

## 4. MATEMAATILISED MUDELID

algebralised  
transsendentsed

funktsioonid  
võrrandid  
võrrandisüsteemid

diferentsiaalvõrrandid  
harilikud (täistuletistega)  
osatuletistega

integraalvõrrandid

integrodiferentsiaalvõrrandid  
mälu (ajaloo) arvestamine

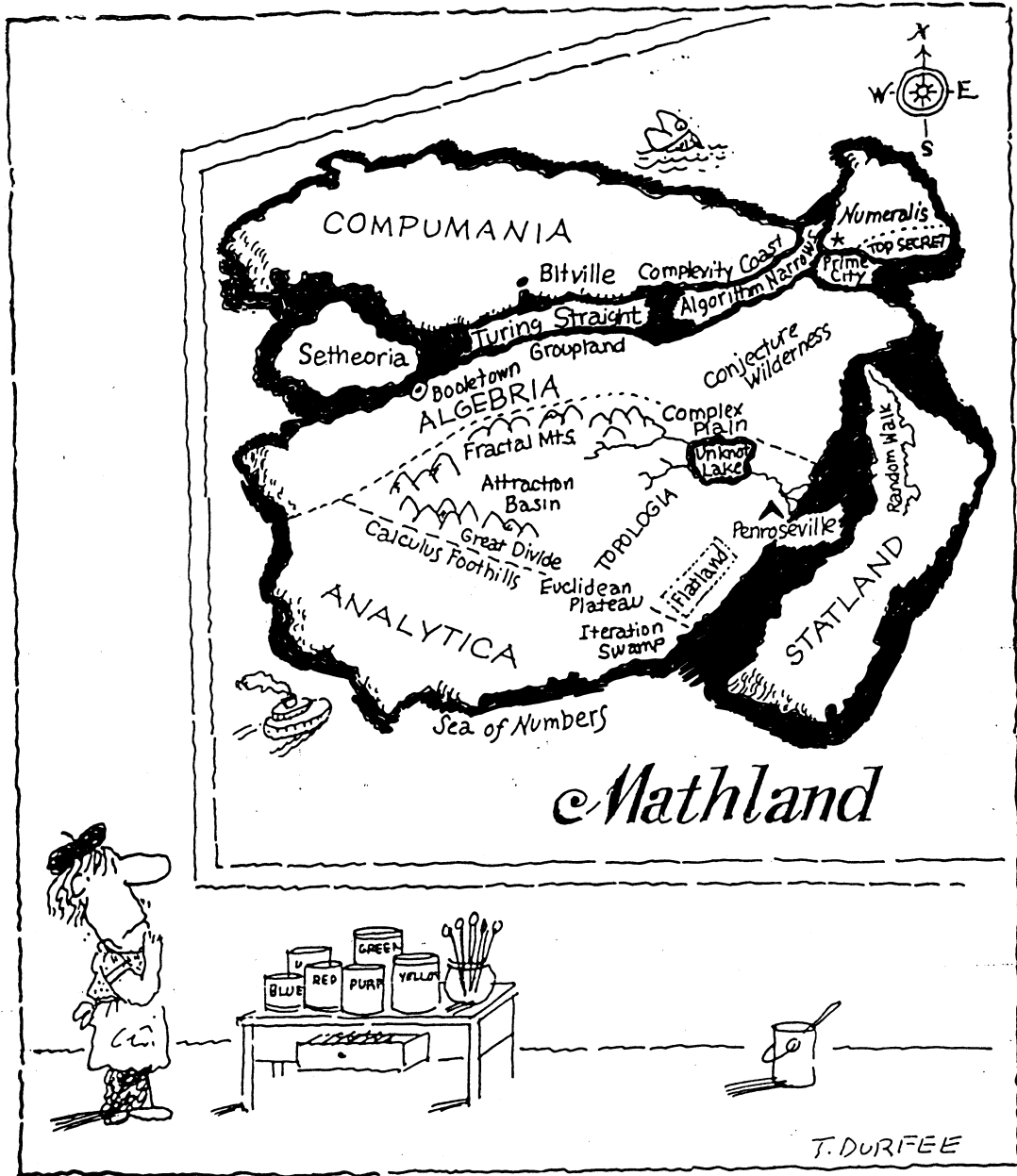
diferentsvõrrandid (kujutised)

