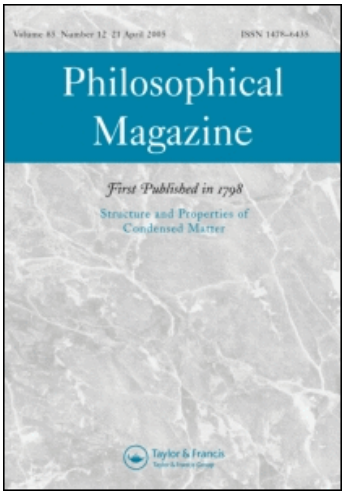

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Waves in microstructured materials and dispersion

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The dispersive effects due to the presence of microstructure in solids are studied. The basic mathematical model is derived following Mindlin's theory. In the one-dimensional case the governing equations of a linear system are presented. An approximation using the slaving principle indicates a hierarchy of waves. The corresponding dispersion relations are compared with each other. The choice between the models can be made on the basis of physical effects described by dispersion relations.

1. Introduction

The complex dynamic behaviour of microstructured materials at high speeds of deformation cannot be explained by the classical theory of continuous media. This concerns granular materials, polycrystalline solids, ceramic composites, functionally graded materials, alloys, damaged materials, etc. Such materials are characterized by the existence of intrinsic space-scales in matter, like the lattice period, the size of a crystallite or a grain, the distance between the microcracks, etc., that introduce some scale-dependence into the governing equations. Within the theories of continua the problems of irregularities of media were foreseen a long time ago by Cosserats and Voigt, and more recently by Mindlin [1] and Eringen [2], among others. Elegant mathematical theories of continua with voids, planar or ordinary spin, vector microstructure, Cosserat continua, micromorphic continua, etc. have since been developed; see the overviews in [3, 4]. Clearly every irregularity (or inclusion) creates an additional stress field around itself. Consequently the most general approach in modelling should be the presentation of all the conservation laws and constitutive equations taking such a stress field into account. This approach is described in detail by Maugin [5] using the concept of pseudomomentum and material inhomogeneity force.

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The theory should be complemented by experimental evidence. Experiments, however, can provide only indirect data on physical effects related to the behaviour of materials with microstructure. There seems to be a gap between the theoretical models and experimental studies in this field. The obvious reason for such a mismatch is that the macroscopic theories involve physical parameters (or even functions) that cannot be determined directly from the experiments. In addition, the influence of nonlinearities causes non-additivity of physical effects. Therefore the physical effects must be carefully studied. The list of structurally sensitive effects is long, including complicated dispersion, emergence of solitary waves, stress-induced phase transitions, kinetic localization of damage, attenuation characterized by asymmetric resonance curves, etc. The situation becomes even more intricate when dealing with ferromagnetic effects, physical–chemical reactions, molecular crystals and nanomaterials.

Of special interest are those phenomena which originate from a combination of several physical reasons. Dispersion and nonlinearity is such a pair of effects that could lead to the emergence of solitary waves due to the balance between them. The celebrated Korteweg–de Vries equation includes quadratic nonlinearity and cubic dispersion, and has served for more than 100 years as a model case for such a balance. In microstructured materials the situation can be much more complicated. Nonlinearity of the material may still be described by the approximation of the strain energy starting from the third powers, resulting in quadratic terms in the governing equation [6], but dispersive effects may have a much more complicated nature. Several mathematical models have been proposed for describing dispersive waves. One group of models is based on lattice theory [7, 8], another group on continuum theory [1–6]. A special case of periodic continua has also been thoroughly analysed [9–12]. Nevertheless, there still seem to be discrepancies between various models concerning the dispersion relations that should be clarified. This becomes especially important for applications of non-destructive evaluation (NDE) of material properties. In NDE the main tool involves phase velocity measurements. Therefore one should have a detailed understanding of dispersive effects and the physical parameters responsible for dispersion.

A crucial point in modelling is to choose the theory. The first decision should be made between discrete and continuum models. In the discrete approach the volume elements of the matter are treated as point masses with a proposed topological structure and some interaction between the discrete masses. This gives a good chance to model crystal lattices with certain symmetries, vacancies, impurities, defects, walls, etc. [7, 8, 13]. The governing equations are then deduced following Newton's law, and the key problem is the modelling of forces between the point masses. Starting from the Born–von Karman model for a one-dimensional atomic chain, such models have gained much attention; see [7, 8, 13] and references therein.

From the viewpoint of continua, the straightforward modelling of microstructured solids leads to assigning all the physical properties to every volume element dV in a solid, introducing the dependence on coordinates X^k. Then the governing equations include space-dependent parameters, and the most effective way to solve the governing equations is numerical integration. Another probably much more effective way is to separate macro- and microstructure in continua. Then the conservation laws for both structures should be separately formulated [1, 2, 4],

or the microstructural quantities are separately taken into account in one set of conservation laws [5]. Separating the macro- and microstructure gives two possibilities: either to consider both structures inertial or to suppose the microstructural quantities to behave non-inertially. The first case is exactly what has been done in [1, 2, 4]; the second case leads to the formalism of internal variables [14–16].

Both approaches – discrete and continuous – are closely related. One could, for example, replace the set of n differential equations with its continuum analogy using Taylor-series expansions. The most celebrated case is that of the Zabusky–Kruskal approximation of the Fermi–Pasta–Ulam problem, producing the Korteweg–de Vries equation for the description of waves influenced by quadratic nonlinearity and cubic dispersion.

