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Solitary waves in nonlinear microstructured materials

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Abstract

Dispersive effects due to microstructure of materials combined with nonlinearities give rise to solitary waves. In this paper the existence of solitary wave solutions is proved for a rather general hierarchical governing equation which accounts for nonlinearities on both macro- and microscales. Properties of the waves are established. Waves are asymmetric in the case of the nonlinearity in the microscale. Dispersive effects are due to the scale dependence.

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1. Introduction and problem formulation

Microstructured materials like alloys, crystallites, ceramics, functionally graded materials, etc have gained wide application. The modelling of wave propagation in such materials should be able to account for various scales of microstructure [1–3]. The scale dependence involves dispersive effects and if in addition the material behaves nonlinearly, then dispersive and nonlinear effects may be balanced. As widely known, in this case solitary waves may emerge as a result of this balance.

The existence and emergence of solitary waves in complicated physical problems apart from the model equations of mathematical physics must be analysed with sufficient correctness. Although there exists a possibility of using numerical simulation, the conditions and/or restrictions to be satisfied for the existence of solitary waves should be established analytically. Numerical simulation does not give exact information in this direction. Moreover, the equation to be studied in this paper is not integrable. This is another reason why the analytical results are of great interest.

There are several studies where the governing equations for waves in microstructured solids have been derived and the solitary waves analysed [4–6]. The crucial point, however, is to distinguish between nonlinearities on macro- and microlevel together with proper modelling of dispersive effects. Here we follow the Mindlin model [7] and use the hierarchical approach

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by Engelbrecht and Pastrone [4]. The basic 1D model for longitudinal waves in material possessing microstructure is

$$\rho u_{tt} = \sigma_x, \tag{1}$$

$$I\psi_{tt} = \eta_x - \tau, \tag{2}$$

where u is the macrodisplacement, ψ is the microdeformation, ρ and I are the macrodensity and microinertia, respectively, σ is the macrostress, η is the microstress and τ is the interactive force. We suppose that the material is physically nonlinear, although we have kept linear deformation and material coordinates x. At this stage we neglect possible dissipation and derive only the governing equation involving dispersive and nonlinear effects.

We consider the free energy function W in the following form:

$$W = W_2 + W_3, \tag{3}$$

where W_2 is the simplest quadratic function

$$W_2 = \tfrac{1}{2}au_x^2 + \tfrac{1}{2}B\psi^2 + \tfrac{1}{2}C\psi_x^2 + D\psi u_x \tag{4}$$

and W_3 includes nonlinearities on both the macro- and microlevel

$$W_3 = \tfrac{1}{6}Nu_x^3 + \tfrac{1}{6}M\psi_x^3. \tag{5}$$

Here a, B, C, D, N and M are constants. Then using the formulae

$$\sigma = \frac{\partial W}{\partial u_x}, \qquad \eta = \frac{\partial W}{\partial \psi_x}, \qquad \tau = \frac{\partial W}{\partial \psi} \tag{6}$$

we can rewrite systems (1), (2) in terms of u and ψ (cf [4, 8])

$$\rho u_{tt} = au_{xx} + Nu_x u_{xx} + D\psi_x, \tag{7}$$

$$I\psi_{tt} = C\psi_{xx} + M\psi_x\psi_{xx} - Du_x - B\psi. \tag{8}$$

Let us rewrite this system in dimensionless variables $X = \frac{x}{L}$, $T = \frac{tc_0}{L}$, $U = \frac{u}{U_0}$, where U_0 and L are certain constants (e.g. amplitude and wavelength of an initial excitation) and $c_0^2 = \frac{a}{\rho}$. Introducing the geometric parameters $\delta = \frac{l^2}{L^2}$, $\epsilon = \frac{U_0}{L}$, where l is the scale of the microstructure, this system reads

$$U_{TT} = U_{XX} + \frac{N\epsilon}{\rho c_0^2}U_X U_{XX} + \frac{D}{\rho c_0^2 \epsilon}\psi_X, \tag{9}$$

$$\delta a I^* \psi_{TT} = \delta C^* \psi_{XX} + \delta^{3/2} M^* \psi_X \psi_{XX} - D\epsilon U_X - B\psi, \tag{10}$$

where $I = I^*\rho l^2$, $C = C^* l^2$ and $M = M^* l^3$.

In order to eliminate the microdeformation ψ from (9), (10) we make use of the slaving principle (cf [4, 6, 8]). We deduce from (10) the expression for ψ

$$\psi = -\frac{D\epsilon}{B}U_X + \frac{\delta}{B}(C^*\psi_{XX} - aI^*\psi_{TT}) + \frac{\delta^{3/2}M^*}{B}\psi_X\psi_{XX}$$

and expand ψ into a Taylor series with respect to $\delta^{1/2}$: $\psi = \psi_0 + \delta^{1/2}\psi_1 + \delta\psi_2 + \delta^{3/2}\psi_3 + \cdots$. Then we obtain the following formulae for the first four terms in this expansion:

$$\psi_0 = -\frac{D\epsilon}{B}U_x, \qquad \psi_1 = 0,$$

$$\psi_2 = \frac{D\epsilon}{B^2}(aI^*U_{TT} - C^*U_{XX})_X, \qquad \psi_3 = \frac{D^2M^*\epsilon^2}{2B^3}\left(U_{XX}^2\right)_X.$$

Substituting $\psi_0 + \delta\psi_2 + \delta^{3/2}\psi_3$ for ψ in (9) we arrive at the following hierarchical governing equation for U:

$$U_{TT} = bU_{XX} + \frac{\mu}{2}\left(U_X^2\right)_X + \delta(\beta U_{TT} - \gamma U_{XX})_{XX} - \delta^{3/2}\frac{\lambda}{2}\left(U_{XX}^2\right)_{XX}, \tag{11}$$

where

$$b = 1 - \frac{D^2}{aB}, \qquad \mu = \frac{N\epsilon}{a}, \qquad \beta = \frac{D^2 I^*}{B^2}, \qquad \gamma = \frac{D^2 C^*}{aB^2}, \qquad \lambda = \frac{D^3 M^* \epsilon}{aB^3}. \tag{12}$$

The inequalities

$$0 < b < 1, \qquad \delta, \beta, \gamma > 0 \tag{13}$$

are valid for the coefficients b, δ, β and γ. Equation (11) actually involves hierarchically two wave operators

$$U_{TT} - bU_{XX} - \frac{\mu}{2}\left(U_X^2\right)_X \quad \text{and} \quad \delta\left(\beta U_{TT} - \gamma U_{XX} - \delta^{1/2}\frac{\lambda}{2}U_{XX}^2\right)_{XX} \tag{14}$$

characteristic of the macro- and microstructure, respectively. If the scale parameter δ is small then the influence of microstructure can be neglected. Conversely, if δ is large then the influence of macrostructure is weaker and the wave process is governed by the properties of the microstructure. Clearly, the intermediate case includes both effects.