### Meetodid

analüütilised  
lähendusmeetodid

numbrilised

lihtsustusmeetodid  
agregatsioon



*Mathland*

T. DURFEE

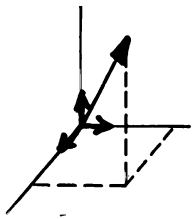
# Põhimõisted

\* skalaar

**a**

temperatuur, aeg

\* vektor

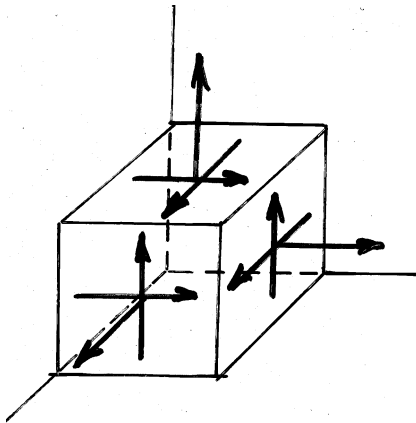


$$\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$$

**i, j, k** ühikvektorid

kiirus, kiirendus

\* tensor



$a_{ij}$ ,  $i, j = 1, 2, 3$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

pinge, deformatsioon

## Algebralised võrrandid ja süsteemid

$$x^n + a_1 x^{n-1} + \dots + a_n = 0$$

...

$$\left\{ \begin{array}{l} a_{11} X_1 + a_{12} X_2 + \dots + a_{1n} X_n = b_1 \\ a_{21} X_1 + a_{22} X_2 + \dots + a_{2n} X_n = b_2 \\ \dots\dots\dots \dots \dots \dots \dots \dots \dots \dots \dots \\ a_{n1} X_1 + a_{n2} X_2 + \dots + a_{nm} X_n = b_n \end{array} \right.$$

$$A \cdot x = B$$

A:  $n \times n$  matriks

x, B - n - vektor

Üle- ja alamääratud süsteemid

Mittelineaarsed võrrandisüsteemid

# Spetsiaalfunktsioonid

## Diraci deltafunktsioon

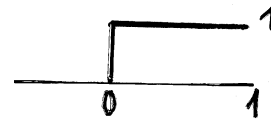
$\delta(x)$  – kirjeldab ühte punkti kontsentreeritud suurust (jõud, laeng, intensiivsus ...)

$$\int_{-\infty}^{\infty} \delta(x) dx = 1$$

diferentseeruv

## Heaviside'i funktsioon

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

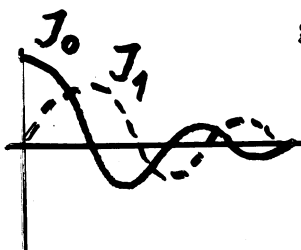


## Besseli funktsioon (id)

$$x^2 y'' + xy' + (x^2 - n^2)y = 0$$

$$y = J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{n+2k}}{k! \Gamma(n+k+1)}$$

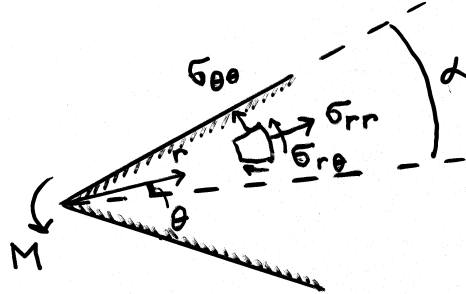
$$\Gamma(x) = \lim_{n \rightarrow \infty} \frac{n! n^{x-1}}{x(x+1)(x+2)\dots(x+n-1)}$$



gammafunktsioon

## Funktsioonid

Näide: elastne kiil, pinged



$$\sigma_{rr} = \frac{2M \sin 2\theta}{(\sin 2\alpha - 2\alpha \cos 2\alpha) r^2}$$

$$\sigma_{r\theta} = \frac{M(\cos 2\alpha - \cos 2\theta)}{(\sin 2\alpha - 2\alpha \cos 2\alpha) r^2}$$

$$\sigma_{\theta\theta} = 0$$

**Näide:** diferentsiaalvõrrandi ilmutamata lahend

$$\dot{x} = kx(x - X_1)(X_2 - x)$$

$$x^\alpha (x - X_1)^\beta (X_2 - x)^{-\gamma} = Ae^{kt}$$

$$A, \alpha, \beta, \gamma \quad - \quad \text{const. (sõltuvad } k, X_1, X_2)$$

faasianalüüs:

$$\dot{X}=0$$

$$kx(x - X_1)(X_2 - x)=0$$

$$\left. \begin{array}{l} x = 0 \\ x = X_1 \\ x = X_2 \end{array} \right\} \text{püsipunkti d}$$

## Harilikud diferentsiaalvõrrandid

1. järku  $\dot{x}(t) = X(t, x(t))$   
näide:  $\dot{x} = x - t$

kõrgemat järku

$$\ddot{x}(t) + a_1 \dot{x}(t) + a_2(x(t), t) = 0$$

süsteemid

$$\dot{x} = X(x, t) \quad X - n - \text{vektor}$$

## Volterra–Lotke

$$\begin{cases} \dot{x} = K_1 A x - K_2 x y \\ \dot{y} = K_2 x y - K_3 B y \end{cases}$$

$x$  – saak

$y$  – röövloom

$K_1, K_2, K_3, A, B$  – const

D.K. Arrowsmith, C.M. Place.  
Ordinary Differential Equations.  
Chapman, London, 1982.