One should also stress the importance of a special case of microstructure, namely periodic (or laminated) structures as they appear in composites. In this case, contrary to functionally graded materials, the periodic layering is important. One of the possibilities to account for such layers is to analyse the frequency spectrum of the time-harmonic Bloch expansion [9]. An important presentation in the form of a dispersive effective medium is given in [10]. This approach was recently generalized for linear [11] and nonlinear [17] cases. The dispersion relation is derived from the Bloch expansion, and then the equation of motion is restored, accounting for dispersive effects caused by periodicity. A model for spatiotemporal periodic (laminated) composites is derived in [12].

We focus here on the clarification of dispersive effects while nonlinearities are only briefly described. A more detailed analysis on consistent modelling of geometrical and physical nonlinearities with a clear distinction between spatial and material coordinates is in progress.

The paper is organized as follows. The basic model and its hierarchical approximation and simplifications are derived in section 2. Some specific models are described in section 3. In section 4 the dispersion relations are derived and the dispersive effects are analysed. A discussion is presented in section 5.

2. The basic model

Here we follow Mindlin [1] who has interpreted the microstructure 'as a molecule of a polymer, a crystallite of a polycrystal or a grain of a granular material'. This microelement is taken as a deformable cell. A rigid cell would lead to the Cosserat model. The displacement u of a material particle in terms of macrostructure is defined by its components $u_i \equiv x_i - X_i$, where x_i, X_i $(i = 1, 2, 3)$ are the components of the spatial and material position vectors, respectively. Within each material volume (particle) there is a microvolume, and the microdisplacement u' is defined by its components $u'_i \equiv x'_i - X'_i$, where the origin of the coordinates x'_i and X'_i moves with the displacement u. The displacement gradient is assumed to be small. This leads to the *basic assumption* of Mindlin [1], that 'the microdisplacement can be expressed as a sum of products of specified functions of x'_i and arbitrary functions of x_i and t'. The first approximation is then

$$u'_j = x'_k \varphi_{kj}(x_i, t). \tag{1}$$

The *microdeformation* is

$$\frac{\partial u_j'}{\partial x_i'} = \partial_i' u_j' = \varphi_{ij}. \tag{2}$$

Further we consider the simplest 1D case and drop the indices i, j dealing with u and φ only. The indices t and x occurring in subsequent formulas indicate differentiation.

The fundamental balance laws for microstructured materials can be formulated separately for macroscopic and microscopic scales [4]. The balance laws can easily be derived from the Lagrangian [1, 18]

$$L = K - W, \tag{3}$$

formed from the kinetic and potential energies

$$K = \frac{1}{2}\rho u_t^2 + \frac{1}{2} I \varphi_t^2, \qquad W = W(u_x, \varphi, \varphi_x), \tag{4}$$

where ρ and I denote the macroscopic density and the micro-inertia, respectively.

The corresponding Euler–Lagrange equations have the general form

$$\begin{aligned} \left(\frac{\partial L}{\partial u_t}\right)_t + \left(\frac{\partial L}{\partial u_x}\right)_x - \frac{\partial L}{\partial u} = 0, \\ \left(\frac{\partial L}{\partial \varphi_t}\right)_t + \left(\frac{\partial L}{\partial \varphi_x}\right)_x - \frac{\partial L}{\partial \varphi} = 0. \end{aligned} \tag{5}$$

Inserting the partial derivatives

$$\begin{aligned} &\frac{\partial L}{\partial u_t} = \rho u_t, \quad \frac{\partial L}{\partial u_x} = -\frac{\partial W}{\partial u_x}, \quad \frac{\partial L}{\partial u} = 0, \\ &\frac{\partial L}{\partial \varphi_t} = I \varphi_t, \quad \frac{\partial L}{\partial \varphi_x} = -\frac{\partial W}{\partial \varphi_x}, \quad \frac{\partial L}{\partial \varphi} = -\frac{\partial W}{\partial \varphi}, \end{aligned} \tag{6}$$

into equations (5) we obtain the equations of motion

$$\rho u_{tt} - \left(\frac{\partial W}{\partial u_x}\right)_x = 0, \quad I \varphi_{tt} - \left(\frac{\partial W}{\partial \varphi_x}\right)_x + \frac{\partial W}{\partial \varphi} = 0. \tag{7}$$

The partial derivatives

$$\sigma = \frac{\partial W}{\partial u_x}, \quad \eta = \frac{\partial W}{\partial \varphi_x}, \quad \tau = \frac{\partial W}{\partial \varphi} \tag{8}$$

are recognized as the macrostress, the microstress and the interactive force, respectively.

The equations of motion (7) now take the form

$$\rho u_{tt} = \sigma_x, \quad I \varphi_{tt} = \eta_x - \tau. \tag{9}$$

These equations can be compared with the analogous equations deduced in a different way in [3].

