_1 = 0.05, \alpha_2 = -0.03, \beta = 0.0111, 0 < t < 20$).

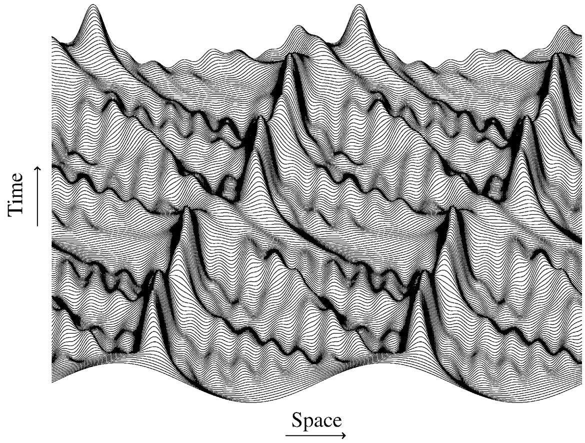

Fig. 2. KdV ensemble and suppressed EA ensemble (time-slice plot over two 2π periods in space, $\alpha_1 = 0.05$, $\alpha_2 = 0.052$, $\beta = 0.0111$, $0 < t < 20$).

can have soliton type solutions in the case of normal as well as mixed dispersion [15,16,18]. In the present paper the normal dispersion case ($d > 0$ and $b < 0$) is studied and logarithmic dispersion parameters

$$d_l = -\log d \quad \text{and} \quad b_l = -\log |b| \tag{6}$$

are used in parallel to d and b.

The KdV435 equation is integrated numerically under localised initial conditions

$$u(x, 0) = A \operatorname{sech}^2 \frac{(x - \vartheta_0)}{\Delta}, \tag{7}$$

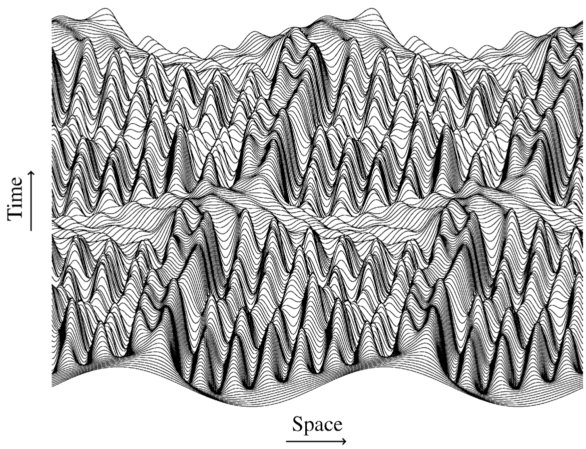

Fig. 3. KdV ensemble and amplified EA ensemble (time-slice plot over two 2π periods in space, $\alpha_1 = 0.05$, $\alpha_2 = 0.0725$, $\beta = 0.0111$, $0 < t < 20$).

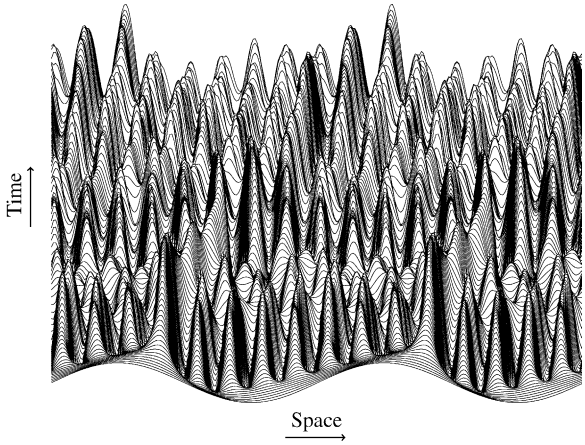

Fig. 4. KdV ensemble and dominating EA ensemble (time-slice plot over two 2π periods in space, $\alpha_1 = 0.05$, $\alpha_2 = 0.07$, $\beta = 0.0111$, $0 < t < 20$).

and periodic boundary conditions

$$u(x + 2nm\pi, t) = u(x, t), \quad n = \pm 1, \pm 2, \quad m > 2 \tag{8}$$

Here A is the amplitude of the localised initial wave, ϑ_0 the initial phase-shift and $\Delta = \sqrt{12d/A}$ is referred to as the width of the soliton in [1]. We are interested whether or not the initial localised wave (7) which is the solution of the KdV equation (1) can propagate with a constant speed, shape and amplitude in media, described by the KdV435 Eq. (5).