P.G. Drazin  
Nonlinear Systems.  
Cambridge Univ. Press, 1992



## Diferentsvõrrandid, kujutised

Difference equations, maps (iterated maps)

$$x_{n+1} = F(x_n, n), \quad n = 0, 1, 2, \dots$$

üldkuju

Kujutis võib olla seotud diferentsiaalvõrrandiga  
näiteks

$$\frac{dx}{dt} = ax - bx^2$$

Valime  $h$ , defineerime  $x_n = x(nh)$ ,  $n = 0, 1, \dots$

$$\frac{dx}{dt} \sim \frac{x_{n+1} - x_n}{h} \quad t = nh \quad \text{jaoks}$$

Siis

$$\frac{x_{n+1} - x_n}{h} = ax_n - bx_n^2$$

ehk

$$x_{n+1} = F(a, b, h, x_n) \quad \text{ligikaudselt}$$

mitmemõõtelised, näiteks

$$\begin{cases} x_{n+1} = y_n \\ y_{n+1} = \mu y_n (1 - x_n) \end{cases} \quad \text{Neimark'i kujutis}$$

$$\begin{cases} x_{n+1} = 1 - ax_n^2 + y_n \\ y_{n+1} = bx_n \end{cases} \quad \text{Hénoni kujutis}$$

## 2. järku osatuletistega diff. võrrandid

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + F(x,y,u,u_x,u_y) = 0$$

↓ täpsed teisendused

**1. hüperboolsed**  $x, y \rightarrow x, t$

$$u_{tt} = a^2 u_{xx}$$

ääretingimused  $u(0,t) = 0, \quad u(1,t) = 0$

algtingimused  $u(x,t_0) = \varphi(x), \quad u_t(x,t_0) = \psi(x)$

**2. paraboolsed**  $x, y \rightarrow x, t$

$$u_{xx} - ku_t = 0$$

$$u(x,0) = \varphi(x) \quad 0 \leq x \leq 1$$

$$\left. \begin{array}{l} u(0,t) = \mu_1(t) \\ u(1,t) = \mu_2(t) \end{array} \right\} \quad 0 \leq t \leq t_1$$

**3. elliptilised**

$$u_{xx} + u_{yy} = 0$$

$$\text{ehk } \Delta u = 0$$

$$-u_1 = f_1 \quad \text{pinnal (joonel) } \Sigma$$

$$-\frac{\partial u}{\partial n} = f_2 \quad \text{pinnal (joonel) } \Sigma$$

## Elastusteooria 3D

$$(\lambda + 2\mu)u_{1,11} + (\lambda + \mu)(u_{2,21} + u_{3,31}) + \mu(u_{1,22} + u_{1,33}) - \rho_0 \frac{\partial^2 u_1}{\partial t^2} = 0$$

$$(\lambda + 2\mu)u_{2,22} + (\lambda + \mu)(u_{3,12} + u_{3,32}) + \mu(u_{2,11} + u_{2,33}) - \rho_0 \frac{\partial^2 u_2}{\partial t^2} = 0$$

$$(\lambda + 2\mu)u_{3,33} + (\lambda + \mu)(u_{1,13} + u_{2,23}) + \mu(u_{3,11} + u_{3,22}) - \rho_0 \frac{\partial^2 u_3}{\partial t^2} = 0$$

$u_1$     $u_2$     $u_3$    – siirde komponendid

$x_1$     $x_2$     $x_3$    – koordinaadid

## Navier- Stokes'i võrrandid hüdrodünaamikas, 3D

$$\rho \left[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right] = - \frac{\partial p}{\partial x} + \mu \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right] + F_x$$
$$\rho \left[ \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right] = - \frac{\partial p}{\partial x} + \mu \left[ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right] + F_z$$
$$\rho \left[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right] = - \frac{\partial p}{\partial x} + \mu \left[ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right] + F_y$$