The simplest potential energy function describing the influence of a microstructure is a quadratic function

$$W = \frac{1}{2}\alpha u_x^2 + A\varphi u_x + \frac{1}{2}B\varphi^2 + \frac{1}{2}C\varphi_x^2 \tag{10}$$

with α, A, B, C denoting material constants. Inserting (10) into (8) and the result into equations (7), the governing equations take the form

$$\begin{aligned} \rho u_{tt} &= \alpha u_{xx} + A\varphi_x, \\ I\varphi_{tt} &= C\varphi_{xx} - Au_x - B\varphi. \end{aligned} \tag{11}$$

This governing system of two second-order equations can also be represented in the form of one fourth-order equation. To this end, equation $(11)_2$ is differentiated once with respect to x, such that the variable φ occurs only in the form of φ_x and derivatives thereof. Now the partial derivative φ_x is determined from $(11)_1$ and can be inserted. The resulting equation has the form

$$u_{tt} = \left(c_0^2 - c_A^2\right)u_{xx} - p^2\left(u_{tt} - c_0^2 u_{xx}\right)_{tt} + p^2 c_1^2\left(u_{tt} - c_0^2 u_{xx}\right)_{xx}, \tag{12}$$

where new material parameters

$$c_0^2 = \frac{\alpha}{\rho}, \quad c_1^2 = \frac{C}{I}, \quad c_A^2 = \frac{A^2}{\rho B}, \quad p^2 = \frac{I}{B} \tag{13}$$

have been introduced. The constants c_0, c_1, c_A are velocities, while p is a time constant. Within equation (12) the wave operator $L_0 = u_{tt} - c_0^2 u_{xx}$ related to the bulk medium without microstructure can be recognized.

An approximation of equation (12) can be obtained if the inherent length-scale l of the microstructure is assumed to be small compared with the wavelength L of the excitation. In addition, the scale of the displacement is characterized by its amplitude U_0. Then we can introduce the following dimensionless variables and parameters:

$$U = \frac{u}{U_0}, \quad X = \frac{x}{L}, \quad T = \frac{c_0 t}{L}, \quad \delta = \left(\frac{l}{L}\right)^2, \quad \varepsilon = \frac{U_0}{L}. \tag{14}$$

We also suppose that $I = \rho l^2 I^*$ and $C = l^2 C^*$, where I^* is dimensionless and C^* has the dimension of stress. The parameter I^* can be interpreted as the ratio of the inherent density of the microstructure and the macroscopic density ρ.

Next, the system (11) is rewritten in its dimensionless form and then the slaving principle [19, 20] is applied. This means that φ is determined in terms of U_X using a series expansion

$$\varphi = \varphi_0 + \delta\varphi_1 + \delta^2\varphi_2 + \cdots. \tag{15}$$

The dimensionless form of equation $(11)_2$ yields

$$\varphi = -\varepsilon\frac{A}{B}U_X - \frac{\delta}{B}(\alpha I^*\varphi_{TT} - C^*\varphi_{XX}), \tag{16}$$

from which the successive terms

$$\varphi_0 = -\varepsilon \frac{A}{B} U_X, \quad \varphi_1 = \varepsilon \frac{A}{B^2} (\alpha I^* U_{XTT} - C^* U_{XXX}), \quad \ldots \tag{17}$$

of the expansion (15) are obtained. Inserting them into the governing equation $(11)_1$ in its dimensionless form, we finally get the single differential equation for U, cf. [21]

$$U_{TT} = \left(1 - \frac{c_A^2}{c_0^2}\right) U_{XX} + \frac{c_A^2}{c_B^2} \left(U_{TT} - \frac{c_1^2}{c_0^2} U_{XX}\right)_{XX}, \tag{18}$$

with $c_B^2 = L^2/p^2 = BL^2/I$. Note that c_B involves the scales L and l, and c_A includes the interaction effects through the parameter A.

Having obtained a clear idea about the scales and developed the corresponding approximation, we can restore the dimensions in order to compare the result with equation (12). Equation (18) yields

$$u_{tt} = \left(c_0^2 - c_A^2\right) u_{xx} + p^2 c_A^2 \left(u_{tt} - c_1^2 u_{xx}\right)_{xx}. \tag{19}$$

In this approximation the u_{tttt} term is absent while the coefficients of u_{ttxx} are different in (12) and (19).

One could also think about further simplifications of (19) neglecting one or another of the higher derivatives. Without further justification we present the two alternate simplifications

$$u_{tt} = \left(c_0^2 - c_A^2\right) u_{xx} + p^2 c_A^2 u_{ttxx}, \tag{20}$$

$$u_{tt} = \left(c_0^2 - c_A^2\right) u_{xx} - p^2 c_A^2 c_1^2 u_{xxxx}. \tag{21}$$

Our analysis of the dispersion curves in section 4 shows that, unlike the approximation (19), these further simplifications are not acceptable.