2.3. Numerical technique

For numerical solution of model equations the discrete Fourier transform (DFT) based pseudospectral methods [19] are used, i.e., space derivatives are approximated making use of DFT and for integration with

Table 1
The number of solitary waves N in the dominating EA ensemble against the ratio α_2/α_1 for $\beta = 0.0111$ and $\beta = 0.0055$

α_2/α_1	N
$\beta = 0.0111$	
1.11	9
1.40	8
1.80	7
2.52	6
3.47	5
$\beta = 0.0055$	
0.93	14
1.07	13
1.27	12
1.50	11
1.81	10
2.26	9

respect to time standard ODE solvers (Runge–Kutta–Fehlberg or implicit Adams methods, for example) are used. For the DFT standard FFT or FFTW algorithms are applied. For analyses of numerical results discrete spectral analysis is applied, i.e., in order to characterise the space-time behaviour of the solution Fourier transform related spectral quantities are used [20].

3. Results and discussion

3.1. HKdV equation

Numerical solutions for the HKdV equation (2) are found over wide range of material parameters β, α_1 and α_2. Possible solution types and subtypes are introduced and analysed in [21]. In the case of the *first solution type* a train of interacting solitons forms like in the case of the KdV equation. Solution of this type is called a KdV ensemble (Fig. 1). In the case of the *second solution type* the KdV ensemble and an ensemble of nearly equal amplitude solitary waves (EA ensemble, shortly) emerge simultaneously. The EA ensemble can be suppressed (Fig. 2) or amplified (Fig. 3). In some cases the rate of the amplification is so high that the EA ensemble starts to dominate over the KdV ensemble (Fig. 4). This is the case we focus at here. Solitary waves in the EA ensemble propagate nearly with the same speed and there are no interactions between themselves — solitary waves from the EA ensemble interact only with solitons from the KdV ensemble. Through these interactions solitary waves from the EA ensemble conserve their speed, shape and amplitude, i.e., these interactions are elastic and one can say that two solitonic structures have formed simultaneously. The solutions of the HKdV equation (2) have symmetry

$$u(x, t, \alpha_1, \alpha_2) = -u(-x, t, -\alpha_1, -\alpha_2) \tag{9}$$

in the $\alpha_1 - \alpha_2$ plane and therefore one can speak about ensembles of positive as well as negative solitons in the case of the present equation.

We concentrate now our attention to the amplification and domination of the EA ensemble. This phenomenon can take place for $\beta > 0$ and $\alpha_1\alpha_2 > 0$. The domination of the EA ensemble and the

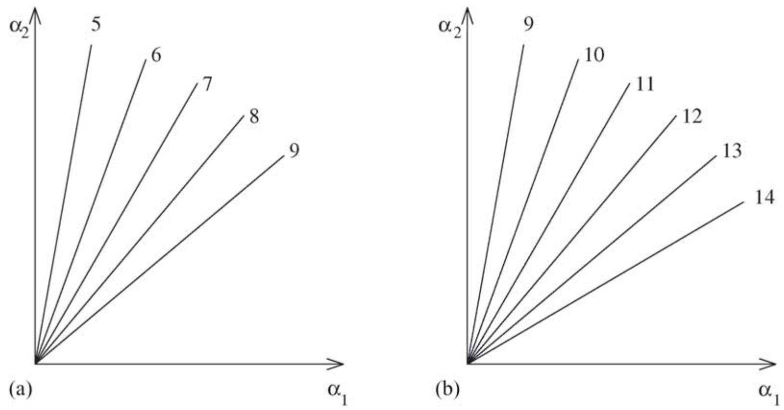

Fig. 5. The number of solitary waves in the dominating EA ensemble and corresponding straight lines α_2/α_1 = constant in the $\alpha_1 - \alpha_2$ plane: (a) $\beta = 0.0111$; (b) $\beta = 0.0055$. The numbers of solitary waves are indicated at each line.