Although the nonlinearities due to macro- and microstructure are identical in systems (9), (10), their influence in the single governing equation (11) is different. The nonlinear term $\frac{\mu}{2}\left(U_X^2\right)_X$, related to the macrostructure, compensates the dispersion caused by the higher-order derivatives and leads to the existence of solitary waves. On the other hand, the nonlinearity in the microscale, expressed by $\delta^{3/2}\frac{\lambda}{2}\left(U_{XX}^2\right)_{XX}$, deforms the solitary wave.

For future analysis we rewrite (11) by means of lower-case letters:

$$u_{tt} = bu_{xx} + \frac{\mu}{2}\left(u_x^2\right)_x + \delta(\beta u_{tt} - \gamma u_{xx})_{xx} - \delta^{3/2}\frac{\lambda}{2}\left(u_{xx}^2\right)_{xx}. \tag{15}$$

The related equation for the deformation $v = u_x$ reads

$$v_{tt} = bv_{xx} + \frac{\mu}{2}(v^2)_{xx} + \delta\left(\beta v_{tt} - \gamma v_{xx}\right)_{xx} - \delta^{3/2}\frac{\lambda}{2}\left(v_x^2\right)_{xxx}. \tag{16}$$

Travelling wave solutions of (16) have the form

$$v(x,t) = w(x - ct), \tag{17}$$

where c is a free parameter (velocity of the wave) and $w = w(\xi)$ satisfies the equation

$$(c^2 - b)w'' - \frac{\mu}{2}(w^2)'' - \delta(\beta c^2 - \gamma)w^{IV} + \delta^{3/2}\frac{\lambda}{2}[(w')^2]''' = 0. \tag{18}$$

We treat equation (18) in the classical sense requiring the solution to be four times continuously differentiable. We are interested in solitary wave solutions, i.e. solutions, which are nontrivial ($w \not\equiv 0$) and vanish at infinity. According to the latter requirement we complement equation (18) with the conditions

$$w(\xi), w'(\xi), w''(\xi) \to 0 \qquad \text{as} \quad |\xi| \to \infty. \tag{19}$$

The aim of this paper is to give a mathematically rigorous explanation of the existence and properties of solitary waves in materials characterized by equation (18). In section 2 we derive a canonical description for the problem, and in section 3 prove the existence of solitary waves and establish their basic properties. Section 4 deals with physical and geometrical properties for both normal and anomalous dispersion. A short discussion is presented in section 5. The proofs of lemmas are given in two appendices.

2. Canonical description of the problem

Let us start by integrating twice (18). This leads to the equation

$$(c^2 - b)w - \frac{\mu}{2}w^2 - \delta(\beta c^2 - \gamma)w'' + \delta^{3/2}\frac{\lambda}{2}[(w')^2]' = C_1\xi + C_2, \tag{20}$$

where C_1 and C_2 are arbitrary constants. In view of (19) we have $C_1 = C_2 = 0$. Therefore, under condition (19), equation (18) is equivalent to the equation of the second order

$$(c^2 - b)w - \frac{\mu}{2}w^2 - [\delta(\beta c^2 - \gamma) - \delta^{3/2}\lambda w']w'' = 0. \tag{21}$$

Further, we multiply this equation by w' to obtain $w''[\delta(\beta c^2 - \gamma)w' - \delta^{3/2}\lambda(w')^2] = \left[(c^2 - b)w - \frac{\mu}{2}w^2\right]w'$ and integrate once again taking (19) into account. This results in the equation

$$\frac{\delta(\beta c^2 - \gamma)}{2}(w')^2 - \frac{\delta^{3/2}\lambda}{3}(w')^3 = \frac{c^2 - b}{2}w^2 - \frac{\mu}{6}w^3. \tag{22}$$

In order to study (22) we assume the inequalities

$$\beta c^2 - \gamma \neq 0, \qquad c^2 - b \neq 0, \qquad \mu \neq 0. \tag{23}$$

These relations are necessary for the existence of the solitary wave solution. Namely, the following lemma holds.

Lemma 1. *If (18) has a solitary wave solution then (23) are valid.*

Proof of lemma 1 is shifted to appendix A of the paper.
Dividing by $\beta c^2 - \gamma \neq 0$ in (22) we obtain the equation of the first order

$$(w')^2 - \frac{2\delta^{1/2}\lambda}{3(\beta c^2 - \gamma)}(w')^3 = \frac{c^2 - b}{\delta(\beta c^2 - \gamma)}w^2 - \frac{\mu}{3\delta(\beta c^2 - \gamma)}w^3. \tag{24}$$

We can make some immediate conclusions from this equation. Noting that $w, w' \to 0$ as $|\xi| \to \infty$ we see that the asymptotic relation

$$(w')^2 \sim \frac{c^2 - b}{\delta(\beta c^2 - \gamma)}w^2 \qquad \text{as} \quad |\xi| \to \infty \tag{25}$$

is valid for the solution. This leads to the additional necessary solvability condition

$$\frac{c^2 - b}{\beta c^2 - \gamma} > 0. \tag{26}$$

Now we introduce the following three parameters which, as we will see later on, have certain physical or geometrical meaning:

$$\kappa := \sqrt{\frac{c^2 - b}{\delta(\beta c^2 - \gamma)}}, \qquad A := \frac{3(c^2 - b)}{\mu}, \qquad \Theta := 2\left[\frac{c^2 - b}{\beta c^2 - \gamma}\right]^{3/2}\frac{\lambda}{\mu}. \tag{27}$$

In terms of these parameters equation (24) has the form

$$(w')^2 - \frac{\Theta}{\kappa A}(w')^3 = \kappa^2 w^2\left(1 - \frac{w}{A}\right). \tag{28}$$

Thus, the solution depends upon κ, A and Θ.