3. Specific models

3.1. *Lattice theory*

The basic ideas of propagating waves in lattices have been analysed in [7]. The interest in models has been increased due to the importance of smaller scales in technology; see [8, 13] and references therein. The simplest 1D model is that of Born and von Karman which describes the longitudinal wave in an infinite elastic chain of particles of mass m, placed at a uniform distance a in equilibrium and linked by identical springs of stiffness k. The longitudinal motion in such a chain is described by the infinite set of equations [7, 13]

$$m \frac{d^2 u_n}{dt^2} = k(\xi_{n+1} - \xi_n), \quad \xi_n = u_n - u_{n-1}, \tag{22}$$

where u_n denotes the displacement of the nth particle. The continuum limit is obtained by introducing a displacement field $u = u(x, t)$ and replacing the differences

in (22) by spatial derivatives. This leads to the standard wave equation

$$u_{tt} - c_0^2 u_{xx} = 0, \quad c_0^2 = \frac{ka^2}{m}. \tag{23}$$

Two generalizations are possible, leading to nonlinearity and to dispersion. The equations $(21)_1$ can be replaced by equations of the more general type

$$\frac{d^2 u_n}{dt^2} = -\frac{\partial \phi}{\partial u_n}, \tag{24}$$

where $\phi = \phi(\xi_n, \xi_{n+1})$ is the potential of interactive forces. This leads to nonlinear equations, in general. On the other hand, the passage to the continuum limit can employ a Taylor expansion including higher-order derivatives. If this is applied to the linear model (22) the dispersive wave equation

$$u_{tt} - c_0^2 u_{xx} - \frac{1}{12} c_0^2 a^2 u_{xxxx} = 0 \tag{25}$$

is obtained. It has been shown [13] that even higher-order derivatives can be taken into account.

3.2. *Periodic structures*

Periodic microstructure is quite typical for composite and functionally graded materials. A suitable model may be constructed by homogenization of periodic media that approximates well the dispersive nature of waves [10]. Homogenization is possible when the shortest wavelength of the initial excitation is several times larger than the characteristic length of the microstructure and the time-scale is large. Santosa and Symes [10] have used the Bloch wave expansion for that, i.e. the solution of an eigenvalue problem is used. Considering a medium with cell size $2\pi p$ it is assumed that $\epsilon = p/\lambda \ll 1$, where λ is the lower limit of the Fourier components of the initial disturbance. Then the dispersion relation

$$\omega(k) = \Omega_1 k + \frac{1}{6}\epsilon^2 \Omega_3 k^3 + \cdots \tag{26}$$

may be constructed, where ω and k are the frequency and the wavenumber, respectively. For a two-component structure with densities ρ_1, ρ_2 and moduli μ_1, μ_2, one has the velocities $c_1^2 = \mu_1/\rho_1$, $c_2 = \mu_2/\rho_2$ and the impedances $z_1 = (\rho_1\mu_1)^{1/2}$, $z_2 = (\rho_2\mu_2)^{1/2}$. Then the coefficients of the expansion (26) become

$$\Omega_1^2 = \frac{(\theta/\mu_1 + (1-\theta)/\mu_2)^{-1}}{\theta\rho_1 + (1-\theta)\rho_2}, \quad \Omega_3 = \Omega_3(z_i, c_i, \theta, \Omega_1), \tag{27}$$

where θ denotes the volume fraction of the mixture. The wave equation corresponding to (26) is

$$u_{tt} = \Omega_1^2 u_{xx} + \frac{1}{3}\Omega_1 \Omega_3 \varepsilon^2 u_{xxxx}. \tag{28}$$

Santosa and Symes [10] have shown that in their case $\Omega_1 < c_1$ and $\Omega_3 < 0$.

It has been shown [12] that in periodic media with moving boundaries of cells (laminates in general), the equation of motion, contrary to equation (28), also includes terms like u_{xt} and u_{xxxt}.

3.3. *Periodic structures with micro-inertia*

A special case of periodic structures is considered by Wang and Sun [22], who have used a continuum model for macromotion also involving micro-inertia terms in the energy densities. Assuming the linear approximation of the microdisplacement field, the macrodisplacement equation of motion yields

$$\rho u_{tt} = L u_{xx} + H u_{xxtt}, \tag{29}$$

where

$$\begin{aligned} \rho &= \rho_1 f_1 + \rho_2 f_2, \quad L = \frac{\mu_1 \delta_1 \delta_2}{\delta_2 f_1 + \gamma \delta_1 f_2}, \\ H &= \frac{1}{6} \frac{f_1 f_2 (\gamma \delta_1 - \delta_2)(p f_1^2 - f_2^2)}{(\delta_2 f_1 + \gamma \delta_1 - f_2)(p f_1 - f_2)} (d_1 + d_2)^2 \rho_2. \end{aligned} \tag{30}$$

The periodic structure consists of alternating layers of widths d_1, d_2 with densities ρ_1, ρ_2, elastic moduli μ_1, μ_2 and Poisson's ratios ν_1, ν_2. Further, the ratios $\gamma = \mu_1/\mu_2$, $p = \rho_1/\rho_2$ and, for $i = 1, 2$, the abbreviations $\delta_i = 2(1 - \nu_i)/(1 - 2\nu_i)$, $f_i = d_i/(d_1 - d_2)$ have been introduced.

4. Dispersion

We assume the solution of the governing equations in the form

$$u(x, t) = \hat{u} e^{i(kx - \omega t)} \tag{31}$$

with wavenumber k and frequency ω. Introducing this to equation (12), the dispersion relation

$$\omega^2 = \left(c_0^2 - c_A^2\right) k^2 + p^2 \left(\omega^2 - c_0^2 k^2\right)\left(\omega^2 - c_1^2 k^2\right) = 0 \tag{32}$$

is obtained. The parameters involved are the time constant p and three characteristic velocities c_0, c_A, c_1. Instead of c_A the velocity $c_R = (c_0^2 - c_A^2)^{1/2}$ could be introduced as a parameter, since it has an obvious meaning for the wave process. Waves of very low frequencies ($\omega \ll p^{-1}$) are propagated at the velocity c_R. The velocity c_A does not occur explicitly as a limit velocity.