number of solitary waves in the EA ensemble is determined by the ratio of dispersion parameters α_2/α_1. The domination of the EA ensemble can take place only in the very narrow neighbourhood of certain straight lines $\alpha_2/\alpha_1 =$ constant. In Table 1 the characteristic values of the ratio α_2/α_1, and corresponding number of solitary waves N in the EA ensemble are presented while in Fig. 5 straight lines are shown for $\beta = 0.0111$ and $\beta = 0.0055$. In Fig. 6 the waveprofile minima $\min_{t,x} u(x, t)$ and maxima $\max_{t,x} u(x, t)$ are plotted against α_2 for fixed value of $\alpha_1 = 0.05$ for $\beta = 0.0111$ (Fig. 6a) and $\beta = 0.0055$ (Fig. 6b). These results indicate the emergence of resonant solitary waves at certain values of α_2. These values depend on the value of β.

It is clear (see Fig. 6 and Table 1) that if the EA ensemble is dominating then waveprofiles are stretched in the positive as well as in the negative direction. The most distinctive domination within our range of parameters corresponds to the number of solitary waves $N = 9$ ($\alpha_2/\alpha_1 = 1.11$) for $\beta = 0.0111$ and $N = 13$ ($\alpha_2/\alpha_1 = 1.07$) for $\beta = 0.0055$. In very simple words the phenomenon of the EA ensemble domination can be described as a certain resonance phenomenon — if we fix the value of α_1 (for example $\alpha_1 = 0.05$ in Figs. 5 and 6) then for certain values of α_2 the EA ensemble is dominating and for certain values it

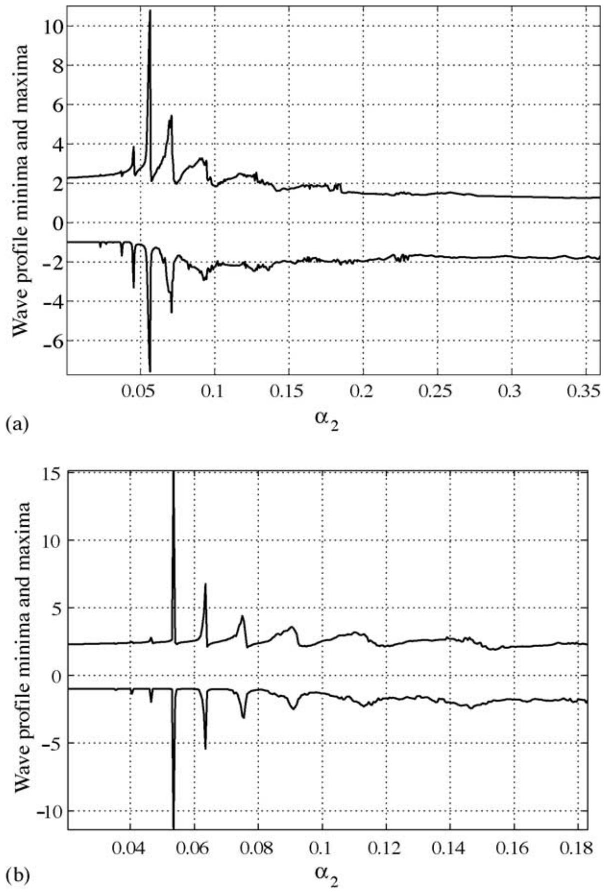

Fig. 6. Waveprofile minima and maxima against the dispersion parameter α_2: (a) $\beta = 0.0111$, $\alpha_1 = 0.05$ and $\alpha_2 = \{0 : 0.0005 : 0.3595\}$; (b) $\beta = 0.0055$, $\alpha_1 = 0.05$ and $\alpha_2 = \{0.02 : 0.0005 : 0.1830\}$.

is suppressed. Furthermore, during one amplification–suppression cycle (α_2 is assumed to increase), the EA ensemble looses the most right solitary wave (see Fig. 3 where the highest KdV ensemble soliton has a double peak). It could be conjectured that the resonance is caused by the feedback between two KdV-systems resulting in the amplification of an EA ensemble. Despite of similarity to cnoidal waves, the EA ensemble is a different structure with clearly dominating spectral amplitude [21].