The parameter κ is the exponential decay rate of the solution. This follows if we compare (25) with the definition of κ. Then we obtain

$$(w')^2 \sim \kappa^2 w^2 \qquad \text{as} \quad |\xi| \to \infty, \tag{29}$$

which implies

$$\ln|w(\xi)| \sim -\kappa|\xi| \qquad \text{as} \quad |\xi| \to \infty. \tag{30}$$

The inverse of decay rate $1/\kappa$ is usually referred to as the width of the wave because it is proportional to the width of the observable support of the wave. Later on we will see that A is the amplitude of the wave. The third parameter Θ is related to the asymmetry of the wave. The size of the parameter Θ which depends on the ratio of coefficients of nonlinear terms of the wave equation $\frac{\lambda}{\mu}$, is crucial for the existence of the solitary wave. We will study it closely in section 3 making use of the geometry of trajectories of the equation on the phase plane. The physical background will be explained in section 4.

To simplify the study of (28), we introduce new variables

$$y = \frac{1}{A}w, \qquad \zeta = \kappa\xi. \tag{31}$$

Then equation (28) is reduced to the following canonical equation for $y(\zeta)$:

$$(y')^2 - \Theta(y')^3 = y^2 - y^3. \tag{32}$$

Here $y' = \frac{\mathrm{d}y}{\mathrm{d}\zeta}$. For further integration, this equation has first to be solved with respect to y':

$$y' = Q(y). \tag{33}$$

Then the inverse of the solution $y(\zeta)$ is expressed in the form

$$\zeta = \int \frac{\mathrm{d}y}{Q(y)}. \tag{34}$$

Unfortunately, an analytical integration of (33) is very complicated because Q contains an inverse of a cubic function in terms of another cubic function. To the authors' knowledge, it cannot be integrated within known functions. Nevertheless, a simple particular case occurs when the nonlinearity in the microscale is absent, i.e. $\lambda = 0$. Then $\Theta = 0$ and we obtain a symmetric bell-shaped solitary wave in the explicit form

$$y(\zeta) = \cosh^{-2}\left(\frac{\zeta}{2}\right) \quad \Longrightarrow \quad w(\xi) = A\cosh^{-2}\left(\frac{\kappa\xi}{2}\right). \tag{35}$$

This case has been thoroughly studied in [6, 9, 10].

Another equivalent representation of equation (18) can be derived, too. Namely, we can write equation (21) in the form of the autonomous system of the first order

$$w' = p, \qquad p' = \frac{(c^2 - b)w - \frac{\mu}{2}w^2}{\delta(\beta c^2 - \gamma) - \delta^{3/2}\lambda p}. \tag{36}$$

Here the denominator $\delta(\beta c^2 - \gamma) - \delta^{3/2}\lambda p = \delta(\beta c^2 - \gamma) - \delta^{3/2}\lambda w'$ is not identically zero. Indeed, otherwise by (23) we had the relations $w' \equiv \frac{\beta c^2 - \gamma}{\delta^{1/2}\lambda} \neq 0$, and $(c^2 - b)w - \frac{\mu}{2}w^2 \equiv 0 \Longrightarrow w' \equiv 0$, which contradict each other.

In canonical variables (36) reads

$$y' = z, \qquad z' = \frac{y\,(2 - 3y)}{2 - 3\Theta z}. \tag{37}$$

Systems (36) and (37) are suitable for numerical solution of the problem. The form (37) will be used in analysis, too.

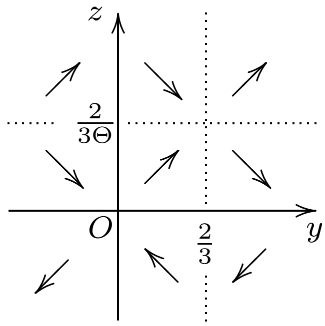

Figure 1. Phase portrait of (37).

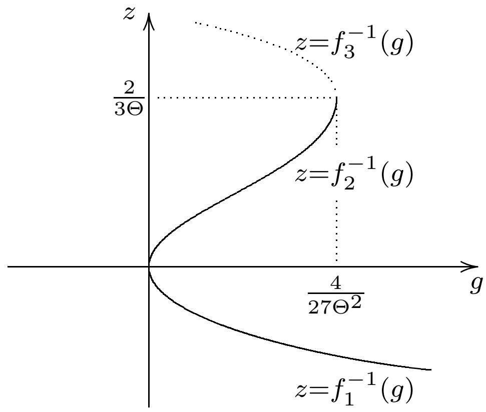

Figure 2. Function $f^{-1}(g)$.

3. Existence and basic properties of canonical solitary waves

Let us begin with the case $\Theta \geqslant 0$. Since the solitary wave solution of (32) satisfies the conditions $y, y' \to 0$ as $|\zeta| \to \infty$, the trajectory (phase curve) T of (37), corresponding to this solution, must satisfy the following condition:

$$T \text{ is a closed curve containing } O = (0, 0). \tag{38}$$

To see the location of such a trajectory on the phase plane, we denote by $\vec{\phi} = (\phi_1, \phi_2)$ the right-hand side vector of system (37), i.e. $\phi_1 = z, \phi_2 = \frac{y(2-3y)}{2-3\Theta z}$, and observe that the zeros $y = 0, y = \frac{2}{3}, z = 0$ and the singularity $z = \frac{2}{3\Theta}$ of ϕ_1, ϕ_2 divide the phase plane into nine subregions. The vector $\vec{\phi}$ preserves the orientation in each of these subdomains. The corresponding phase portrait is shown in figure 1. Due to the orientation of $\vec{\phi}$, a trajectory T with property (38) can be located only in the quarter $y \geqslant 0, z < \frac{2}{3\Theta}$.