In order to reduce the number of independent variables we introduce dimensionless quantities

$$\xi = p c_0 k, \quad \eta = p\omega, \tag{33}$$

and dimensionless parameters

$$\gamma_1 = c_1/c_0, \quad \gamma_A = c_A/c_0. \tag{34}$$

Using these new quantities the full dispersion relation (32) assumes the form

$$\eta^2 = (1 - \gamma_A^2)\xi^2 + (\eta^2 - \xi^2)(\eta^2 - \gamma_1^2\xi^2). \tag{35}$$

In the same way, the approximate differential equation (19) yields the dimensionless dispersion relation

$$\eta^2 = (1 - \gamma_A^2)\xi^2 - \gamma_A^2(\eta^2 - \gamma_1^2\xi^2)\xi^2. \tag{36}$$

Eventually the simplified differential equations (20) and (21) yield

$$\eta^2 = (1 - \gamma_A^2)\xi^2 - \gamma_A^2\eta^2\xi^2, \tag{37}$$

$$\eta^2 = (1 - \gamma_A)\xi^2 + \gamma_A^2\gamma_1^2\xi^4, \tag{38}$$

respectively.

The behaviour of the dispersion curves can be described as follows. The full dispersion relation (35) represents two branches which, in general, are distinct (see figure 1). Only in the exceptional case $\gamma_A = 0$, $\gamma < 1$ will the branches meet in one point. The upper, or 'optical', branch starts at $\eta = 1$ with zero slope, while the lower, or 'acoustical', branch starts at the origin with slope $\gamma_R = c_R/c_0 = (1 - c_A)^{1/2}$.

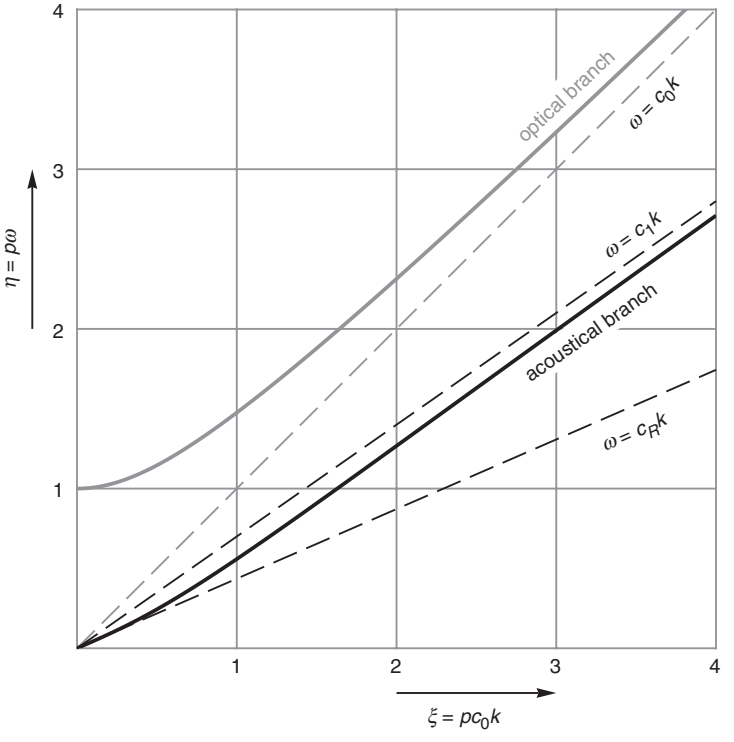

Figure 1. Dispersion curves for $\gamma_1 = 0.7$ and $\gamma_A = 0.9$.

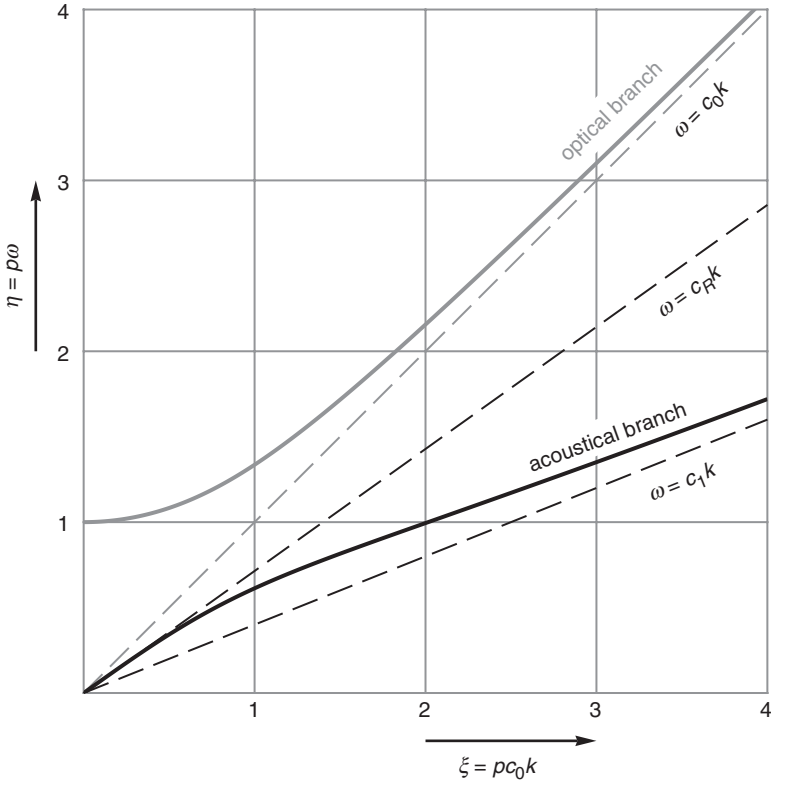

Figure 2. Dispersion curves for $\gamma_1 = 0.7$ and $\gamma_A = 0.9$.