3.2. KdV435 equation

Making use of results of numerous numerical experiments with different values of logarithmic dispersion parameters and amplitude of the initial localised wave, two main solution types can be distinguished. In what follows, the typical examples are presented for $d_l = 0.8$ and $b_l = 2.0$ below.

In the case of *the first type* the initial localised wave decays to a wave-train. Figs. 7a and 8a demonstrate the behaviour of waveprofile minima $u_{\min}(t) = \min_x u(x, t)$ and maxima $u_{\max}(t) = \max_x u(x, t)$ against

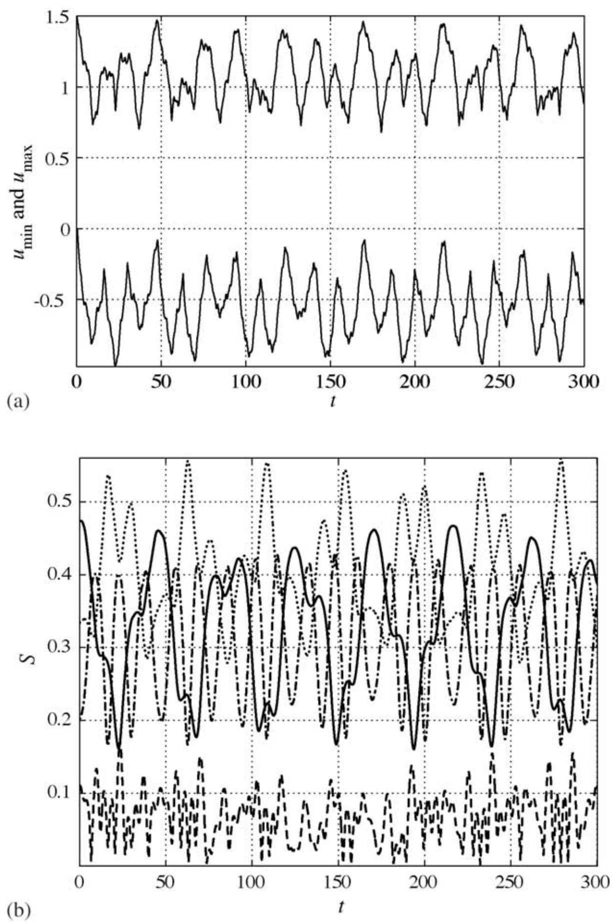

Fig. 7. Solution type 1a (amplitude of the initial localised wave $A = 1.5 < A^*$): (a) waveprofile minima and maxima against time; (b) spectral amplitudes $S_1, \ldots, S_4$ against time.