The equation of T is by (32) $z^2 - \Theta z^3 = y^2 - y^3$. We are going to study this equation in the quarter $y \geqslant 0, z < \frac{2}{3\Theta}$. To this end we express it as $z = f^{-1}(y^2 - y^3)$ where f^{-1} is the inverse of $f(z) = z^2 - \Theta z^3$. To see more clearly the behaviour of this function we introduce the intermediate variable g and split the equation $z = f^{-1}(y^2 - y^3)$ into two subsequent relations

$$z = f^{-1}(g), \qquad g = y^2 - y^3.$$

The components $f^{-1}(g)$ and $g(y)$ are graphed in figures 2 and 3.

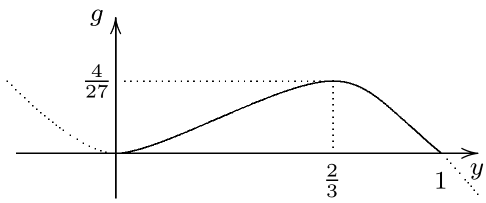

Figure 3. Function $g = y^2 - y^3$.

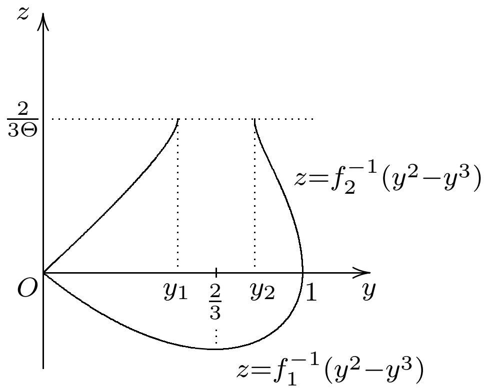

Figure 4. T in the case $\Theta > 1$.

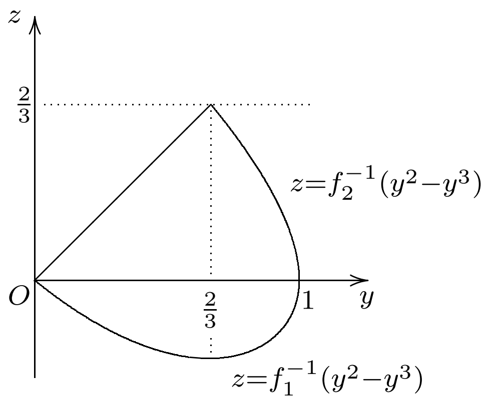

Figure 5. T in the case $\Theta = 1$.

The inverse f^{-1} has three branches f_1^{-1}, f_2^{-1} and f_3^{-1}. The latter one is not related to the solitary wave because it falls beyond the singularity line $z = \frac{2}{3\Theta}$. The remaining branches f_1^{-1} and f_2^{-1} are defined for non-negative values of g. This together with the above inequality $y \geqslant 0$ restricts the domain of g to [0, 1]. The branch f_1^{-1} forms the curve $z = f_1^{-1}(y^2 - y^3)$, which connects the points $(0, 0)$ and $(1, 0)$ and passes through the lower half-plane $z < 0$ (lower parts of the trajectories in figures 4–6).

Concerning another branch f_2^{-1}, three different particular cases can occur:

(1) $\Theta > 1$. Let us compare the range of g which is $\left[0, \frac{4}{27}\right]$ with the domain of f_2^{-1} which is $\left[0, \frac{4}{27\Theta^2}\right]$. Since $\frac{4}{27\Theta^2} < \frac{4}{27}$, the whole range of g does not go in the domain

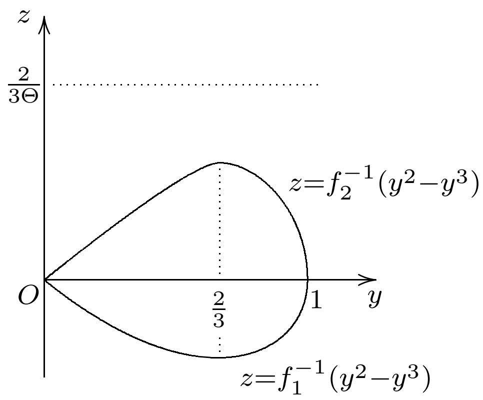

Figure 6. *T* in the case $0 \leqslant \Theta < 1$.

of f_2^{-1}. From figure 3 we see that restricting the range of g to set $\left[0, \frac{4}{27\Theta^2}\right]$ restricts the domain of g to a certain union of the form $[0, y_1] \cup [y_2, 1]$, where y_1 and y_2 are some numbers in intervals $\left(0, \frac{2}{3}\right)$ and $\left(\frac{2}{3}, 1\right)$, respectively. Summing up, the composition $z = f_2^{-1}(g(y)) = f_2^{-1}(y^2 - y^3)$ is defined for $y \in [0, y_1] \cup [y_2, 1]$ but not for $y \in (y_1, y_2)$. This case is illustrated in figure 4. System (37) has not a trajectory with property (38), hence the solitary wave does not exist.

(2) $\Theta = 1$. Then the curve $z = f_2^{-1}(y^2 - y^3)$ connects the points $(0, 0)$ and $(1, 0)$ and passes through the upper half-plane $z > 0$ (see figure 5). Function $z = f_2^{-1}(y^2 - y^3)$ has a maximum $\left(\frac{2}{3}, \frac{2}{3}\right)$ which is located on the singularity line $z = \frac{2}{3}$. To see the behaviour of the curve at this point we observe that the equation of the trajectory has the form $z^2 - z^3 = y^2 - y^3$, which admits a particular linear solution $z = y$ passing through $(0, 0)$. This implies that the curve $z = f_2^{-1}(y^2 - y^3)$ is the straight line $z = y$ to the left of $y = \frac{2}{3}$. It has positive slope at the maximum $\left(\frac{2}{3}, \frac{2}{3}\right)$, hence $z = f_2^{-1}(y^2 - y^3)$ is not smooth at $\left(\frac{2}{3}, \frac{2}{3}\right)$. The function $\frac{dz}{dy}$ is discontinuous, hence y'' is discontinuous. Four times continuously differentiable solitary wave solution does not exist. However, the solution exists in a certain generalized sense.