In the short-wave limit, $\xi \gg 1$, the branches asymptotically approach the lines $\eta = \xi$ and $\eta = \gamma_1 \xi$.

Since the free energy (10) should be positive definite, we always have $\gamma_A > 0$. There is, however, no physical restriction on the magnitude of γ_1. Figure 1 shows an example where $\gamma_R < \gamma_1 < 1$. The qualitative behaviour of the acoustical branch becomes different, if $\gamma_1 < \gamma_R < 1$, as seen from figure 2. For $\gamma_1 > 0$, which means that $c_1 > c_0$, the role of the asymptotic lines $\eta = \xi$ and $\eta = \gamma_1 \xi$ is exchanged so that the upper branch always tends to the higher slope, whichever it is.

The dispersion relation (19) provides an approximation for the acoustical branch only. The curve starts at $\xi = 0$ with the slope γ_R and, for $\xi \to \infty$, tends asymptotically to the line $\eta = \gamma_1 \xi$ provided $\gamma_A > 0$. As seen from figure 3, the full and approximate dispersion relations agree pretty well. The special feature of this approximation is that it can be used over the whole range of wavenumbers, since it does not represent a short-wave or long-wave approximation. The underlying assumption is that the influence of the microstructure is small. The approximation becomes worse if the parameter γ_A tends to zero and, for $\gamma_A = 0$, degenerates to the non-dispersive wave represented by $\eta = \xi$.

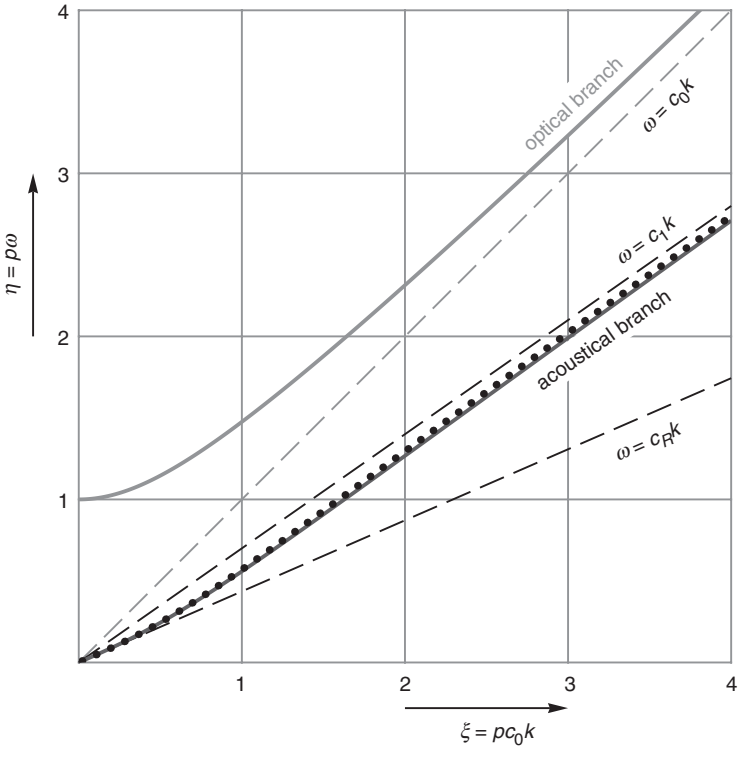

Figure 3. Dispersion curves for $\gamma_1 = 0.7$ and $\gamma_A = 0.9$: (——) full dispersion relation (35); (• • • • • •) approximate dispersion relation (36).

The simplified cases (37) and (38) give rather distorted results. The dispersion curves deviate strongly from the correct course (see figure 4).

5. Discussion

The wave speed is clearly influenced by the microstructure. This is described by all the models except the straightforward lattice theory (cf. (23)). The dispersion is affected by the inertia of the microstructure and the wave velocity in the pure microstructure. Various models described in sections 3 and 4 have different accuracy. The full model (12) and its approximation (19), which retains the character of double dispersion (terms u_{ttxx} and u_{xxxx}), agree in the description of the lower branch of the dispersion curve pretty well (see figure 3). The simplified models that concentrate only on one effect could be used only with special attention to wave and material parameters, because in the long run the dispersive properties are not satisfactorily described.