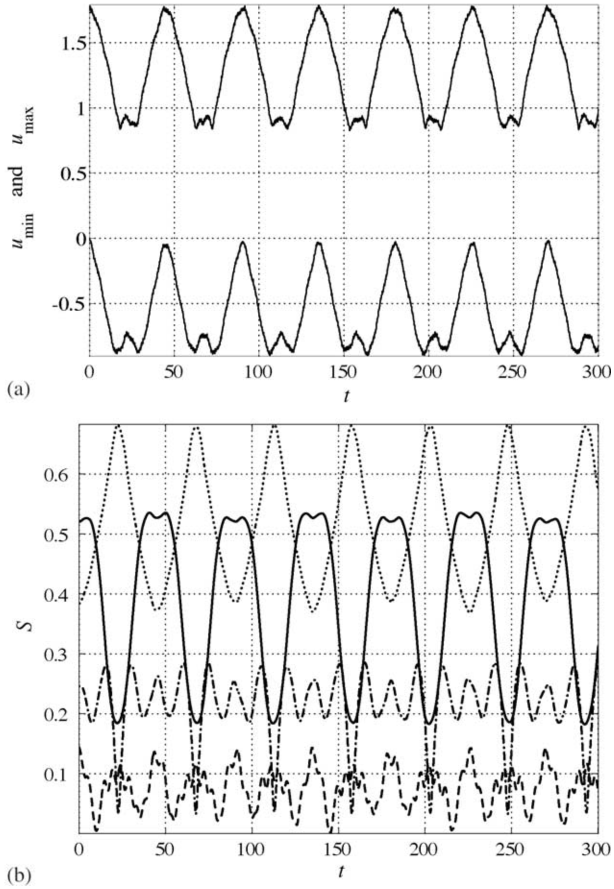

Fig. 8. Solution type 1b (amplitude of the initial localised wave $A = 1.75 < A^{*}$): (a) waveprofile minima and maxima against time; (b) spectral amplitudes $S_1, \ldots, S_4$ against time.

time. It is clear that in these cases the initial excitation $0 < u \leq A$ is completely destroyed and the waveprofile is stretched in the negative as well as in the positive direction. In Fig. 8a one can find that the behaviour of quantities $u_{\max}(t)$ and $u_{\min}(t)$ is (quasi)periodic in time, but in Fig. 7a one cannot detect such a periodicity. This phenomenon is verified in Figs. 7b and 8b by time-dependences of spectral amplitudes $S_\omega = 2|U(\omega, t)|/n,\ \omega = 1, \ldots, n/2$, (the quantity $U(\omega, t)$ is the DFT of the solution $u(x, t)$). We refer the nonperiodic case as a solution type 1a and the periodic case — as 1b.

Time-slice plot in Fig. 9a demonstrates that for a certain value of the amplitude A, an initial wave really can propagate without essential changes in the amplitude, shape and speed. In this case quantities $u_{\max}(t)$ and $u_{\min}(t)$ are not absolutely constant, but oscillate and deviate from initial values $u_{\max}(0) = A$ and $u_{\min}(0) = 0$ by a small extent (see Fig. 9b where corresponding time dependences are presented). This case is called *the second type* solution and it can be realised if the amplitude of the initial wave A is higher than a certain critical value A^{*}. In the case considered in the present paper, i.e. for $d_l = 0.8$ and $b_l = 2.0$ the critical value $1.86 < A^{*} < 1.87$, i.e., for $A = 1.86$ the initial solitary wave is destroyed, but for $A = 1.87$ it propagates with nearly constant amplitude, shape and speed. It means