(3) $0 \leqslant \Theta < 1$. Then $\frac{4}{27\Theta^2} > \frac{4}{27}$. The curve $z = f_2^{-1}(y^2 - y^3)$ connects the points $(0, 0)$ and $(1, 0)$ and passes through the band $0 < z < \frac{2}{3\Theta}$. This case is presented in figure 6. The trajectory *T*, defined as the union of the curves $z = f_1^{-1}(y^2 - y^3)$ and $z = f_2^{-1}(y^2 - y^3)$, has property (38). Let us consider the Cauchy problem for (37) with the initial conditions $y(0) = 1, z(0) = 0$. This has a solution, which by the relation $z^2 \sim y^2$ as $y \to 0$, following from the equation $z^2 - \Theta z^3 = y^2 - y^3$, satisfies the conditions $y(\zeta), z(\zeta) = y'(\zeta) \to 0$ as $|\zeta| \to \infty$. Therefore, $y(\zeta)$ is the solitary wave solution. Other solitary wave solutions can be derived from $y(\zeta)$ by the simple argument shift $\zeta \mapsto \zeta + C$, C-constant. Since the right-hand side of (37) is infinitely differentiable for $y \geqslant 0, z < \frac{2}{3\Theta}$, the solution $y(\zeta)$ is infinitely differentiable.

The qualitative behaviour (increase, decrease, concavity and convexity intervals, etc) of $y(\zeta)$ follows immediately from the related properties of *T*. Namely,

y has amplitude equal to 1;

y increases for $\zeta < 0$ and decreases for $\zeta > 0$;

there exist $\zeta_1 < 0$ and $\zeta_2 > 0$ such that y is concave for $\zeta < \zeta_1, \zeta > \zeta_2$ and convex for $\zeta_1 < \zeta < \zeta_2$;

$y(\zeta_1) = y(\zeta_2) = \frac{2}{3}$.

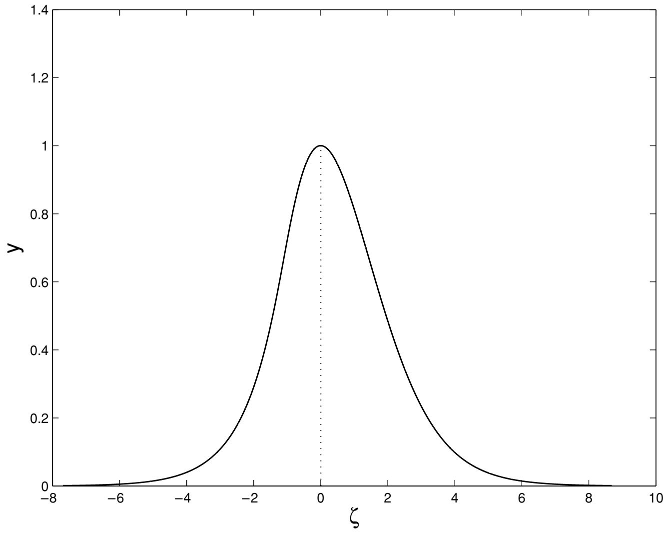

Figure 7. Canonical wave in the case $\Theta = 0.9$.

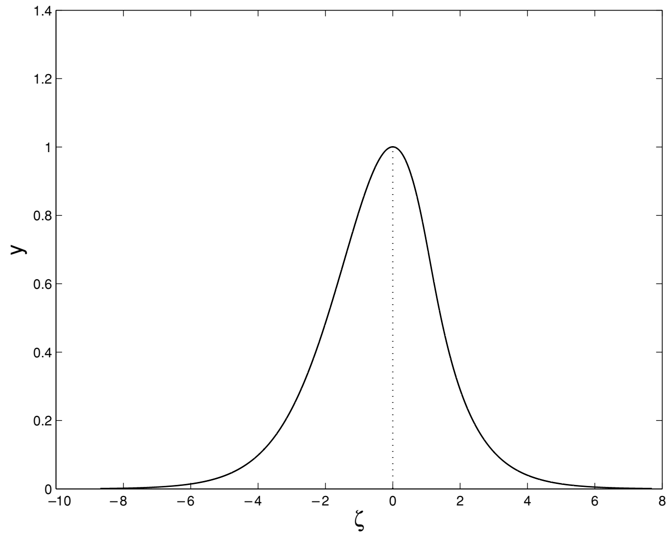

Figure 8. Canonical wave in the case $\Theta = -0.9$.

From (32) we easily see that the $y(\zeta)$ solves (32) if and only if $y(-\zeta)$ solves (32) with Θ replaced by $-\Theta$. Therefore, the solution corresponding to $\Theta < 0$ is the reflection over the line $\zeta = 0$ of the solution corresponding to $-\Theta > 0$. Two examples of the solitary wave solutions, computed numerically by means of the second-order Adams–Bashforth method, are depicted in figures 7 and 8.

Summing up, the general existence condition for the canonical solitary wave is

$$|\Theta| < 1. \tag{39}$$

We remark that the wave is asymmetric when $\Theta \neq 0$. To measure the asymmetry, let us introduce some additional notation. We note that the solution $y(\zeta)$ is strictly monotone to the left and right of the amplitude point $\zeta = 0$. Therefore, for any $y \in (0, 1)$ it has two inverses: $\zeta^-(y) < 0$ and $\zeta^+(y) > 0$. The quantities $|\zeta^-(y)|$ and $|\zeta^+(y)|$ can be interpreted as the front and rear half-lengths of the wave at the fixed level y. The asymmetry of the wave at level $y \in (0, 1)$ can be measured by the ratio of the half-lengths

$$\frac{|\zeta^+(y)|}{|\zeta^-(y)|}. \tag{40}$$

We note that this ratio is increasing in Θ. Namely, the following lemma is valid.