What follows clearly from the approximate differential equation (19) is the hierarchical nature [23] of the wave process in microstructured materials. Indeed, if the wavelength is large (small wavenumbers), then the wave velocity is dictated by *macrostructural* properties (cf. figure 3), and if the wavelength is small (larger

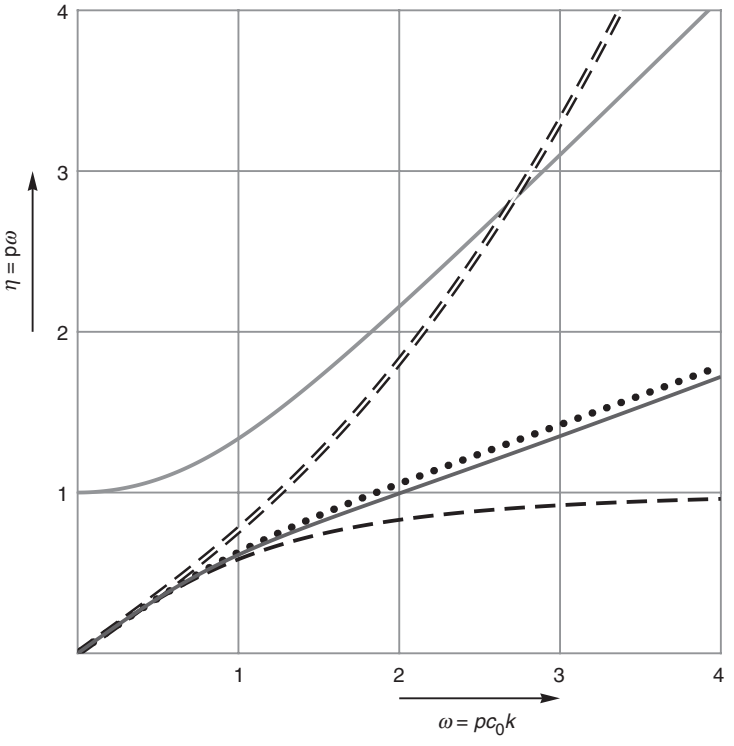

Figure 4. Dispersion curves for $\gamma_1 = 0.4$ and $\gamma_A = 0.7$: (——) full dispersion relation (35); (• • • • • •) approximate dispersion relation (36); (– – –) simplified dispersion relation (37); (= = =) simplified dispersion relation (38).

wavenumbers) then *microstructural* properties are important. This means that either the operator $L_1 = u_{tt} - (c_0^2 - c_A^2)u_{xx}$ or the operator $L_2 = u_{tt} - c_1^2 u_{xx}$ prevails, respectively. Such hierarchical systems have been analysed earlier [21, 24].

Numerical simulation supports the analysis. The finite volume method [25] is used, which permits us to assign all the physical parameters to every volume element in a material. As an example, the wave propagation in metal–ceramic composites is considered. The elastic properties of the metal matrix and ceramic reinforcement are the following [26]: Young's moduli 70 GPa and 420 GPa, Poisson's ratios 0.3 and 0.17, and densities 2800 kg m^{-3} and 3100 kg m^{-3}, respectively. A 2D specimen was dynamically loaded at the left boundary of the specimen (see figure 5) between 40 and 160 space steps.

The stress initiated at $x = 0$ was $\sigma(0, t) = \sigma_0 \sin^2 \pi(t - 2t_r)/t_r$ with $\sigma_0 = 125$ MPa and $t_r = 10$. The initial wavelength corresponded to 20 space steps. In figure 5 the wavefronts in pure ceramics (figure 5a) and in pure metal (figure 5b) are shown. In case of metal–matrix composites the volume fraction $f = V_c/V$ characterizes the material. Here V_c is the volume of ceramic particles and V is the total volume. In figure 6 the wavefronts in such composites are shown for various volume fractions. The wave speed clearly depends on the volume fraction, i. e. on the density distribution.