$u$ ,    $v$ ,    $w$    – kiiruse komponendid

$x$ ,    $y$ ,    $z$    – koordinaadid

$p$    – surve

$\mu$    – viskoosus

$F_x$ ,    $F_y$ ,    $F_z$    – välisjõud

$$\rho \left[ \frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} \right] = -\nabla p + \mu \nabla^2 \mathbf{U} + \mathbf{F}$$

## Hetkväärtused ja mälu

$$G(t) = A \cdot F(t), \quad A = \text{const.}$$

hetkväärtuste vaheline seos

$$G(t) = A \cdot F(t) + \int_0^t R(t - \tau) F(\tau) d\tau$$

võtab arvesse ajal  $\tau$  ka eelmisi väärtusi intervallis  $[0, t]$

$$G(t) = A \cdot F(t) + R(t) * F(t)$$

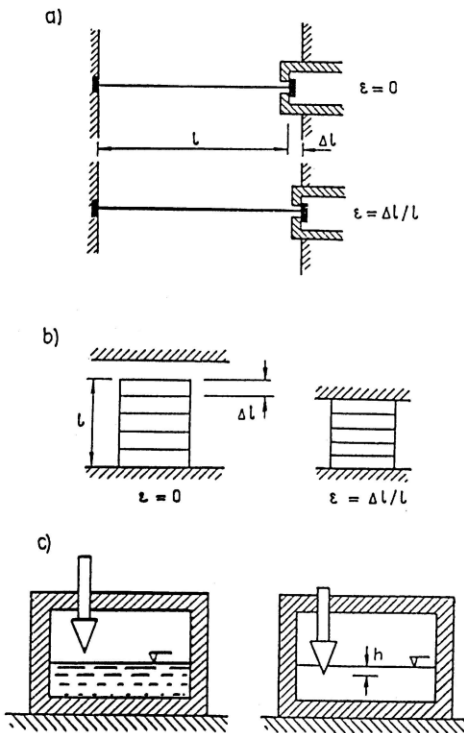
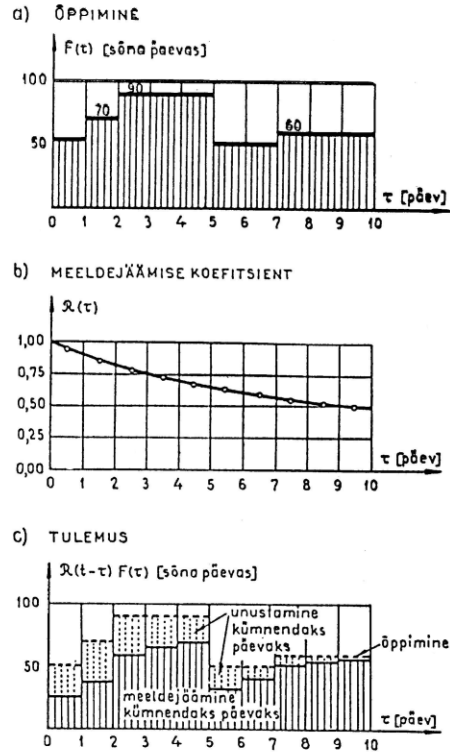
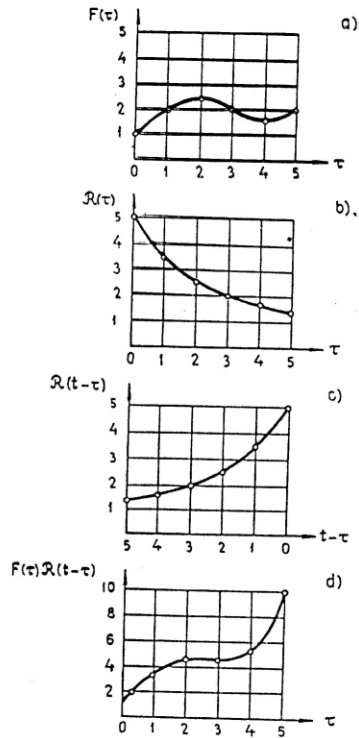
sidumiintegral