Lemma 2. *For any* $y \in (0, 1)$ *the equality*

$$\frac{|\zeta^+(y)|}{|\zeta^-(y)|} = F_y(\Theta) \tag{41}$$

holds where $F_y(\Theta)$ *is an increasing function of* Θ *in the interval* $(-1, 1)$ *and* $F_y(0) = 1$.

Proof of lemma 2 is included in appendix B of the paper.

We remark that in the case $0 < \Theta < 1$ the asymmetry is greater than one, i.e. the front half-length is greater than the rear half-length (figure 7). In contrast, if $-1 < \Theta < 0$ then the asymmetry is smaller than one, i.e. the front half-length is smaller than the rear half-length (figure 8). In the intermediate case $\Theta = 0$ the wave is symmetric and expressed by $y(\zeta) = \cosh^{-2}\left(\frac{\zeta}{2}\right)$.

4. Physical and geometrical properties of solitary waves in general form

Let us return to equation (28) with parameters A, κ and Θ. According to the results of the previous section, it admits a solitary wave solution if and only if $|\Theta| < 1$. This solution has by (31) the form

$$w(\xi) = Ay_\Theta(\kappa\xi), \tag{42}$$

where y_Θ is the canonical solitary wave corresponding to given Θ.

Clearly, $A = \frac{3(c^2-b)}{\mu}$ is the amplitude of the wave. Depending on the signs of $c^2 - b$ and μ, waves with positive and negative amplitudes may occur. The absolute value of the amplitude is increasing in c^2.

Another important parameter is $\Theta = 2\left[\frac{c^2-b}{\beta c^2-\gamma}\right]^{3/2}\frac{\lambda}{\mu}$, which is related to the asymmetry. The sign of Θ equals the sign of $\mu\lambda$. Therefore, the following subcases may occur:

(1) In the case $A > 0$, $\mu\lambda > 0$ the wave has the shape of the canonical wave in figure 7.
(2) In the case $A > 0$, $\mu\lambda < 0$ the wave has the shape of the canonical wave in figure 8.
(3) In the case $A < 0$ the wave is the reflection over the line $w = 0$ of the wave corresponding to the amplitude $-A$.

In all mentioned cases the wavefunction $w(\xi)$ is strictly monotone to the left and right of the amplitude point $\xi = 0$. Therefore, it has two inverses: $\xi^-(w) < 0$ and $\xi^+(w) > 0$ which are defined for any w between 0 and A. Let us fix some relative level $y \in (0, 1)$ and consider the front and rear half-lengths of the wave at this relative level, namely the quantities $|\xi^-(yA)|$ and $|\xi^+(yA)|$. The asymmetry of the wave at the relative level $y \in (0, 1)$ is the ratio

$$\frac{|\xi^+(yA)|}{|\xi^-(yA)|}. \tag{43}$$

Since the relation $\xi^{\pm}(yA) = \frac{1}{\kappa}\zeta^{\pm}(y)$ holds between the inverses of non-canonical and canonical solutions, by lemma 2 the asymmetry on the relative level y is expressed by the formula

$$\frac{|\xi^+(yA)|}{|\xi^-(yA)|} = F_y(\Theta) = F_y\left(2\left[\frac{c^2-b}{\beta c^2-\gamma}\right]^{3/2}\frac{\lambda}{\mu}\right). \tag{44}$$

The asymmetry depends on the velocity, the coefficients of linear terms in (11) b, β, γ, and also on the ratio of the coefficients of nonlinear terms in micro- and macroscale $\frac{\lambda}{\mu}$. The latter two coefficients have different influences for the wave process. The nonlinearity in macroscale balances the dispersion, hence opens the possibility for the solitary wave. The nonlinearity in microscale rather disturbs this balance. As a result, the ratio $\frac{\lambda}{\mu}$ affects the shape of the wave. The bigger the ratio $\frac{\lambda}{\mu}$, the bigger the asymmetry. The balance between nonlinearity and dispersion collapses at the critical value $|\Theta| = 1 \Leftrightarrow \left|\frac{\lambda}{\mu}\right| = 2\left[\frac{\beta c^2-\gamma}{c^2-b}\right]^{3/2}$.

To give an insight to the dependence of the asymmetry and the width of the wave on the velocity, we have to distinguish the cases of normal and anomalous dispersion. The dispersion relation of (11) is $\omega^2 - bk^2 + \delta(\beta\omega^2k^2 - \gamma k^4) = 0$, where $\omega = \omega(k)$ and k are the frequency and the wave number, respectively [11]. This yields the formulae

$$c_g(k) = c_{ph}(k) + \frac{\delta k^2[\gamma - \beta b]}{c_{ph}(k)(\delta\beta k^2+1)^2} \tag{45}$$

for the phase and group velocities $c_{ph} = \frac{\omega(k)}{k}$ and $c_g = \omega'(k)$. From these formulae we see that the cases $c_{ph} > c_g$ (normal dispersion) and $c_{ph} < c_g$ (anomalous dispersion) correspond to the relations $\frac{\gamma}{\beta} < b$ and $\frac{\gamma}{\beta} > b$, respectively. In addition, there exists an intermediate case $c_{ph} = c_g$ when $\frac{\gamma}{\beta} = b$ and dispersion is absent. The latter case, although being mathematically correct, is not possible physically in a dispersive medium.