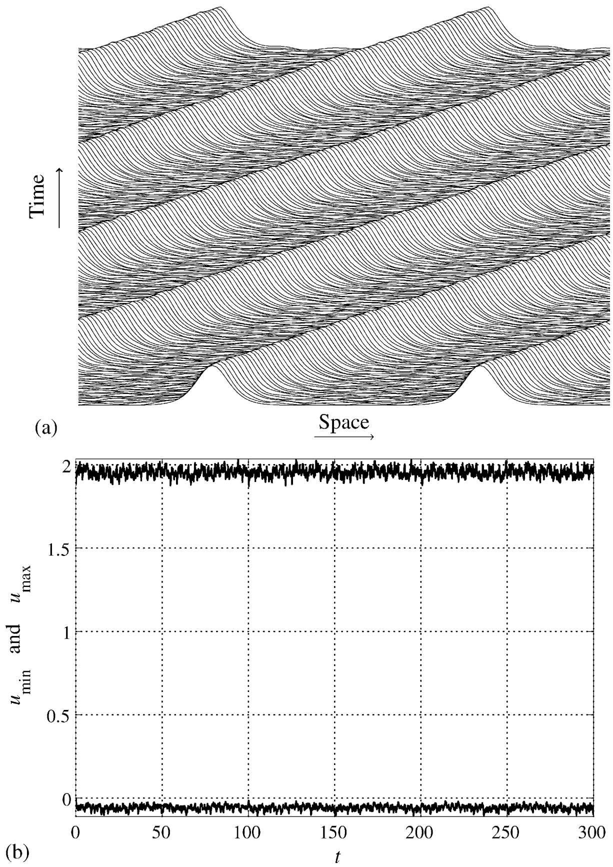

Fig. 9. Solution type 2 (amplitude of the initial localised wave $A = 1.88 > A^*$): (a) time-slice plot over two space periods; (b) waveprofile minima and maxima against time.

that the balance between nonlinearity and dispersion can occur only up from the certain initial energy level.

4. Concluding remarks

The wave phenomena in microstructured materials are strongly influenced by dispersive and nonlinear effects. In principle, both effects are more complicated than those described by the standard KdV equation. Here two possible generalisations are studied — (i) the HKdV equation built up by two different KdV equations and (ii) the KdV435 equation involving also the quartic nonlinearity and the fifth-order dispersion. Both generalisations include some similar terms (see Section 1) that is the reason to analyse them together. The formation of solitary wave structures from an harmonic input and the stability of localised waves are the main physical phenomena to understand. Based on numerical analysis the main results of this study are described below.

4.1. HKdV equation and harmonic input

The wave propagation in dilatant granular materials is modelled by the HKdV equation (2). In the case of harmonic initial conditions two solitonic structures — a KdV ensemble and an EA ensemble — can emerge simultaneously [21]. Furthermore, as shown here, for certain combination of values of parameters β, α_1 and α_2, the EA ensemble can be amplified or suppressed. In some cases the EA ensemble starts to dominate over the KdV ensemble. In the present paper the main attention was paid to the amplification–suppression phenomenon of the EA ensemble. Making use of results of many numerical experiments including cases $\beta > 0.0111$ and $\beta < 0.0055$, the following can be concluded:

- The smaller the value of the parameter β the higher the maximum number of solitary waves in the dominating EA ensemble, the higher their maximum amplitude and the wider the region where the domination can take place. However, simulations in the $\beta = 0$ limit result in standard KdV-soliton train.
- In all cases, however, the higher the value of α_2 (for fixed α_1) the smaller the number of solitary waves in the EA ensemble and the less distinctive the domination.
- The number of solitary waves in the EA ensemble can be properly determined only if the EA ensemble is amplified to a certain extent — otherwise some EA solitary waves are hidden under the highest soliton of the KdV ensemble.
- A resonance phenomenon for EA ensembles can occur in certain domains in the three-dimensional space of parameters β, α_1 and α_2 (see Fig. 6).

4.2. KdV435 equation and localised input

The KdV-like evolution Eq. (5) including higher-order nonlinear and dispersive terms is used to model the wave propagation in microstructured solids (shape-memory alloys particularly). In the present paper, Eq. (5) was solved numerically under initial conditions that correspond to the one soliton solution of the KdV equation (1). In addition to our earlier results related to harmonic excitations [15,16,18] one can conclude the following:

- If the amplitude of the initial localised wave is higher than a certain critical value then it can travel with a constant speed and without significant changes in its amplitude.
- If the amplitude is lower than the critical value then the initial localised wave decays to a wave-train. In some cases the behaviour of the wave-train can be (quasi)periodic in time and in these cases the phenomena of recurrence and superrecurrence can also be evident.
- As a rule, the amplitude of the initial localised wave has a “ideal” value near the critical one. In this case the amplitude of the solitary wave oscillates during the propagation only by very small extent and the radiating tail behind the solitary wave is practically undetectable.
- The higher the amplitude $A > A^*$ the higher and narrower are the solitary waves, the faster they are going to the right, the more distinctive is the tail and the more oscillations of their amplitude can be seen. This is an important result showing the difference from the standard KdV soliton.

Several questions in this paper remain open and research will be continued.

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