Volterra integralvõrrand

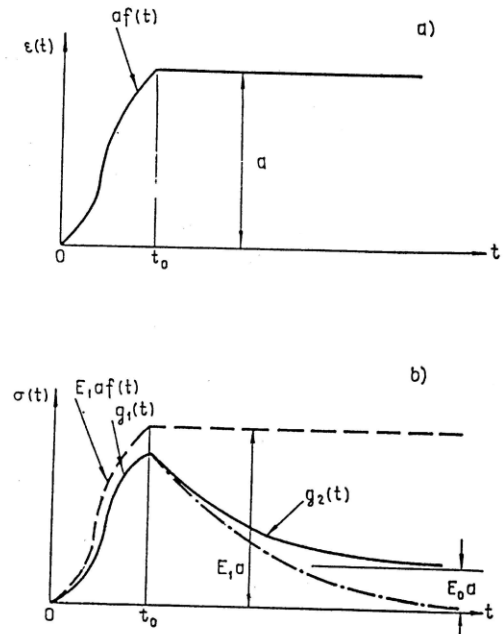
reoloogia, nn. relaksatsioonieffekt

bioloogilised protsessid viiteajaga

# Näide: mälu



Joonis 6. Relaksatsioonikatte skeeme : a - keele või trossi tõmbedeformatsioon; b - prisma või silindri surve-deformatsioon; c - koonuse süvistamine mörtil



Joon. 7. Standardse viskoelastse keha relaksatsioon: a - deformatsioonidiagramm; b - pingediagramm (täisjoon - standardne viskoelastne keha, punktirjoon - elastne keha elastsusmooduliga  $E_1$ , punkt-kriipsjoon - viskoelastne keha, mille puhul  $E_0=0$ ).

## Integro–differentiaalfõrrandid

$$\rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} = \mathbf{G}(0) \frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \int_0^\infty \mathbf{G}'(s) \frac{\partial^2}{\partial \mathbf{x}^2} \mathbf{u}(\mathbf{x}, t - s) ds$$

$\mathbf{G}(\cdot)$  – relaksatsioonifunktsioon

$$\mathbf{G}'(\cdot) = \frac{d}{ds} \mathbf{G}(\cdot)$$

$\rho_0$  – tihedus,  $\mathbf{u}$  – siire

## Olulised matemaatilised mudelid

lainevõrrand 
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Schrödingeri võrrand 
$$i \hbar \frac{\partial}{\partial t} \psi = \hat{H} \psi$$

Fourier teisendus 
$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) \exp(-2\pi i x \xi) dx$$

Maxwelli võrrandid 
$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{H} = 0$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}$$

**Lisa**

## **TYPES OF EVOLUTION EQUATIONS**

$$u_t + uu_x + du_{xxx} = 0$$

### **Korteweg-de Vries**

- surface gravity waves in shallow water
- internal waves
- acoustic waves in bubbly liquids
- voidage slugs in fluidized beds
- magma flows
- hydromagnetic plasma waves
- ion-acoustic waves in cold plasma
- waves in transmission lines

$$u_t + u^p u_x + du_{xxx} = 0$$

### **modified Korteweg-de Vries**



$$u_t + 6uu_x + u_{xxx} + \alpha c_1 u^2 u_x + \alpha c_2 u_x u_{xx} + \\ + \alpha c_3 uu_{xxx} + \alpha c_4 u_{xxxxx} = 0$$

- surface waves: progression from shallow-water waves to deep-water waves
- internal waves in density-stratified fluids
- surface waves subject to a shear flow

$$u_t - uu_x + u^3 u_x + du_{xxx} + bu_{xxxxx} = 0$$

- waves in microstructured alloys (austenitic - martensitic type)

$$u_t + uu_x + bu_{xxxxx} = 0$$

- magneto-acoustic waves
- gravitational waves influenced by surface tension
- waves in nonlinear circuits

$$u_t + uu_x + du_{xxx} + bu_{xxxxx} = 0$$

- capillary-gravity waves
- waves in chains of coupled nonlinear oscillators
- magneto-sound waves in plasmas
- interfacial waves in a two-fluid system

hierarchy of dispersive waves:

$$u_t + uu_x + d_1 u_{xxx} + b(u_t + uu_x + d_2 u_{xxx})_{xx} = 0$$

$u_{xxx}$

$u_{xxxxx}$

- waves in granular materials

## Näiteid CENSist:

$$u_t + kuu_x + du_{xxx} = 0$$

$$u_t + [P(u)]_x + du_{xxx} + bu_{xxxx} = 0$$

$$P(u) = -\frac{1}{2}u^2 + \frac{1}{4}u^4$$

$$u_t + kuu_x + du_{xxx} + r(v_x^2)_{xx} = 0$$

$$(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0$$

$$u_{\xi t} + f(u)u_{\xi} + g(u) = 0$$

$$f(u) = k_0 + k_1u + k_2u^2, g(u) = g_0u$$

$$u_{tt} - c_1^2u_{xx} + m(u_{tt} - c_2^2u_{xx})_{xx} = 0$$