Let us start with the case of normal dispersion when $\frac{\gamma}{\beta} < b$. In view of (27)

$$\frac{1}{\kappa} = (\delta\beta)^{1/2}\left[1 - \frac{b-\frac{\gamma}{\beta}}{c^2-\frac{\gamma}{\beta}}\right]^{-1/2}, \qquad \Theta = \frac{2\lambda}{\mu\beta^{3/2}}\left[1 - \frac{b-\frac{\gamma}{\beta}}{c^2-\frac{\gamma}{\beta}}\right]^{3/2}. \tag{46}$$

Due to the inequality $\frac{\gamma}{\beta} < b$ the term $1 - \frac{b-\frac{\gamma}{\beta}}{c^2-\frac{\gamma}{\beta}}$ is increasing in c^2. Consequently, the width $1/\kappa$ decreases in c^2. Further, in the case $\mu\lambda > 0$ the parameter Θ increases in c^2. This by (44) and the monotonicity of F_y (see lemma 2) implies that the asymmetry is increasing in c^2. In the opposite case $\mu\lambda < 0$ the asymmetry is decreasing in c^2. The solvability condition $|\Theta| = 2\left[\frac{c^2-b}{\beta c^2-\gamma}\right]^{3/2}\left|\frac{\lambda}{\mu}\right| < 1$ defines the range for the velocity. The range is different in three subcases $0 \leqslant q < \frac{\gamma}{b}$, $\frac{\gamma}{b} \leqslant q \leqslant \beta$ and $\beta < q$, where

$$q = \left(\frac{2\lambda}{\mu}\right)^{2/3}. \tag{47}$$

Namely,

$$\begin{aligned}
&c^2 \in \left(0; \frac{\frac{\gamma}{b}-q}{\beta-q}b\right) \cup (b;\infty) && \text{if} \quad 0 \leqslant q < \frac{\gamma}{b},\\
&c^2 \in (b;\infty) && \text{if} \quad \frac{\gamma}{b} \leqslant q \leqslant \beta,\\
&c^2 \in \left(b; \frac{q-\frac{\gamma}{b}}{q-\beta}b\right) && \text{if} \quad \beta < q.
\end{aligned}$$

Secondly, let us consider the case of anomalous dispersion when $\frac{\gamma}{\beta} > b$. Then the term $1 - \frac{b-\frac{\gamma}{\beta}}{c^2-\frac{\gamma}{\beta}}$ in (46) is decreasing in c^2. This implies that the width $1/\kappa$ increases in c^2 and the asymmetry decreases in c^2 when $\mu\lambda > 0$ and increases in c^2 when $\mu\lambda < 0$. The range for c^2 is as follows.

$$c^2 \in (0; b) \cup \left(\frac{\frac{\gamma}{b} - q}{\beta - q} b; \infty \right) \qquad \text{if} \quad 0 \leqslant q < \beta,$$
$$c^2 \in (0; b) \qquad \text{if} \quad \beta \leqslant q \leqslant \frac{\gamma}{b},$$
$$c^2 \in \left(\frac{q - \frac{\gamma}{b}}{q - \beta} b; b \right) \qquad \text{if} \quad \frac{\gamma}{b} < q.$$

Summing up, both in the cases of normal and anomalous dispersion the size of the range depends on the ratio $\left|\frac{\lambda}{\mu}\right|$. The bigger the ratio $\left|\frac{\lambda}{\mu}\right|$, the smaller the range. If $\mu > 0$ then the amplitude is positive for $c^2 > b$ and negative for $c^2 < b$. Conversely, if $\mu < 0$ then the amplitude is negative for $c^2 > b$ and positive for $c^2 < b$.

5. Discussion

It has been shown that the complicated nonlinear and dispersive effects in microstructured solids may lead to stationary solutions, i.e. may be balanced. This is the precondition for the existence of solitary waves, which has been *proved analytically*. The classical Korteweg–de-Vries (KdV) model with quadratic nonlinearity and cubic dispersion leads to *symmetric* solitary wave soliton [12]. In our case, the background of nonlinearities and dispersive effects is more complicated (see equation (11)) resulting in an *asymmetric* solitary wave if $\lambda \neq 0$ (figures 7, 8). Such asymmetric solitary waves are known to exist in various physical systems. For example, long SH waves in a seismically active layer of the Earth's crust represent a case of weak energy influx and are described by a perturbed KdV-equation admitting also asymmetric solitary waves [13]. Strain waves in rods where combined dissipative/amplification effects due to various embedding or geometric conditions may occur, may also lead to asymmetric solitary waves [6, 14].

The governing equation (11) is actually more general than just describing waves in microstructured solids. The hierarchical properties of equation (11) reflect a general role of mixed higher order derivatives (in the present case ascribed to inertia properties of microstructure). A similar effect is also evident in generalized rod models [6, 14] but absent in lattice models [15]. What makes the present case interesting is the mechanism of nonlinearities due to various scales to be balanced with complicated dispersion.

The existence of solitary waves gives evidence of deep simplicity of complex systems. The analytically established structural parameters of solitary waves can be used for determining the properties of media, i.e. posing and solving inverse problems. First, the existence of symmetric solitary waves gives additional conditions between the parameters of the governing equation (see [6] and references therein). Second, the asymmetry (and also the amplitude and width) of solitary waves, analysed in this paper, gives clearly additional information on properties of microstructured solids that can be used in nondestructive testing (NDT) of material properties. The experimental studies of strain waves in microstructured materials [16] have demonstrated the asymmetry of solitary waves. In this case tungsten–epoxy composites were used with reference samples made of aluminium. Based on derived strict conditions, more detailed studies to elaborate concrete algorithms of NDT are in progress.

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Appendix A. Proof of lemma 1

First we prove the inequality $\beta c^2 - \gamma \neq 0$. Let us suppose that (18) has a solitary wave solution and $\beta c^2 - \gamma = 0$. Then (21) has the form

$$-\delta^{3/2}\lambda w' w'' = \left(c^2 - b - \frac{\mu}{2} w\right) w. \qquad (A.1)$$

Evidently, $\lambda \neq 0$. Otherwise w is a constant, hence does not satisfy (19). Since $w(\xi) \to 0$ as $|\xi| \to \infty$, the function $|w|$ has an absolute maximum at some point $\xi_1 \in \mathbb{R}$. We have $w'(\xi_1) = 0$. This by (A.1) implies $w(\xi_1) = \nu := 2(c^2 - b)/\mu$. Further, since $w'(\xi) \to 0$ as $|\xi| \to \infty$, $|w'|$ has an absolute maximum at a point $\xi_2 \in \mathbb{R}$. We have $w''(\xi_2) = 0$ and by (A.1) either $w(\xi_2) = \nu$ or $w(\xi_2) = 0$. In the former case $|w|$ attains values greater than $|\nu|$ in a neighbourhood of ξ_2, because $w'(\xi_2) \neq 0$. This is in contradiction with the proved statement that $|\nu|$ is the absolute maximum of $|w|$. In the latter case w changes the sign at ξ_2. Then, in view of $w(\xi) \to 0$ as $|\xi| \to \infty$, there exist $\xi_3, \xi_4 \in \mathbb{R}$, such that $w(\xi_3) < 0$ and $w(\xi_4) > 0$. Moreover, $w'(\xi_3) = w'(\xi_4) = 0$. Relation (A.1) implies $w(\xi_j) \in \{0; \nu\}$, $j = 3, 4$. But this contradicts the inequalities $w(\xi_3) < 0$ and $w(\xi_4) > 0$. Consequently, the supposition $\beta c^2 - \gamma = 0$ is wrong. We obtain $\beta c^2 - \gamma \neq 0$.