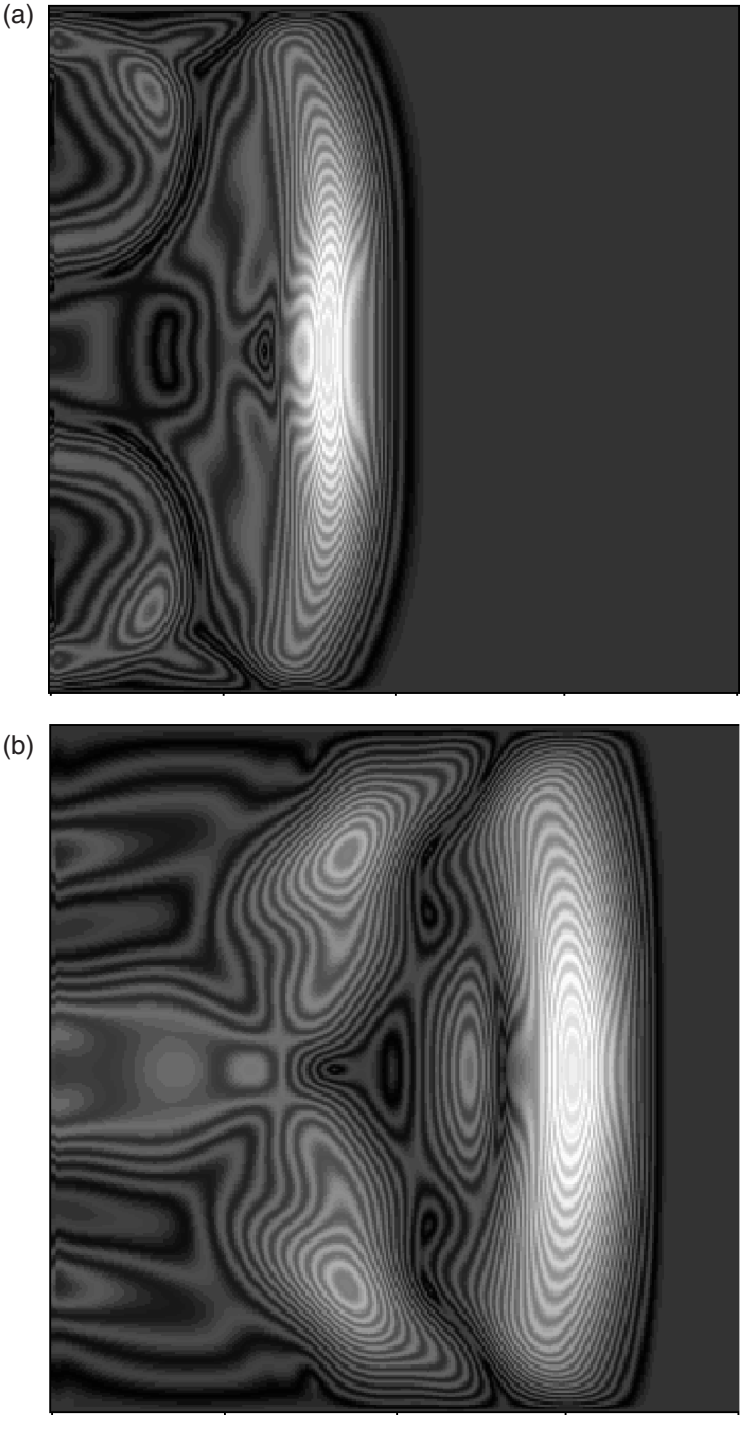

Figure 5. Wavefronts at $t=180$: (a) pure ceramics; (b) pure metal.

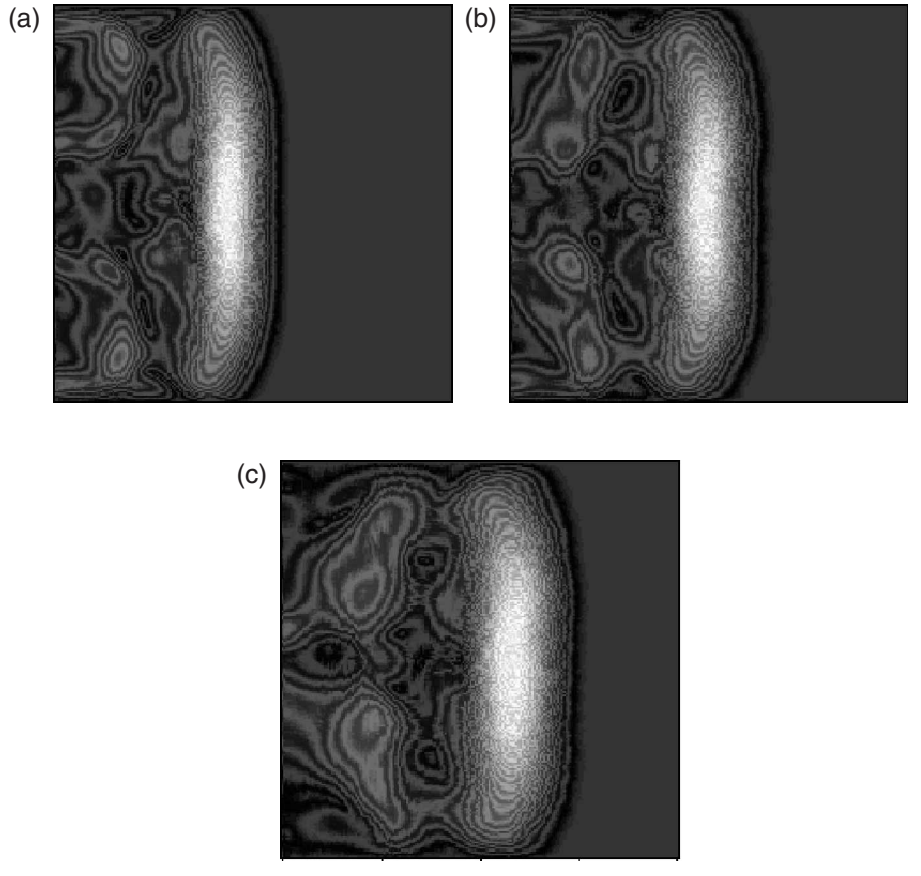

Figure 6. Wavefronts at $t=180$ for metal–metal composite: (a) $f=0.75$, (b) $f=0.50$, (c) $f=0.25$.

The free energy function (10) includes quadratic terms only and therefore leads to the linear dispersive model that permits to analyse the accuracy of dispersion relations. If the free energy function also includes cubic terms then a physically nonlinear theory can be constructed. In order to also include geometrical nonlinearity, the deformation tensor should be taken in its full form. The crucial point is whether to include nonlinearities only in the macroscale or only in the microscale or both.

A preliminary analysis has shown that the full governing system (11) in the first approximation towards nonlinear theory will then be enlarged to involve terms $u_x u_{xx}$ and $\varphi_x \varphi_{xx}$. If the approximated equation (19) in terms of macrodisplacement u only is used, then terms like $u_x u_{xx}$ and $(u_{xx}^2)_{xx}$ appear (cf. [21]). Studies on a possible balance between dispersive and nonlinear effects are in progress.

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