Next let us prove the inequality $c^2 - b \neq 0$. Suppose that (18) has a solitary wave solution and $c^2 - b = 0$. Then (18) reads

$$-\frac{\mu}{2} w^2 - [\delta(\beta c^2 - \gamma) - \delta^{3/2}\lambda w']w'' = 0. \qquad (A.2)$$

This implies

$$\operatorname{sign} w'' = \operatorname{sign}[\delta(\beta c^2 - \gamma) - \delta^{3/2}\lambda w']. \qquad (A.3)$$

Since $w(\xi) \to 0$ as $|\xi| \to \infty$, function w'' must at least twice change the sign on the line $(-\infty, \infty)$. Therefore, there exist points $\hat{\xi}_1 \neq \hat{\xi}_2$ such that

$$w''(\hat{\xi}_1) = \delta(\beta c^2 - \gamma) - \delta^{3/2}\lambda w'(\hat{\xi}_1) = w''(\hat{\xi}_2) = \delta(\beta c^2 - \gamma) - \delta^{3/2}\lambda w'(\hat{\xi}_2) = 0 \qquad (A.4)$$

and w'', $\delta(\beta c^2 - \gamma) - \delta^{3/2}\lambda w'$ differ from 0 between $\hat{\xi}_1$ and $\hat{\xi}_2$. By (A.4) and Rolle's theorem there exists a point $\hat{\xi}_3$ between $\hat{\xi}_1$ and $\hat{\xi}_2$ such that $[\delta(\beta c^2 - \gamma) - \delta^{3/2}\lambda w']' = 0$ at $\hat{\xi}_3$. From this relation we obtain $\lambda w''(\hat{\xi}_3) = 0$. Observing that $\lambda \neq 0$ (otherwise from (A.3) we had monotone w', which cannot approach zero as $|\xi| \to \infty$), we see that $w''(\hat{\xi}_3) = 0$. But since $\hat{\xi}_3$ is located between $\hat{\xi}_1$ and $\hat{\xi}_2$, the function w'' differs from zero at $\hat{\xi}_3$. We reached a contradiction. Consequently, the supposition $c^2 - b = 0$ is wrong. We obtain $c^2 - b \neq 0$.

It remains to prove $\mu \neq 0$. Again, we suppose that (18) has a solitary wave solution but $\mu = 0$. Then from (22) we obtain

$$w^2 = \frac{\delta(\beta c^2 - \gamma)}{c^2 - b}(w')^2 - \frac{2\delta^{3/2}\lambda}{3(c^2 - b)}(w')^3. \qquad (A.5)$$

In view of (A.5) function w equals zero in every stationary point. But on the other hand, due to the relation $w(\xi) \to 0$ as $|\xi| \to \infty$ function w must have at least one stationary point $\bar{\xi}$ such that $w(\bar{\xi}) \neq 0$. We reached a contradiction. Thus, $\mu \neq 0$. The proof is complete.

Appendix B. Proof of lemma 2

Let us differentiate the equation of the trajectory $z^2 - \Theta z^3 = y^2 - y^3$ with respect to Θ:

$$2z\frac{\mathrm{d}z}{\mathrm{d}\Theta} - 3\Theta z^2\frac{\mathrm{d}z}{\mathrm{d}\Theta} - z^3 = 0 \tag{B.1}$$

and solve this equation for $\frac{\mathrm{d}z}{\mathrm{d}\Theta}$:

$$\frac{\mathrm{d}z}{\mathrm{d}\Theta} = \frac{z^2}{2 - 3\Theta z}. \tag{B.2}$$

Since $z < \frac{2}{3\Theta}$ on the trajectory T (figure 6), the inequality $\frac{\mathrm{d}z}{\mathrm{d}\Theta} > 0$ holds for any $z \neq 0$. Recalling that $z = y'$, we see that the derivative $y'(\xi)$ is increasing in Θ for any $\xi \neq 0$, because $\xi = 0$ is the single stationary point of the solitary wave solution $y(\xi)$ (figure 7). This yields that derivatives of the inverses of the solution $\zeta^{-'}(y)$ and $\zeta^{+'}(y)$ are decreasing in Θ for any $y \neq 1$. Observing in addition the signs of these inverses we have the following relations:

$$\begin{aligned}&\zeta^{-'}(y) \text{ is positive and increases in } \Theta\\ &-\zeta^{+'}(y) \text{ is positive and decreases in } \Theta.\end{aligned} \tag{B.3}$$

Thus we can express the asymmetry at the level $y \in (0, 1)$ as follows:

$$\frac{|\zeta^+(y)|}{|\zeta^-(y)|} = \left|\frac{\int_1^y \zeta^{+'}(s)\,\mathrm{d}s}{\int_1^y \zeta^{-'}(s)\,\mathrm{d}s}\right| = \frac{\int_1^y [-\zeta^{+'}(s)]\,\mathrm{d}s}{\int_1^y \zeta^{-'}(s)\,\mathrm{d}s}.$$

By virtue of (B.3), the asymmetry is increasing in Θ. Let us denote the function, which assigns to any value of Θ from $(-1, 1)$ the asymmetry, by $F_y(\Theta)$. We have proved that $F_y(\Theta)$ is increasing. In the case $\Theta = 0$ the solution is symmetric (see (35)), hence $F_y(0) = 1$. The lemma is proved.

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